Exercise 1. Recall the definitions:

\[ \theta(p, \varepsilon) = \mathbb{P}_{p,\varepsilon}[x \leftrightarrow \Gamma] = \mathbb{P}_{p,\varepsilon}[C \cap \Gamma \neq \emptyset], \]

\[ \theta_r(p, \varepsilon) = \mathbb{P}_{p,\varepsilon}[C_r \cap \Gamma \neq \emptyset], \]

\[ \chi(p, \varepsilon) = \mathbb{E}_{p,\varepsilon}[|C|1_{\{x \not\in \Gamma\}}]. \]

Show that

\[ \theta_r(p, \varepsilon) = 1 - \sum_{n=1}^{\infty} \mathbb{P}_p[|C_r| = n](1 - \varepsilon)^n, \]

\[ \theta(p, \varepsilon) = 1 - \sum_{n=1}^{\infty} \mathbb{P}_p[|C| = n](1 - \varepsilon)^n, \]

\[ \chi(p, \varepsilon) = \sum_{n=1}^{\infty} \mathbb{P}_p[|C| = n] \cdot n(1 - \varepsilon)^n. \]

Deduce that

\[ \chi = (1 - \varepsilon) \frac{\partial \theta}{\partial \varepsilon}, \]

\[ \frac{\partial \theta_r}{\partial \varepsilon} = \sum_{n=1}^{\infty} \mathbb{P}_p[|C_r| = n]n(1 - \varepsilon)^{n-1}. \]

Solution to Exercise 1. The series expansion follows from

\[ 1 - \theta(p, \varepsilon) = \sum_{C : |C| = n} \mathbb{P}_p[C = C] \cdot \mathbb{P}[C \cap \Gamma = \emptyset] = \sum_{n=1}^{\infty} \mathbb{P}_p[|C| = n] \cdot (1 - \varepsilon)^n. \]

Similarly for \( \theta_r. \)

As for \( \chi, \)

\[ \chi(p, \varepsilon) = \sum_{n=1}^{\infty} n \mathbb{P}_p[|C| = n, C \cap \Gamma = \emptyset] = \sum_{n=1}^{\infty} \mathbb{P}_p[|C| = n]n(1 - \varepsilon)^n. \]

The differential identities follow simply. \( \square \)
Exercise 2. Show that as \( r \to \infty \),
\[
\theta_r \nearrow \theta \quad \text{and} \quad \frac{\partial \theta_r}{\partial p} \to \frac{\partial \theta}{\partial p} \quad \text{and} \quad \frac{\partial \theta_r}{\partial \varepsilon} \to \frac{\partial \theta}{\partial \varepsilon}.
\]

Solution to Exercise 2. The events \( \{ C_r \cap \Gamma \neq \emptyset \} \) increase to \( \{ C \cap \Gamma \neq \emptyset \} \), which gives the first limit. This also gives convergence of
\[
\mathbb{P}_{p+\delta}[C_r \cap \Gamma \neq \emptyset] - \mathbb{P}_p[C_r \cap \Gamma \neq \emptyset] \to \mathbb{P}_{p+\delta}[C \cap \Gamma \neq \emptyset] - \mathbb{P}_p[C \cap \Gamma \neq \emptyset]
\]
for any \( \delta > 0 \), so the \( \partial p \)-derivatives converge as well.

Note that the random variables \( 1_{\{|C_r| = n\}} \) are bounded uniformly by 1 and converge to \( 1_{\{|C| = n\}} \). Thus, the expansions for the \( \partial \varepsilon \)-derivatives also converge. \( \square \)

Exercise 3. Prove that
\[
\mathbb{P}_{p,\varepsilon}[|C_r \cap \Gamma| = 1] = \varepsilon \frac{\partial \theta_r}{\partial \varepsilon}.
\]

Solution to Exercise 3. The usual expansion gives
\[
\mathbb{P}_{p,\varepsilon}[|C_r \cap \Gamma| = 1] = \sum_{n=1}^{\infty} \mathbb{P}_p[|C_r| = n] n \varepsilon (1 - \varepsilon)^{n-1},
\]
which is of the required form. \( \square \)