Probability

Solutions to Mock Exam

Question 1.

- $\mathbb{P}[B|A] = \mathbb{P}[B \cap A]/\mathbb{P}[A]$. So we need to calculate $\mathbb{P}[B \cap A]$. The inclusion-exclusion principle gives that

$$\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[B \cap A],$$

so $\mathbb{P}[B \cap A] = \frac{1}{2} + \frac{1}{3} - \frac{3}{4} = \frac{1}{12}$. Thus, $\mathbb{P}[B|A] = (1/12)/(1/2) = 1/6$.

- Let $X = A \cup B$ and $Y = A \cup C$. So

$$\mathbb{P}[A \cup B \cup C] = \mathbb{P}[X \cup Y] = \mathbb{P}[X] + \mathbb{P}[Y] - \mathbb{P}[X \cap Y] = \mathbb{P}[A \cup B] + \mathbb{P}[A \cup C] - \mathbb{P}[(A \cup B) \cap (A \cup C)].$$

Or

$$\mathbb{P}[(A \cup B) \cap (A \cup C)] = \frac{1}{3} + \frac{1}{4} - \frac{1}{2} = \frac{1}{12}.$$

Note that for any set S we have that

$$A^c \cap (A \cup S) = (A^c \cap A) \cup (A^c \cap S) = A^c \cap S,$$

so

$$(A \cup B) \cap (A \cup C) = [(A \cup B) \cap (A \cup C) \cap A] \biguplus [(A \cup B) \cap (A \cup C) \cap A^c]$$
$$= A \biguplus [(A^c \cap B) \cap (A^c \cap C)] = A \biguplus (A^c \cap B \cap C).$$

Thus,

$$\frac{1}{12} = \mathbb{P}[(A \cup B) \cap (A \cup C)] = \mathbb{P}[A] + \mathbb{P}[C \cap B \cap A^c] = \frac{1}{12} + \mathbb{P}[C \cap B \cap A^c].$$

So $\mathbb{P}[C \cap B \cap A^c] = 0$.

- First note that since $B \subset A \cup B$ we have that

$$\mathbb{P}[A \cup B] - \mathbb{P}[B] = \mathbb{P}[(A \cup B) \setminus B] = \mathbb{P}[A \cap B^c].$$

Next, we have that A, B are independent if and only if A, B^c are independent; indeed, since $A = (A \cap B^c) \uplus (A \cap B)$,

$$\mathbb{P}[A] - \mathbb{P}[A \cap B^c] = \mathbb{P}[A \cap B].$$

So $\mathbb{P}[A \cap B^c] = \mathbb{P}[A]\,\mathbb{P}[B^c]$ if and only if $\mathbb{P}[A \cap B] = \mathbb{P}[A]\,\mathbb{P}[B]$.

Finally, we conclude that A, B are independent, if and only if A, B^c are independent, which is if and only if $\mathbb{P}[A \cup B] - \mathbb{P}[B] = \mathbb{P}[A]\,\mathbb{P}[B^c]$, which is if and only if $\mathbb{P}[A \cup B] = \mathbb{P}[A]\,\mathbb{P}[B^c] + \mathbb{P}[B]$.

$\square$

Question 2.

- We use the formula from class for absolutely continuous random variables:

$$\mathbb{E}[Z] = \int_0^\infty (\mathbb{P}[Z > t] - \mathbb{P}[Z \leq -t])dt.$$

Since Z is continuous, and symmetric we get that

$$\mathbb{P}[Z \leq -t] = \mathbb{P}[Z < -t] = \mathbb{P}[-Z > t] = 1 - F_{-Z}(t) = 1 - F_Z(t).$$

So,

$$\mathbb{E}[Z] = \int_0^\infty (1 - F_Z(t) - (1 - F_Z(t)))dt = 0.$$

- Using linearity of expectation we have that

$$\mathbb{E}[X_n - Y_n] = \mathbb{E}[X_n] - \mathbb{E}[Y_n] = 0.$$

Since X_n, Y_n are independent

$$\mathrm{Var}[X_n - Y_n] = \mathrm{Var}[X_n] + \mathrm{Var}[-Y_n] = \mathrm{Var}[X_n] + \mathrm{Var}[Y_n] = \frac{2}{\lambda^2}.$$

- For each n let $Z_n = X_n - Y_n$. So $\mathbb{E}[Z_n] = 0$ and $\mathrm{Var}[Z_n] = 2\lambda^{-2}$. Also, $(Z_n)_n$ is an independent sequence of random variables. The central limit theorem now tells us that

$$\frac{\lambda}{\sqrt{2n}} \sum_{k=1}^n Z_k \xrightarrow{\mathcal{D}} N(0,1).$$

That is,

$$\mathbb{P}[\lambda \sum_{k=1}^n Z_k \leq t\sqrt{2n}] \to \mathbb{P}[N(0,1) \leq t].$$

Specifically, for $t = 0$ we get that

$$\mathbb{P}[\sum_{k=1}^n X_k \leq \sum_{k=1}^n Y_k] = \mathbb{P}[\sum_{k=1}^n Z_k \leq 0] \to \mathbb{P}[N(0,1) \leq 0].$$

So we only have to show that $\mathbb{P}[N(0,1) \leq 0] = 1/2$. This follows by using a change of variables $u = -s$ so $ds = -du$ and

$$\mathbb{P}[N(0,1) \leq 0] = \int_{-\infty}^0 \frac{1}{\sqrt{2\pi}} e^{-s^2/2} ds = -\int_\infty^0 \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du = \mathbb{P}[N(0,1) > 0].$$

Since $1 = \mathbb{P}[N(0,1) \le 0] + \mathbb{P}[N(0,1) > 0]$ we get that $\mathbb{P}[N(0,1) \le 0] = \mathbb{P}[N(0,1) > 0] = 1/2$.

□

Question 3.

- Since the range of both X and Y is $\{1, 2, \ldots\}$, the range of $X - Y$ is $\mathbb{Z}$.
- We use the fact that $(\{Y = k\})_{k=1}^{\infty}$ are pairwise disjoint events such that $\mathbb{P}[\bigcup_{k=1}^{\infty} \{Y = k\}] = 1$. So for any event A,

$$\mathbb{P}[A] = \sum_{k=1}^{\infty} \mathbb{P}[A \cap \{Y = k\}].$$

For $z \ge 0$ we have that

$$\begin{aligned} f_{X-Y}(z) = \mathbb{P}[X = Y + z] &= \sum_{k=1}^{\infty} \mathbb{P}[X = k + z, Y = k] \\ &= \sum_{k=1}^{\infty} \mathbb{P}[X = k + z]\,\mathbb{P}[Y = k] = \sum_{k=1}^{\infty} p(1-p)^{k+z-1} q(1-q)^{k-1} \\ &= pq(1-p)^z \sum_{k=1}^{\infty} [(1-p)(1-q)]^{k-1} = \frac{pq}{1-(1-p)(1-q)}(1-p)^z. \end{aligned}$$

Similarly, for $z < 0$ we have

$$\begin{aligned} f_{X-Y}(z) &= \sum_{k=1}^{\infty} \mathbb{P}[X = k, Y = k - z] = \sum_{k=1}^{\infty} \mathbb{P}[X = k]\,\mathbb{P}[Y = k - z] \\ &= \sum_{k=1}^{\infty} p(1-p)^{k-1} q(1-q)^{k-z-1} = \frac{pq}{1-(1-p)(1-q)}(1-q)^{-z}. \end{aligned}$$

Finally, note that $1 - (1-p)(1-q) = p + q - pq$.

□

Question 4.

- Define $Y = X\mathbf{1}_{\{X \le m\}}$. Note that if $X(\omega) \le m$ then $Y(\omega) = X(\omega)$. Thus, $\mathbb{P}[Y = X] \ge \mathbb{P}[X \le m] = 1$. Also, since $m > 0$, then for any $\omega \in \Omega$, if $X(\omega) \le m$, then $Y(\omega) = X(\omega) \le m$, and if $X(\omega) > m$ then $Y(\omega) = 0 \le m$. Thus, $Y \le m$.
- For any λ,

$$\mathbb{P}[X \le \lambda] = \mathbb{P}[X \le \lambda, X = Y] + \mathbb{P}[X \le \lambda, X \ne Y] = \mathbb{P}[Y \le \lambda, X = Y] + \mathbb{P}[Y \le \lambda, X \ne Y] = \mathbb{P}[Y \le \lambda],$$

where we have used the fact that

$$\mathbb{P}[X \le \lambda, X \ne Y] = 0 = \mathbb{P}[Y \le \lambda, X \ne Y].$$

- Since $\mathbb{P}[X - Y = 0] = 1$ we get that $\mathbb{E}[X - Y] = 0$ so $\mathbb{E}[X] = \mathbb{E}[Y]$.
- For any $\lambda < m$ we have

$$\mathbb{P}[X \le \lambda] = \mathbb{P}[Y \le \lambda] = \mathbb{P}[m - Y \ge m - \lambda].$$

Since $m - Y \ge 0$ we have using Markov's inequality,

$$\mathbb{P}[X \le \lambda] = \mathbb{P}[m - Y \ge m - \lambda] \le \frac{\mathbb{E}[m - Y]}{m - \lambda} = \frac{m - \mathbb{E}[X]}{m - \lambda}.$$

□

Question 5.

- Calculate:

$$\mathrm{Cov}[X + Y, X - Y]\,\mathrm{Cov}[X, X - Y] + \mathrm{Cov}[Y, X - Y] = \mathrm{Cov}[X, X] - \mathrm{Cov}[X, Y] + \mathrm{Cov}[Y, X] - \mathrm{Cov}[Y, Y]$$

$$= \mathrm{Var}[X] - \mathrm{Var}[Y] = 0.$$

- Let X have range $\{1, -1\}$ and density $f_X(1) = f_X(-1) = 1/2$. So $\mathbb{E}[X] = 0$ and $\mathrm{Var}[X] = \mathbb{E}[X^2] = 1$. Let Z be independent of X have range $\{-1, 1\}$ and density $f_Z(1) = f_X(-1) = 1/2$. Let $Y = XZ$. Since X, Z are independent we have that $\mathbb{E}[Y] = \mathbb{E}[XZ] = \mathbb{E}[X]\,\mathbb{E}[Z] = 0$ and $\mathrm{Var}[Y] = \mathbb{E}[Y^2] = \mathbb{E}[X^2 Z^2] = 1$.

  Now $X + Y = X(1 + Z)$ and $X - Y = X(1 - Z)$. Note that

$$\mathbb{P}[X+Y = 2, X-Y = 2] = 0 \ne \frac{1}{4}\cdot\frac{1}{4} = \mathbb{P}[X = 1, Z = 1]\cdot\mathbb{P}[X = 1, Z = -1] = \mathbb{P}[X+Y = 2]\cdot\mathbb{P}[X-Y = 2].$$

So $X + Y, X - Y$ are not independent.

□