1.1. Functions of Random Variables

Exercise 1.1. *Let* $U \sim U[0,1]$. *Show that* $X = \frac{1}{1-U}$ *is absolutely continuous, and calculate the density of* X.

Solution (exercise 1). For any positive t,

$$F_X(t) = \mathbb{P}[X \leq t] = \mathbb{P}[U \leq 1 - 1/t] = \begin{cases} 0 & t < 1 \\ 1 - 1/t & t \geq 1 \end{cases}$$

If $t < 0$ then

$$F_X(t) = \mathbb{P}[X \leq t] = 0.$$

We can take

$$f_X(s) = \begin{cases} 0 & s < 1 \\ s^{-2} & s \geq 1 \end{cases}$$

□

Exercise 1.2. *Let* $U \sim U[0,1]$. *Let* $X = \sin(\pi U)$. *Calculate* $\mathbb{P}[X \leq 1/2]$.

Solution (exercise 2). The solution to $\sin(\pi u) = 1/2$ is $\pi u = \pi/6$ (use equilateral triangle cut in half). sin is monotone increasing up to $\pi/2$ and decreasing from $\pi/2$ to π. Thus, $\sin(\pi u) \leq 1/2$ if and only if $\pi u \in [0, \pi/6] \cup [\pi - \pi/6, \pi]$.

So,

$$\mathbb{P}[X \leq 1/2] = \mathbb{P}[U \in [0, 1/6] \uplus U \in [5/6, 1]] = \tfrac{1}{6} + \tfrac{1}{6} = \tfrac{1}{3}.$$

□

Exercise 1.3. *X is absolutely continuous with density*

$$f_X(t) = \begin{cases} 0 & t \not\in [0,5] \\ ct^2 & 0 < t < 5 \end{cases}$$

Find c. *Find the distribution function of* X, F_X. *Find* t *such that* $\mathbb{P}[X < t] = 1/3$.

Exercise 1.4. *Let* $X \sim \mathrm{Exp}(\lambda)$. *Let* $Y = \lfloor X \rfloor$. *Show that* Y *is discrete. What is the probability density function of* Y*?*

Show that $Y \sim \mathrm{Geo}(p)$. *What is* p*?*

Solution (exercise 4). It is simple that $\mathbb{P}[Y \in \mathbb{N}] = 1$. So Y is discrete.

For any $n \in \mathbb{N}$,

$$\mathbb{P}[Y = n] = \mathbb{P}[n \le X < n+1] = \int_n^{n+1} \lambda e^{-\lambda s} ds = e^{-\lambda n} - e^{-\lambda(n+1)} = e^{-\lambda n} \cdot \left(1 - e^{-\lambda}\right).$$

So $p = 1 - e^{-\lambda}$. □

Exercise 1.5. *Let $X \sim N(0, \sigma)$. What is $\mathbb{P}[X < 0]$? What is $\mathbb{P}[X > 0]$?*

Let $Y = \mathrm{sign}(X) = \mathbf{1}_{\{X>0\}} - \mathbf{1}_{\{X<0\}}$. Show that Y is discrete and compute the probability density function of Y. Let $Z = \frac{1+Y}{2}$. What is the distribution of Z?

Solution (exercise 5). Y is ± 1 with probability 1/2 each, so $Z \sim \mathrm{Ber}(1/2)$.

Indeed,

$$\mathbb{P}[X > 0] + \mathbb{P}[X < 0] = 1 - \mathbb{P}[X = 0] = 1$$

and since f_X is symmetric around 0,

$$\mathbb{P}[X > 0] = \int_0^\infty f_X(t)dt = \int_0^\infty f_X(-t)dt = \int_{-\infty}^0 f_X(t)dt = \mathbb{P}[X < 0].$$

□

Exercise 1.6. *Let $X \sim N(20, 5)$. The price of a stock has the distribution of $X^+ = X \vee 0 = \max\{X, 0\}$. If I bought the stock at 17 what is the distribution of my profit?*

Is Y discrete? Is Y continuous?

Solution (exercise 6). If Y is the profit, then $Y = X^+ - 17$. So

$$\mathbb{P}[Y \le t] = \mathbb{P}[X^+ \le t + 17]$$

if $t \le 17$ this is 0. For $t > -17$,

$$\begin{aligned}\mathbb{P}[Y \le t] &= \mathbb{P}[X^+ \le t + 17] = \mathbb{P}[X \le 0] + \mathbb{P}[0 < X \le t + 17] \\ &= \mathbb{P}[X \le t + 17] = \int_{-\infty}^{t+17} f_X(s)ds = \int_{-\infty}^{t} f_X(s + 17)ds.\end{aligned}$$

Y is not continuous because $\mathbb{P}[Y = -17] = \mathbb{P}[X^+ = 0] = \mathbb{P}[X \le 0] > 0$.

Y is not discrete because if R is the (countable) range of Y, then

$$1 = \mathbb{P}[Y \in R] = \mathbb{P}[X^+ \in R + 17] = \mathbb{P}[X \in (R+17)^+] \leq \sum_{r \in R} \mathbb{P}[X = (r+17)^+] = 0,$$

where $(R+17)^+ = \{(r+17) \vee 0 \ : \ r \in R\}$. □

Exercise 1.7. *Let $X \sim U[-1,1]$. Show that $Y = e^X$ is absolutely continuous and compute the density of Y.*

Solution (exercise 7). For any $u \in [-1,1]$, $e^u \leq t$ if and only if $t \geq e^{-1}$ and $u \leq \log t$. So if $t < e^{-1}$ then $\mathbb{P}[Y \leq t] = 0$. If $e^{-1} \leq t \leq e$ then

$$\mathbb{P}[Y \leq t] = \mathbb{P}[X \leq \log t] = \frac{1 + \log t}{2}.$$

If $t > e$ then $\mathbb{P}[Y \leq t] = 1$. So

$$F_Y(t) = \begin{cases} 0 & t < e^{-1} \\ \frac{1+\log t}{2} & t \in [e^{-1}, e] \\ 1 & t > e \end{cases}$$

If we take

$$f_Y(s) = \begin{cases} \frac{1}{2s} & s \in [e^{-1}, e] \\ 0 & s \notin [e^{-1}, e], \end{cases}$$

then

$$F_Y(t) = \int_{-\infty}^{t} f_Y(s) ds.$$

□

Exercise 1.8. *Let X be absolutely continuous with density*

$$f_X(t) = \begin{cases} \frac{1}{2}\cos t & t \in [-\frac{\pi}{2}, \frac{\pi}{2}] \\ 0 & \textit{otherwise} \end{cases}$$

Show that $Y = \sin X$ is absolutely continuous, and calculate the density of Y.

Solution (exercise 8). Since $\mathbb{P}[X \in [-\frac{\pi}{2}, \frac{\pi}{2}]] = 1$ we have that $\mathbb{P}[Y \in [-1, 1]] = 1$. Now, for any $t \in [-1, 1]$,

$$F_Y(t) = \mathbb{P}[Y \leq t] = \mathbb{P}[X \leq \arcsin t] = \int_{-\infty}^{\arcsin t} f_X(s) ds$$
$$= \int_{-\pi/2}^{\arcsin t} \tfrac{1}{2} \cos s ds = \tfrac{1}{2} \cdot (\sin(\arcsin t) - \sin(-\pi/2)) = \tfrac{t+1}{2}.$$

So for $f_Y(s) = \frac{1}{2} \cdot \mathbf{1}_{[-1,1]}$, we have that f_Y is the density of Y, which is the same as saying that $Y \sim U[-1, 1]$. $\square$