Introduction to Probability

Exercise sheet 1

This exercise sheet is to review some facts from set theory.

Recall the definitions:

For $A, B \subset \Omega$:

$$A \cup B = \{\omega \ : \ \omega \in A \text{ or } \omega \in B\} \qquad \text{and} \qquad A \cap B = \{\omega \ : \ \omega \in A \text{ and } \omega \in B\}.$$

$$A \setminus B = \{\omega \ : \ \omega \in A, \omega \notin B\} \qquad \text{and} \qquad A^c = \Omega \setminus A.$$

$$A \triangle B = (A \setminus B) \cup (B \setminus A).$$

$$A \times B = \{(a, b) \ : \ a \in A, b \in B\}.$$

Exercise 1. Prove or give a counter-example:

(a) $A \setminus B \subset A$.

(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(c) $A \cup B = A \cup (B \setminus A)$.

(d) $A \setminus (B \setminus C) = (A \setminus B) \setminus C$.

(e) $(A \setminus B) \setminus C = (A \setminus B) \setminus (B \setminus C)$.

(f) $A \setminus (B \setminus C) = (A \setminus B) \cup (B \cap C)$.

(g) $(A \cup B) \cap A = (A \cap B) \cup A$.

(h) $A \triangle B = (A \cap B^c) \cup (B \cap A^c)$.

Exercise 2. Show the following:

(a) $(A \setminus B) \cup B = A$ if and only if $B \subseteq A$.

(b) $A \cup B = A \cap B$ if and only if $A = B$.

(c) $(A \cap B) \cup C = A \cap (B \cup C)$ if and only if $C \subseteq A$.

(d) $A \triangle B = (A \cup B) \setminus (A \cap B)$.

(e) $A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C)$.

Exercise 3 (De-Morgan)**.** Show that:

(a) $\left(\bigcup_n A_n\right)^c = \bigcap_n A_n^c$.

(b) $\left(\bigcap_n A_n\right)^c = \bigcup_n A_n^c$.

Exercise 4. For any $k \geq 1$ define $B_k = \bigcup_{n=1}^k A_n$. Define $C_k = A_k \setminus B_{k-1}$, where $B_0 = \emptyset$.

Show that

(a) (C_k) are mutually disjoint; *i.e.* for any $n \neq m$, $C_n \cap C_m = \emptyset$.

(b) For any n,

$$\bigcup_{k=1}^{n} C_k = \bigcup_{k=1}^{n} A_k.$$

(c) $\bigcup_n C_n = \bigcup_n A_n$.

Exercise 5. Assume that $B \subseteq A \subseteq C$. Let X be a set such that $A \cap X = B$ and $A \cup X = C$. Prove that $X = B \cup (C \setminus A)$.

Exercise 6. Let A be a finite set of size $|A| = n$. Let $B, C \subseteq A$ be non-empty subsets such that $B \cap C = \emptyset$ and $B \cup C = A$. What is the maximal size of $B \times C$? What is the minimal size?

Exercise 7. Let $A_j \subset \Omega_j$ for $j = 1, 2, \ldots, n$ and let $\Omega = \Omega_1 \times \cdots \times \Omega_n$. For any j, let $B_j = \Omega_1 \times \cdots \times \Omega_{j-1} \times A_j^c \times \Omega_{j+1} \times \cdots \times \Omega_n$. Show that

$$(A_1 \times A_2 \times \cdots \times A_n)^c = B_1 \cup B_2 \cup \cdots \cup B_n.$$

Exercise 8. Consider Ω_j for $j = 1, 2, \ldots, n$ and let $\Omega = \Omega_1 \times \cdots \times \Omega_n$. For every j, let $(A_m^{(j)})_{m \in \mathbb{N}}$ be a collection of subsets of Ω_j. For every m, let $B_m = A_m^{(1)} \times \cdots \times A_m^{(n)}$. Show that

$$\bigcap_{m \in \mathbb{N}} B_m = \big(\bigcap_m A_m^{(1)}\big) \times \cdots \times \big(\bigcap_m A_m^{(n)}\big).$$

Exercise 9. Let $0 < \alpha < 1$. Prove that

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}.$$

Hint: Consider the finite sum $\sum_{n=0}^{N} \alpha^n$ and multiply it by $(1 - \alpha)$.