Introduction to Probability

Exercise sheet 2

Exercise 1. Let A, B, C be events in a probability space. Find an experesion for the following:

(a) Only A occurs.

(b) A and B occur, but not C.

(c) At least one of A, B, C occurs.

(d) At least two of A, B, C occur.

(e) None of A, B, C occur.

(f) All of A, B, C occur.

(g) At most one of A, B, C occurs.

(h) At most two of A, B, C occur.

(i) Exactly two of A, B, C occur.

Exercise 2. A fair die and a fair coin are tossed.

- Propose a suitable sample space Ω.
- Describe the events:
  (a) The number on the die is even.
  (b) The coin falls tails.
  (c) The number on the die is odd or the coin falls heads.
  (d) The number on the die is even and the coin is tails, or the number on the die is odd and the coin is heads.

Exercise 3. A fair die and a fair coin are tossed. All possible outcomes are equally likely.

Compute the probability of the events from the previous exercise.

Exercise 4. A coin with one side red and one side blue is tossed four times.

- Propose a suitable sample space Ω.
- Describe the following events:
  (a) The first toss falls blue.
  (b) Both the first toss and the second are red.
  (c) There are exactly three red tosses.
  (d) Blue comes out at least once.

Exercise 5. A coin with one side red and one side blue is tossed four times. All outcomes are equally likely.

Compute the probabilities of the events in the previous exercise.

Exercise 6. Three fair dice ar tossed, all outcomes being equally likely. Propose a probability space, and compute the probability that the sum of the numbers on the dice is at most 4.

Exercise 7. Three urns contain balls. Urn A contains 2 black balls and 2 red balls. Urn B contains 3 white balls and 1 black ball. Urn C contains 1 black, 1 white and 1 red ball.

An urn is chosen, and then two balls are removed, one after the other.

The probability to chose urn A is $1/2$, the probability to chose urn B is $1/4$ and the probability to chose urn C is $1/4$. In each urn any ball is equally likely to be removed.

(a) Propose a probability space.
(b) Describe the following evens and calculate their probabilities.
(c) At least one black ball is removed.
(d) The second ball removed is white.
(e) At least one black ball is removed and the second ball removed is white.
(f) Both balls removed are the same color.

Exercise 8. Propose a sample space for the following experiment: A die is thrown repeatedly until a 6 is observed.

Exercise 9. An urn contains black and white balls. n balls are removed one after the other. A_j is the event that the j-th ball removed is white. Describe using set operations on the sets $A_1, \ldots, A_n$ the following events:

(a) All n balls removed are white.
(b) At least one ball removed is white.
(c) Exactly one ball removed is white.
(d) At least two balls removed are white.
(e) All n balls removed are the same color.

Exercise 10. On her way to work, Shir passes three traffic lights. The sample space $\Omega = \{0, 1, 2, 3\}$ describes the number of red lights Shir encounters. The probability of 1 red light is $1/10$, of 2 red lights is $2/10$, and of no red lights is $2/5$.

Calculate the following probabilities:

(a) Shir encounters 3 red lights.

(b) Shir encounters at least 1 red light.

(c) Shir encounters at least 1 green light.

(d) The number of green lights encountered is even.

(e) At most one green light is encountered.

Exercise 11. Let $\Omega = \mathbb{N} = \{0, 1, 2, \ldots, \}$. Define a probability measure on $(\Omega, 2^\Omega)$ by

$$\mathbb{P}(\{n\}) = C \cdot 2^{-n}.$$

What is C?

What is the probability of the event $\{n \in \Omega \ : \ n > 4\}$?

What is the probability of the event $\{n \in \Omega \ : \ n \text{ is even } \}$?

Exercise 12. Let A, B, C be events in some probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

(a) Show that

$$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C).$$

(b) Show that

$$\mathbb{P}(A^c \cap B^c) = 1 - \mathbb{P}(A) - \mathbb{P}(B) + \mathbb{P}(A \cap B).$$