Introduction to Probability

Exercise sheet 4

Exercise 1. Let X be a random variable on some probability spac $(\Omega, \mathcal{F}, \mathbb{P})$. Write the distribution function of the following random variables as an expression of F_X:

(a) $-X$.

(b) $X^+ := \max\{X, 0\}$.

(c) $X^- := \max\{-X, 0\}$.

Exercise 2. Let X be a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\phi : (\mathbb{R}, \mathcal{B}) \to (\mathbb{R}, \mathcal{B})$ be a measurable function. Show that $Y = \phi \circ X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B})$ is a random variable.

Exercise 3. Let $X \sim \mathrm{Bin(n,p)}$. What is the density of X^2?

Exercise 4. Let X be a random variable with distribution function F_X. Show that for any $a < b \in \mathbb{R}$,

- $\mathbb{P}(a < X \leq b) = F_X(b) - F_X(a)$.
- $\mathbb{P}(a < X < b) = F_X(b^-) - F_X(a)$.
- $\mathbb{P}(X = x) = F_X(x) - F_X(x^-)$. Deduce that F_X is continuous at x if and only if $\mathbb{P}(X = x) = 0$.

Exercise 5. Let $X \sim \mathrm{Exp}(1)$. What is the density of $Y = \log X$?

Exercise 6. Let $X \sim U([0,1])$. What is the density of X^2?

Exercise 7. Let $X_1, X_2, \ldots, X_n$ be n independent random variables, each disributed Bernoulli-p. Let $Y = X_1 + X_2 + \cdots + X_n$. What is the density of Y?

Exercise 8. Let B, C be independent random variables. Define the polynomial $p(x) := x^2 + Bx + C$. What is the probability that p has exactly on root, when:

(a) B, C are all independent Bernoulli-p.

(b) $B \sim \mathrm{Geo}(p)$ and $C = X^2$ for $X \sim \mathrm{Poi}(\lambda)$.

Exercise 9. Construct two probability spaces $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\Omega', \mathcal{F}', \mathbb{P}')$, and two joint distributions of random variables (X, Y) on $(\Omega, \mathcal{F}, \mathbb{P})$ and (X', Y') on $(\Omega', \mathcal{F}', \mathbb{P}')$ such that the following holds:

- The distribution $\mathbb{P}_X = \mathbb{P}'_{X'}$ and the distribution $\mathbb{P}_Y = \mathbb{P}'_{Y'}$.
- However, $\mathbb{P}_{(X,Y)} \neq \mathbb{P}'_{(X',Y')}$.