Introduction to Probability

Exercise sheet 5

Exercise 1. Let X be a random variable on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let Y be the random variable defined by $Y(\omega) = c$ for all $\omega \in \Omega$ ($c \in \mathbb{R}$ is some constant). Show that X, Y are independent.

Exercise 2. Let X be an absolutely continuous random variable. Let $Y = -X$. Show that Y is absolutely continuous with density $f_Y(s) = f_X(-s)$ for all $s \in \mathbb{R}$.

Exercise 3. Show that if $X = (X_1, \ldots, X_d)$ is a jointly discrete random variable, then also each of $X_1, \ldots, X_d$ are discrete random variables.

Exercise 4. Let $X_1, \ldots, X_d$ be discrete random variables. Show that $X_1, \ldots, X_d$ are mutually independent, if and only if

$$\forall\ (t_1, \ldots, t_d) \in \mathbb{R}^d \qquad f_{(X_1,\ldots,X_d)}(t_1, \ldots, t_d) = f_{X_1}(t_1) \cdot f_{X_2}(t_2) \cdots f_{X_d}(t_d).$$

Exercise 5. Let X, Y be discrete random variables. Show that

$$f_{X,Y}(x, y) = f_{X|Y}(x|y) f_Y(y),$$

for all y such that $f_Y(y) > 0$.

Exercise 6. Let $Z = (X, Y)$ be an absolutely continuous 2-dimensional random variable with density

$$f_Z(t, s) = \frac{1}{2\pi} \exp\left(-\frac{t^2 + s^2}{2}\right).$$

- Show that $X \sim N(0, 1)$ and that $Y \sim N(0, 1)$.
- Show that X, Y are independent.

Exercise 7. Let $Z = (X, Y)$ be a 2-dimensional absolutely continuous random variable with density given by

$$f_Z(t, s) = \begin{cases} \frac{1}{t} & 0 < s \le t \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

- Show that f_Z is indeed a density.
- Calculate the marginal densities, f_X, f_Y.
- What is $f_{Y|X}(\cdot|t)$? What is the distribution of $Y|X = t$?

Exercise 8. Let $Z = (X, Y)$ be a 2-dimensional absolutely continuous random variable with density given by

$$f_Z(t,s) = \begin{cases} 1 & t \in [0,1/2]\ ,\ s \in [0,1], \\ 1 & t \in [1/2,1]\ ,\ s \in [1,2], \\ 0 & \text{otherwise.} \end{cases}$$

- Show that f_Z is indeed a density.
- Calculate the marginal distribution functions and densities, F_X, f_X, F_Y, f_Y.
- Are X, Y independent?

Exercise 9. Let $X \sim \mathrm{Bin}(n,p)$ and $Y \sim \mathrm{Ber}(p)$, such that X, Y are independent. Show that $X + Y \sim \mathrm{Bin}(n+1,p)$. (*Hint:* Show Pascal's triangle formula $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.)

Conclude that for $Z = \sum_{k=1}^{n} Y_k$ where $Y_1, \ldots, Y_n$ are mutually independent random variables with $Y_j \sim \mathrm{Ber}(p)$, we have that $Z \sim \mathrm{Bin}(n,p)$.