Introduction to Probability

Exercise sheet 6+ *(additional exercises)*

Exercise 1. Let X, Y be independent random variables, each with distribution $X \sim \text{Geo}(p)$ and $Y \sim \text{Geo}(q)$. Let $Z = \min\{X, Y\}$.

- Use the formula
$$\mathbb{E}[Z] = \sum_{k=0}^{\infty} \mathbb{P}[Z > k]$$
together with independence to show that
$$\mathbb{E}[Z] = \sum_{k=0}^{\infty} \mathbb{P}[X > k] \cdot \mathbb{P}[Y > k].$$
- Conclude that
$$\mathbb{E}[Z] = \frac{1}{1 - (1-p)(1-q)}.$$

(*)Exercise 2. Generalize the previous exercise to d independent random variables: Let $X_1, \dots, X_d$ be d-independent random variables, all with the same distribution, taking values in $\mathbb{N}$. Let $Z = \min\{X_1, \dots, X_d\}$. Then,
$$\mathbb{E}[Z] = \sum_{k=0}^{\infty} (\mathbb{P}[X_1 > k])^d.$$
So if $X_j \sim \text{Geo}(p)$ for all j, then
$$\mathbb{E}[Z] = \frac{1}{1 - (1-p)^d}.$$

Exercise 3. Let $X \sim \text{Geo}(p)$. Let $V = \mathbb{E}[X^2]$. Show that V satisfies the equation
$$V = 1 + (1-p)V + \frac{2(1-p)}{p}.$$
Conclude that $V = \frac{2-p}{p^2}$.

Exercise 4. At every minute a card is removed randomly from a deck of 52, all cards equally likely, and then the card is returned to the deck. The removals at every minute are all independent.

Let X be the number of minutes until a king is removed. What is $\mathbb{E}[X]$?

What is the expected number of minutes needed to remove two kings?

Exercise 5. Let $X \sim \text{Geo}(p)$. Let $m \in \mathbb{N}$ and set $Y = \max\{X, m\}$. What is $\mathbb{E}[Y]$?

Exercise 6. Let $U \sim U[0,1]$. Set $X = \cos(2\pi U)$ and $Y = \sin(2\pi U)$. Show that $\mathbb{E}[X] = \mathbb{E}[Y] = 0$. Show that $\mathrm{Cov}[X,Y] = 0$.

Are X, Y independent?

Exercise 7. Let X be a random variable with $0 < \mathrm{Var}[X] < \infty$. Show that

$$\mathbb{P}[|X - \mathbb{E}[X]| < 3\sqrt{\mathrm{Var}[X]}] \geq \frac{8}{9}.$$

Exercise 8. Let $\zeta = \sum_{k=1}^{\infty} k^{-2}$. Show that

$$f_X(k) = \begin{cases} \frac{1}{2\zeta k^2} & 0 \neq k \in \mathbb{Z} \\ 0 & k = 0 \end{cases}$$

is a density.

Let X be a discrete random variable with range $\mathbb{Z}$ and density f_X. Show that $\mathbb{E}[X]$ is not defined (X does not have expectation).