Introduction to Probability

Exercise sheet 6

Exercise 1. Let $(A_k)_k$ be a sequence of events (not necessarily disjoint). Let

$$X = \sum_{k=0}^{\infty} a_k \mathbf{1}_{A_k},$$

for positive numbers $a_k > 0$.

Show that

$$\mathbb{E}[X] = \sum_{k=0}^{\infty} a_k \, \mathbb{P}[A_k].$$

Exercise 2. Let X be a discrete random variable, with range $\mathbb{Z}$ such that $\mathbb{E}[X]$ exists. Show that

$$\mathbb{E}[X] = \sum_{k=0}^{\infty} (\mathbb{P}[X > k] - \mathbb{P}[X < -k]).$$

Exercise 3. Show that if X, Y are independent absolutely continuous random variables, then for any two measurable functions $g, h : \mathbb{R} \to \mathbb{R}$

$$\mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)] \cdot \mathbb{E}[h(Y)].$$

Exercise 4. Let $X \sim N(0, \sigma)$.

(a) Let $G(t) = -e^{-t^2/2}$ and $g(t) = te^{-t^2/2}$. Show that $G'(t) = g(t)$.

(b) Use integration by parts to show that for all $n \geq 2$,

$$\mathbb{E}[X^n] = \sigma^2 (n-1) \, \mathbb{E}[X^{n-2}].$$

Conclude that for even $n \geq 2$,

$$\mathbb{E}[X^n] = \sigma^n \prod_{k=1}^{n/2} (2k-1).$$

(c) Show that $-X \sim N(0, \sigma)$.

(d) Show that $|\mathbb{E}[X^n]| < \infty$ and $\mathbb{E}[(-X)^n] = \mathbb{E}[X^n]$, for any $n \in \mathbb{N}$.

(e) Deduce that if n is odd, then $\mathbb{E}[X^n] = 0$.

(f) What is $\mathbb{E}[X^4]$? $\mathbb{E}[X^6]$?

Exercise 5. Let $X \sim N(100, 10)$. Calculate $\mathbb{E}[(X - 100)^8]$. Calculate $\mathbb{E}[(X - 50)^2]$.

Exercise 6. Let $X \geq 0$ be a non-negative random variable. In this exercise we will show that

$$\mathbb{E}[X] = 0 \qquad \text{if and only if} \qquad \mathbb{P}[X = 0] = 1. \tag{1}$$

- First assume that X is discrete, and prove (1).
- Now let X be a general non-negative random variable. Consider $X_n = 2^{-n}\lfloor 2^n X \rfloor$. $X_n \nearrow X$ (why?). Show that $\mathbb{E}[X_n] = 0$ for all n.
- Deduce that for all n, $\mathbb{P}[X_n = 0] = 1$.
- Show that
$$\{X > 0\} = \lim_{n\to\infty} \{X_n > 0\}.$$
Deduce that $\mathbb{P}[X > 0] = 0$, and so $\mathbb{P}[X = 0] = 1$.
- For the "if" part: assume that $\mathbb{P}[X = 0] = 1$. Show that $\mathbb{E}[X] = 0$ by splitting into the discrete case, and approximating.
- Conclude that $\mathrm{Var}[X] = 0$ if and only if $\mathbb{P}[X = \mathbb{E}[X]] = 1$.

Exercise 7. Let X, Y be random variables with finite second moment. Show that if $\mathrm{Var}[X] = 0$ then $\mathrm{Cov}[X, Y] = 0$. Show that if $\mathbb{E}[X^2] = 0$ then $\mathbb{E}[XY] = 0$.

Exercise 8. Let $X_1, \ldots, X_n$ be random variables with finite second moment. Show that

$$\mathrm{Var}[X_1 + \cdots + X_n] = \sum_{k=1}^n \mathrm{Var}[X_k] + 2\sum_{j<k} \mathrm{Cov}[X_j, X_k].$$

Deduce the Pythagorian Theorem: If $X_1, \ldots, X_n$ are all pairwise uncorrelated, then

$$\mathrm{Var}[X_1 + \cdots + X_n] = \sum_{k=1}^n \mathrm{Var}[X_k].$$

Exercise 9. Prove a generalization of the arithmetic-geometric mean inequality:

Let $p_1, \ldots, p_n$ be numbers in $[0, 1]$ such that $\sum_{k=1}^n p_k = 1$. Let $a_1, \ldots, a_n$ be any positive numbers. Then,

$$\sum_{k=1}^n p_k a_k \geq \prod_{k=1}^n a_k^{p_k}.$$

Hint: Let X be a discrete random variable with density $f_X(a_k) = p_k$. Use Jensen's inequality on $-\log \mathbb{E}[X]$.