Introduction to Probability

Exercise sheet 7

Exercise 1. Let X, Y be random variables with finite second moment. Let A be the matrix

$$A = \begin{bmatrix} \mathrm{Cov}[X,X] & \mathrm{Cov}[X,Y] \\ \mathrm{Cov}[Y,X] & \mathrm{Cov}[Y,Y] \end{bmatrix}$$

- Show that $\det(A) = 0$ if and only if there exist numbers $a, b \in \mathbb{R}$ such that $\mathbb{P}[Y = aX + b] = 1$. (*Hint:* Consider $\mathrm{Var}[X' - Y']$ where $X' = \frac{X - \mathbb{E}[X]}{\sqrt{\mathrm{Var}[X]}}$ and $Y' = \frac{Y - \mathbb{E}[Y]}{\sqrt{\mathrm{Var}[Y]}}$).
- Define

$$\rho(X,Y) = \frac{\mathrm{Cov}[X,Y]}{\sqrt{\mathrm{Var}[X]\,\mathrm{Var}[Y]}}.$$

  Show that $|\rho(X,Y)| \leq 1$.
- Show that $|\rho(X,Y)| = 1$ if and only if there exist numbers $a, b \in \mathbb{R}$ such that $\mathbb{P}[Y = aX + b] = 1$.

Exercise 2. In this exercise we show that convergence in L^q implies convergence in L^p if $0 < p < q$.

Let $(X_n)_n, X$ be a sequence of random variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $0 < p < q$. Show that if (X_n) converges to X in L^q then (X_n) converges to X in L^p.

Exercise 3. Suppose that $X_n \xrightarrow{L^p} X$ and $X_n \xrightarrow{L^q} Y$. Show that $\mathbb{P}[X = Y] = 1$.

Suppose that $X_n \xrightarrow{\text{a.s.}} X$ and $X_n \xrightarrow{L^p} Y$. Show that $\mathbb{P}[X = Y] = 1$.

Suppose that $X_n \xrightarrow{P} X$ and $X_n \xrightarrow{P} Y$. Show that $\mathbb{P}[X = Y] = 1$.

Exercise 4. Show that if $X_n \xrightarrow{P} X$ then there exists a subsequence $(X_{n_k})_k$ such that $X_{n_k} \xrightarrow{\text{a.s.}} X$.

(*Hint:* Show that for every $k \geq 1$, there exists $n_k > n_{k-1}$ such that $\mathbb{P}[|X_{n_k} - X| > 2^{-k}] < 2^{-k}$. Now use Borel-Cantelli.)

Exercise 5. Prove the following generalization of the Law of Large Numbers: Let $X_1, X_2, \ldots,$ be mutually independent and identically distributed random variables with $\mathbb{E}[X_n] = \mu$ and $\mathbb{E}[X_n^2] < \infty$. Let $S_N = \sum_{n=1}^{N} X_n$. Then,

$$\frac{S_N}{N} \xrightarrow{\text{a.s.}} \mu.$$

Exercise 6. Prove the monotone convergence theorem for a.s. convergence:

Let $(X_n)_n$ be a sequence of non-negative random variables such that $X_n \xrightarrow{\text{a.s.}} X$. Assume further that $(X_n)_n$ is a non-decreasing monotone sequence. Prove that $\mathbb{E}[X_n] \nearrow \mathbb{E}[X]$.

(*Hint:* Consider $A = \{\omega \ : \ X_n(\omega) \to X(\omega)\}$. What do you know about $\mathbb{E}[X_n \mathbf{1}_{A^c}]$?)