Probability

Solutions to Exam 1, Fall 2012

Question 1.

- For every $\omega \in \Omega$ we have that

$$\inf_n Z_n(\omega) \leq t \iff \forall\, m > 0\ \exists\, n\ :\ Z_n(\omega) \leq t + \frac{1}{m}.$$

Thus,

$$\left\{\inf_n Z_n \leq t\right\} = \left\{\omega\ :\ \inf_n Z_n(\omega) \leq t\right\} = \bigcap_m \bigcup_n \left\{\omega\ :\ Z_n(\omega) \leq t + m^{-1}\right\} = \bigcap_m \bigcup_n \left\{Z_n \leq t + m^{-1}\right\}.$$

Since Z_n are all random variables, $\left\{Z_n \leq t + m^{-1}\right\}$ are events for all t, m, n, so $\{\inf_n Z_n \leq t\}$ is also an event for any t. Thus, $\inf_n Z_n$ is a measurable function, that is a random variable.

In a similar way,

$$\left\{\sup_n Z_n \leq t\right\} = \{\omega\ :\ \forall\ n\ Z_n(\omega) \leq t\} = \bigcap_n \{Z_n \leq t\},$$

so $\sup_n Z_n$ is a random variable as well.

- For every n, $Y_n = \inf_{k \geq n} X_k$ is a random variable by the previous bullet. Moreover, for any $\omega \in \Omega$, and any n,

$$Y_n(\omega) = \inf_{k \geq n} X_k(\omega) \leq \inf_{k \geq n+1} X_k(\omega) = Y_{n+1}(\omega),$$

since the second infimum is over a smaller set. Thus $(Y_n)_n$ is a monotone non-decreasing sequence.

This now implies that $Y = \lim_n Y_n = \sup_n Y_n$. This implies that Y is also a random variable by the first bullet.

- Since $X_k \geq 0$ for all k, we have that $Y_n \geq 0$ for all n. Since $(Y_n)_n$ is a monotone non-decreasing sequence of non-negative random variables, we get by monotone convergence that $\mathbb{E}[Y_n] \nearrow \mathbb{E}[Y]$.

- For any $\omega \in \Omega$ and any $k \geq n$ we have that

$$Y_n(\omega) = \inf_{m \geq n} X_m(\omega) \leq X_k(\omega).$$

Thus, $Y_n \leq X_k$ and so $\mathbb{E}[Y_n] \leq \mathbb{E}[X_k]$.

Since this holds for all $k \geq n$ we get that $\mathbb{E}[Y_n] \leq \inf_{k \geq n} \mathbb{E}[X_k]$.

The definition of $\liminf$ now gives that

$$\mathbb{E}[\liminf_n X_n] = \mathbb{E}[Y] = \lim_n \mathbb{E}[Y_n] \leq \lim_n \inf_{k \geq n} \mathbb{E}[X_k] = \liminf_n \mathbb{E}[X_n].$$

□

Question 2. Let X_k denote that change in height at the k-th step. Note that $\mathbb{P}[X_k = 1] = p$ and $\mathbb{P}[X_k = -1] = 1 - p$.

- $\mathbb{E}[X_k] = p - (1-p) = 2p - 1$. $\mathbb{E}[X_k^2] = p + (1-p) = 1$. So $\mathrm{Var}[X] = 1 - (2p-1)^2 = 4p(1-p)$.
- $S_n = \sum_{k=1}^n X_k$, and all A_k are independent. So $\mathbb{E}[S_n] = n(2p-1)$ (linearity) and $\mathrm{Var}[S_n] = 4np(1-p)$ (independence, Pythagoras' Theorem).
- Note

$$\mathbb{P}[S_n \leq \tfrac{n}{4}] = \mathbb{P}\left[\frac{S_n - n(2p-1)}{\sqrt{4np(1-p)}} \leq \frac{\frac{n}{4} - n(2p-1)}{\sqrt{4np(1-p)}}\right].$$

If $\frac{1}{4} = 2p - 1$, that is $p = 5/8$, then using the central limit theorem we have that

$$\mathbb{P}[S_n \leq \tfrac{n}{4}] = \mathbb{P}\left[\frac{S_n - n(2p-1)}{\sqrt{4np(1-p)}} \leq 0\right] \to \mathbb{P}[N(0,1) \leq 0].$$

So we only need to prove that $\mathbb{P}[N(0,1) \leq 0] = 1/2$. This follows by using a change of variables $u = -s$ so $ds = -du$ and

$$\mathbb{P}[N(0,1) \leq 0] = \int_{-\infty}^0 \frac{1}{\sqrt{2\pi}} e^{-s^2/2} ds = -\int_{\infty}^0 \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du = \mathbb{P}[N(0,1) > 0],$$

and since $1 = \mathbb{P}[N(0,1) \leq 0] + \mathbb{P}[N(0,1) > 0]$ we get that $\mathbb{P}[N(0,1) \leq 0] = \mathbb{P}[N(0,1) > 0] = 1/2$.

□

Question 3.

- For $0 \leq j \leq n$ we have that

$$f_X(j) = \sum_{k=j}^{m+j} f_{X,Y}(j,k) = \binom{n}{j} p^j (1-p)^{n-j} \cdot \sum_{k=j}^{m+j} \binom{m}{k-j} p^{k-j} (1-p)^{m-(k-j)}$$
$$= \binom{n}{j} p^j (1-p)^{n-j} \cdot \sum_{r=0}^{m} \binom{m}{r} p^r (1-p)^{m-r} = \binom{n}{j} p^j (1-p)^{n-j}.$$

That is, $X \sim \mathrm{Bin}(n,p)$.

- The range of $Y - X$ is

$$\{k - j \ : \ 0 \le j \le n \ , \ j \le k \le m + j\} = \{0 \le r \le m\} \,.$$

The density is (for $0 \le r \le m$):

$$f_{Y-X}(r) = \mathbb{P}[Y - X = r] = \sum_{j=0}^{n} \mathbb{P}[Y - X = r, X = j] = \sum_{j=0}^{n} f_{X,Y}(j, r + j)$$

$$= \sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \cdot \binom{m}{r} p^r (1-p)^{m-r} = \binom{m}{r} p^r (1-p)^{m-r}.$$

So $Y - X \sim \mathrm{Bin}(m, p)$.

- $X, Y - X$ are independent, since for all $0 \le j \le n, 0 \le r \le m$,

$$\mathbb{P}[X = j, Y - X = r] = f_{X,Y}(j, r+j) = \binom{n}{j} p^j (1-p)^{n-j} \cdot \binom{m}{r} p^r (1-p)^{m-r} = \mathbb{P}[X = j] \cdot \mathbb{P}[Y - X = r].$$

- $\mathbb{E}[X] = np$ and $\mathbb{E}[Y] = mp$ because $X \sim \mathrm{Bin}(n, p)$ and $Y \sim \mathrm{Bin}(m, p)$. Also,

$$\mathrm{Cov}[X, Y] = \mathrm{Cov}[X, Y - X] + \mathrm{Cov}[X, X] = 0 + \mathrm{Var}[X] = np(1 - p),$$

because $X, Y - X$ are independent.

□

Question 4. Let

$$f(s) = \begin{cases} 0 & s \notin [0, 1] \\ 2s & s \in [0, 1] \end{cases}$$

- For X to be absolutely continuous we need F_X to be continuous, specifically at 1, so $C1^2 = 1$, or $C = 1$. Now note that

$$F_X(t) = \begin{cases} 0 = \int_{-\infty}^{t} f(s) ds & t < 0 \\ t^2 = \int_0^t 2s ds = \int_{-\infty}^{t} f(s) ds & 0 \le t < 1 \\ 1 = \int_0^1 2s ds = \int_{-\infty}^{t} f(s) ds & t \ge 1. \end{cases}$$

So X is absolutely continuous with density $f_X = f$.

- $$\mathbb{E}[X] = \int_{-\infty}^{\infty} s f_X(s) ds = \int_0^1 2s^2 ds = \frac{2}{3}.$$

- $$\mathbb{E}[X^2] = \int_0^1 s^2 \cdot 2s ds = \frac{2}{4} = \frac{1}{2}.$$

So $\mathrm{Var}[X] = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$.

□