Question 1.

- Let $W = 2Y$. So $W \geq 0$. Since for all $\omega \in \Omega$, $X_n(\omega) \to X(\omega)$, and for all $n \left| X_n(\omega) \right| \leq Y$, we get that $|X| \leq Y$. Thus, $Z_n \leq |X| + |X_n| \leq 2Y = W$, for all $n$.
- If $\lim_n Z_n(\omega) = 0$ then $\limsup_n Z_n(\omega) = 0$. So
  \[ \Omega = \{ \omega : X(\omega) \to X \} \subset \{ \limsup_n Z_n = 0 \}, \]

so $\limsup_n Z_n(\omega) = 0$ for all $\omega \in \Omega$. Thus, $\mathbb{P}[\limsup_n Z_n = 0] = 1$ and $\mathbb{E}[\limsup_n Z_n] = \mathbb{E}[0] = 0$.
- Since $Z_n \leq W$ and since $\mathbb{E}[W] < \infty$, the expectation of $Z_n$ is finite. Using Fatou’s Lemma for non-negative random variables,
  \[ 0 \leq \limsup_n \mathbb{E}[Z_n] \leq \mathbb{E}[\limsup_n Z_n] = 0. \]

That is, $\mathbb{E}[|X - X_n|] \to 0$. This is the definition of $L^1$ convergence.

\[\square\]

Question 2.

- Define
  \[ g(s) = \begin{cases} 
  s^2 & s \geq 0 \\
  -s^2 & s < 0.
  \end{cases} \]

$g$ is continuous, and so measurable and odd. Let $X$ be a discrete random variable with range $\{-1, 2\}$ and density $\mathbb{P}[X = 2] = \frac{1}{3}, \mathbb{P}[X = -1] = \frac{2}{3}$. So $\mathbb{E}[X] = 0$. Calculate:
  \[ \mathbb{E}[g(X)] = g(2) \cdot \frac{1}{3} + g(-1) \cdot \frac{2}{3} = \frac{4}{3} - \frac{2}{3} = \frac{2}{3} \neq 0. \]

- If $X$ is symmetric then
  \[ \mathbb{P}[X = r] = F_X(r) - F_X(r^-) = F_{-X}(r) - F_{-X}(r^-) = \mathbb{P}[-X = r] = \mathbb{P}[X = -r]. \]
On the other hand, if \( P[X = r] = P[X = -r] = P[-X = r] \) for all \( r \), and \( X \) is discrete, then let \( R \) be the range of \( X, -X \) together. Then \( R \) is countable and
\[
F_{-X}(t) = \sum_{R \ni r \leq t} P[-X = r] = \sum_{R \ni r \leq t} P[X = r] = F_X(t).
\]

- Let \( g \) be odd such that \( E[g(X)] \) exists. Let \( R \) be the range of \( X, -X \) together. If \( P[X = r] > 0 \) then \( P[X = -r] > 0 \) so so \( -R = R \). Then,
\[
E[g(X)] = \sum_{r \in R} g(r) P[X = r] = \sum_{r \in R} g(r) P[X = -r] = -\sum_{r \in R} g(r) P[X = r] = -E[g(X)].
\]

Thus, \( 2 E[g(X)] = 0 \) which implies that \( E[g(X)] = 0 \).

\( \square \)

**Question 3.**

- For every \( r \) such that \( P[X = r] > 0 \) we have that \( f_{X,Y}(r, s) = f_X(r) f_{Y|X}(s|r) = f_X(r) f_{Z,s}(s) \).

Let \( R_X, R_Y \) be ranges for \( X \) and \( Y \) respectively. Now let \( h : \mathbb{R}^2 \to \mathbb{R} \) be the function \( h(x, y) = g(x)y \). Then,
\[
E[g(X)Y] = E[h(X, Y)] = \sum_{r \in R_X, s \in R_Y} h(r, s) f_{X,Y}(r, s) = \sum_{r \in R_X, s \in R_Y} g(r)s f_{X}(r) f_{Z,s}(s) = \sum_{r \in R} g(r) f_{X}(r) E[Z,s].
\]

- The marginal density of \( X \) is: for \( n \geq 1 \),
\[
f_X(n) = \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{n-1-k} \cdot p(1-p)^{n-1} = p(1-p)^{n-1}.
\]

That is, \( X \sim \text{Geo}(p) \).

For \( n \geq 1 \), and for \( 0 \leq k \leq n - 1 \),
\[
f_{Y|X}(k|n) = \frac{f_{X,Y}(n, k)}{f_X(n)} = \frac{\binom{n-1}{k} p^k (1-p)^{n-1-k} \cdot p(1-p)^{n-1}}{p(1-p)^{n-1}} = \binom{n-1}{k} p^k (1-p)^{n-1-k}.
\]

That is, for any \( n \geq 1 \), \( Y|X = n \sim \text{Bin}(n-1, p) \). So \( E[Y|X = n] = (n-1)p \).

- \( E[X] = 1/p \) since \( X \sim \text{Geo}(p) \). Using the first bullet with the function \( g(x) = 1 \),
\[
E[Y] = E[g(X)Y] = \sum_{n=1}^{\infty} p(1-p)^{n-1} E[Y|X = n] = \sum_{n=1}^{\infty} p(1-p)^{n-1} (n-1)p = E[(X-1)p] = 1 - p.
\]

Similarly, for \( g(x) = x \) we get
\[
E[XY] = E[g(X)Y] = \sum_{n=1}^{\infty} np(1-p)^{n-1}(n-1)p = E[X(X-1)p] = pE[X^2] - pE[X] = \frac{2(1-p)}{p}.
\]
Thus,
\[ \text{Cov}[X, Y] = \mathbb{E}[XY] - \mathbb{E}[X] \mathbb{E}[Y] = \frac{2(1 - p)}{p^2} - \frac{1}{p} \cdot (1 - p) = \frac{(2 - p)(1 - p)}{p^2}. \]

\[ \square \]

**Question 4.**

- \( \mathbb{P}[Y \leq t] = \mathbb{P}[X \geq \frac{1}{t}] = \int_{1/t}^{\infty} f_X(s) ds = \begin{cases} 0 & t < 1/2 \\ \int_{1/t}^{2} ds = 2 - 1/t & 1/2 \leq t < 1 \\ \int_{1}^{2} ds = 1 & t \geq 1. \end{cases} \)

- We want a non-negative function \( f \) such that

\[ \mathbb{P}[Y \leq t] = \int_{-\infty}^{t} f(s) ds. \]

Let

\[ f(s) = \begin{cases} 0 & s \notin [1/2, 1] \\ s^{-2} & s \in [1/2, 1]. \end{cases} \]

Then,

\[ \int_{-\infty}^{t} f(s) ds = \begin{cases} 0 & t < 1/2 \\ -s^{-1}\bigg|_{1/2}^{t} = 2 - 1/t & 1/2 \leq t < 1 \\ -s^{-1}\bigg|_{1/2}^{1} = 1 & t \geq 1. \end{cases} = F_Y(t). \]

So \( Y \) is absolutely continuous with density \( f_Y = f \).

- We can calculate the moments of \( Y \) in two ways. The first way is to use the density of \( X \) and \( Y = g(X) \) for \( g(s) = 1/s \):

\[ \mathbb{E}[Y] = \int_{1}^{2} \frac{1}{s} ds = \log 2. \]

\[ \mathbb{E}[Y^2] = \int_{1}^{2} \frac{1}{s^2} ds = \left. -s^{-1}\right|_{1}^{2} = \frac{1}{2}. \]

The other way is to use the density of \( Y \):

\[ \mathbb{E}[Y] = \int_{1/2}^{1} s \cdot s^{-2} ds = \log 2. \]

\[ \mathbb{E}[Y^2] = \int_{1/2}^{1} s^2 \cdot s^{-2} ds = \frac{1}{2}. \]

\[ \square \]