Probability

Solutions to Exam 3, Fall 2012

Question 1.

- Since $\Omega = \{X \leq \lambda\mu\} \uplus \{X > \lambda\mu\}$ we have that $\mathbf{1}_{\{X\leq\lambda\mu\}} + \mathbf{1}_{\{X>\lambda\mu\}} = 1$. Also, for any $\omega \in \Omega$, if $X(\omega) > \lambda\mu$ then $X(\omega)Y(\omega) = 0 \leq \lambda\mu$. If $X(\omega) \leq \lambda\mu$ then $X(\omega)Y(\omega) = X(\omega) \leq \lambda\mu$. Thus, $XY \leq \lambda\mu$.

  Finally, these two together imply that
$$\mathbb{E}[XZ] = \mathbb{E}[X] - \mathbb{E}[XY] \geq \mu - \lambda\mu = (1-\lambda)\mu.$$

- By the Cauchy-Schwarz inequality
$$(\mathbb{E}[XZ])^2 \leq \mathbb{E}[X^2]\,\mathbb{E}[Z^2] = \mathbb{E}[X^2]\cdot\mathbb{P}[X > \lambda\mu],$$
since $Z^2 = Z$.

- We have
$$(1-\lambda)^2\mu^2 \leq \mathbb{E}[XZ] \leq \mathbb{E}[X^2]\cdot\mathbb{P}[X > \lambda\mu].$$

- If $X \sim \mathrm{Exp}(\alpha)$, then
$$e^{-(1-\alpha)} = \mathbb{P}[X > \frac{1-\alpha}{\alpha}] = \mathbb{P}[X > (1-\alpha)\cdot\mathbb{E}[X]] \geq \alpha^2 \cdot \frac{(\mathbb{E}[X])^2}{\mathbb{E}[X^2]} = \frac{\alpha^2}{2}.$$
We have used that $\mathbb{E}[X^2] = \mathrm{Var}[X] + (\mathbb{E}[X])^2 = \frac{1}{\alpha^2} + \frac{1}{\alpha^2} = 2(\mathbb{E}[X])^2$.

$\square$

Question 2.

- Note that
$$\mathbb{E}[(X-c)^2] - \mathbb{E}[(X-\mu)^2] = \mathbb{E}[X^2] + c^2 - 2c\,\mathbb{E}[X] - \mathbb{E}[X^2] - \mu^2 + 2\mu\,\mathbb{E}[X]$$
$$= c^2 - 2c\mu + \mu^2 = (c-\mu)^2 \geq 0.$$

- Note that $\Omega = A \uplus B \uplus C$, so $1 = \mathbf{1}_A + \mathbf{1}_B + \mathbf{1}_C$. Also, for any $\omega \in A$ we have that $X(\omega) \leq s < t$, so $|X(\omega) - s| = s - X(\omega)$ and $|X(\omega) - t| = t - X(\omega)$. Similarly, for $\omega \in B$, $|X(\omega) - s| = X(\omega) - s$ and $|X(\omega) - t| = t - X(\omega)$, and for any $\omega \in C$, $|X(\omega) - s| = X(\omega) - s$ and $|X(\omega) - t| = X(\omega) - t$. Combining all this we get that
$$|X-s|\mathbf{1}_A = (s-X)\mathbf{1}_A \quad \text{and} \quad |X-t|\mathbf{1}_A = (t-X)\mathbf{1}_A,$$
$$|X-s|\mathbf{1}_B = (X-s)\mathbf{1}_B \quad \text{and} \quad |X-t|\mathbf{1}_A = (t-X)\mathbf{1}_B,$$
$$|X-s|\mathbf{1}_C = (X-s)\mathbf{1}_C \quad \text{and} \quad |X-t|\mathbf{1}_C = (X-t)\mathbf{1}_C.$$

Thus,

$$|X-s| = |X-s|(\mathbf{1}_A+\mathbf{1}_B+\mathbf{1}_C) = (s-X)\mathbf{1}_A + (X-s)\mathbf{1}_B + (X-s)\mathbf{1}_C,$$

$$|X-t| = |X-t|(\mathbf{1}_A+\mathbf{1}_B+\mathbf{1}_C) = (t-X)\mathbf{1}_A + (t-X)\mathbf{1}_B + (X-t)\mathbf{1}_C.$$

- Using the above

$$\begin{aligned}\mathbb{E}[|X-s|]-\mathbb{E}[|X-t|] &= \mathbb{E}[((s-X)-(t-X))\mathbf{1}_A + ((X-s)-(t-X))\mathbf{1}_B + ((X-s)-(X-t))\mathbf{1}_C] \\ &= \mathbb{E}[(s-t)\mathbf{1}_A] + \mathbb{E}[(2X-s-t)\mathbf{1}_B] + \mathbb{E}[(t-s)\mathbf{1}_C] \\ &= (s-t)(\mathbb{P}[A]-\mathbb{P}[C]) + \mathbb{E}[(2X-s-t)\mathbf{1}_B] + (s-t)\,\mathbb{E}[\mathbf{1}_B] - (s-t)\,\mathbb{E}[\mathbf{1}_B] \\ &= (s-t)(\mathbb{P}[A]-\mathbb{P}[C]-\mathbb{P}[B]) + 2\,\mathbb{E}[(X-t)\mathbf{1}_B].\end{aligned}$$

- Since $\Omega = A \uplus B \uplus C$ we have that $B \uplus C \subseteq A^c$. Thus, $\mathbb{P}[B]+\mathbb{P}[C] \leq \mathbb{P}[A^c] = 1-\mathbb{P}[A]$. If $\mathbb{P}[A] \geq 1/2$, then $1-\mathbb{P}[A] \leq \mathbb{P}[A]$. That is, $\mathbb{P}[A]-\mathbb{P}[B]-\mathbb{P}[C] \geq 0$.

  Also, for any $\omega \in B$, $X(\omega) < t$. Thus, $(X-t)\mathbf{1}_B \leq 0$. So altogether, if $\mathbb{P}[X \leq s] = \mathbb{P}[A] \geq 1/2$ then for $t > s$,

$$\mathbb{E}[|X-s|]-\mathbb{E}[|X-t|] = (s-t)(\mathbb{P}[A]-\mathbb{P}[C]-\mathbb{P}[B]) + 2\,\mathbb{E}[(X-t)\mathbf{1}_B] \leq 0.$$

- Let $X=-Y$ and $s=-r$. So $\mathbb{P}[X \leq s] = \mathbb{P}[Y \geq r] \geq \frac{1}{2}$. Since $\mathbb{E}[X^2]=\mathbb{E}[Y^2]$, X has finite second moment, and we get using the above that for any $t > s$,

$$\mathbb{E}[|X-s|]-\mathbb{E}[|X-t|] \leq 0.$$

  If we take $t=-p$ for $p<r$ then $t>s$. So

$$\mathbb{E}[|Y-r|] = \mathbb{E}[|X-s|] \leq \mathbb{E}[|X-t|] = \mathbb{E}[|Y-p|].$$

- Since for m we have that $\mathbb{P}[X \geq m] \geq 1/2$ and $\mathbb{P}[X \leq m] \geq 1/2$, combining the above we get that for any $c > m$ or any $c < m$ we have that

$$\mathbb{E}[|X-m|] \leq \mathbb{E}[|X-c|].$$

  For $c=m$ this is obvious.

□

Quesion 3.

- We want to find a density f_Y so that

$$\mathbb{P}[Y \leq t] = \int_{-\infty}^{t} f_Y(s)ds.$$

Since

$$\mathbb{P}[Y \leq t] = \mathbb{P}[\sqrt{X} \geq \tfrac{1}{t}] = \mathbb{P}[X \geq t^{-2}],$$

is we have $t^{-2} \in [0,1]$ then $\mathbb{P}[Y \leq t] = 1 - t^{-2}$.

A good guess would now be

$$f_Y(s) = \begin{cases} 2s^{-3} & \text{if } s \geq 1 \\ 0 & \text{if } s < 1 \end{cases}$$

Indeed, for any $t \geq 1$, we have that $t^{-2} \in [0,1]$, so

$$\int_{-\infty}^{t} f_Y(s)ds = \int_1^t 2s^{-3}ds = -s^{-2}\Big|_1^t = 1 - t^{-2} = \mathbb{P}[Y \leq t].$$

If $t < 1$, since $X \in [0,1]$, we have that $Y = \frac{1}{\sqrt{X}} \geq 1$, and so

$$\int_{-\infty}^{t} f_Y(s)ds = 0 = \mathbb{P}[Y \leq t].$$

So f_Y is the density of Y and Y is absolutely continuous.

- Since Y is absolutely continuous,

$$\mathbb{E}[Y] = \int s f_Y(s)ds = \int_1^{\infty} 2s^{-2}ds = -2s^{-1}\Big|_1^{\infty} = 2.$$

- Similarly,

$$\mathbb{E}[Y^2] = \int_1^{\infty} 2s^{-1}ds = \infty$$

-

□

Question 4.

- First, since for $s \in [0, \pi/2]$ we have that $\cos(s) \geq 0$, f_X is non-negative. Second,

$$\int f_X(s)ds = \int_0^{\pi/2} \cos(s)ds = \sin(s)\Big|_0^{\pi/2} = 1,$$

so f_X is indeed a density.

- Since $\frac{\partial}{\partial x}(x\sin(x) + \cos(x)) = x\cos(x)$, we have that

$$\mathbb{E}[X] = \int_0^{\pi/2} s\cos(s)ds = (s\sin(s) + \cos(s))\Big|_0^{\pi/2} = \frac{\pi}{2} - 1.$$

- If $Y = \frac{1}{X}$, since $X \in [0, \pi/2]$ we have that $Y \in [\frac{2}{\pi}, \infty)$. That is, for any $t < \frac{2}{\pi}$,

$$\mathbb{P}[Y \leq t] = 0 = \int_{-\infty}^{t} f_Y(s) ds.$$

For $t \geq \frac{2}{\pi}$ we have that $\frac{1}{t} \in [0, \pi/2]$, so

$$\begin{aligned}\mathbb{P}[Y \leq t] = \mathbb{P}[X \geq \tfrac{1}{t}] &= \int_{1/t}^{\pi/2} \cos(s) ds \\ &= \sin(s)\Big|_{1/t}^{\pi/2} = 1 - \sin(1/t).\end{aligned}$$

But also, since

$$\frac{\partial}{\partial x} \sin(1/x) = -\cos(1/x) \cdot x^{-2},$$

we get that for $t \geq \frac{2}{\pi}$,

$$\begin{aligned}\int_{-\infty}^{t} f_Y(s) ds = \int_{2/\pi}^{t} \cos(1/s) \cdot s^{-2} ds &= -\sin(1/s)\Big|_{2/\pi}^{t} \\ &= 1 - \sin(1/t) = \mathbb{P}[Y \leq t].\end{aligned}$$

So f_Y is indeed the density of Y.

$\square$