Probability

Solutions Exam A, 2017

Solution to Q1.

(A) By linearity of expectation

$$\mathbb{E}[W] = \sum_{k=1}^{n} a_k \, \mathbb{P}[Z = k] = \frac{1}{n} \sum_{k=1}^{n} a_k.$$

Also,

$$\mathbb{E}[W^2] = \mathbb{E} \sum_{j,k=1}^{n} a_j a_k \mathbf{1}_{\{Z=j\}} \mathbf{1}_{\{Z=k\}}.$$

Since

$$\mathbf{1}_{\{Z=j\}} \mathbf{1}_{\{Z=k\}} = \mathbf{1}_{\{Z=j\} \cap \{Z=k\}} = \begin{cases} \mathbf{1}_{\{Z=k\}} & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

we get that

$$\mathbb{E}[W^2] = \mathbb{E} \sum_{k=1}^{n} a_k^2 \mathbf{1}_{\{Z=k\}} = \frac{1}{n} \sum_{k=1}^{n} a_k^2.$$

Since $\mathbb{E}[W^2] - (\mathbb{E}[W])^2 = \mathrm{Var}[W] \geq 0$, we have that

$$\frac{1}{n} \sum_{k=1}^{n} a_k^2 \geq \Big(\frac{1}{n} \sum_{k=1}^{n} a_k \Big)^2,$$

which is equivalent to

$$\sum_{k=1}^{n} a_k^2 \geq \frac{1}{n} \Big(\sum_{k=1}^{n} a_k \Big)^2.$$

Since $(\{X=k\})_{k=1}^n$ are an almost-partition of the space, using the law of total probability

$$\begin{aligned}\mathbb{P}[X=Y] &= \sum_{k=1}^n \mathbb{P}[X=Y, X=k] = \sum_{k=1}^n \mathbb{P}[X=k, Y=k] \\ &= \sum_{k=1}^n \mathbb{P}[X=k]\cdot\mathbb{P}[Y=k] = \sum_{k=1}^n f_X(k)^2.\end{aligned}$$

(B) Since $(\{X=k\})_{k=1}^n$ are an almost-partition of the space, using the law of total probability

$$\begin{aligned}\mathbb{P}[X=Y] &= \sum_{k=1}^n \mathbb{P}[X=Y, X=k] = \sum_{k=1}^n \mathbb{P}[X=k, Y=k] \\ &= \sum_{k=1}^n \mathbb{P}[X=k]\cdot\mathbb{P}[Y=k] = \sum_{k=1}^n f_X(k)^2 = \frac{1}{n}.\end{aligned}$$

(C) As before, since $(\{X=k\})_{k=1}^n$ are an almost-partition of the space, using the law of total probability

$$\begin{aligned}\mathbb{P}[X=Y] &= \sum_{k=1}^n \mathbb{P}[X=Y, X=k] = \sum_{k=1}^n \mathbb{P}[X=k, Y=k] \\ &= \sum_{k=1}^n \mathbb{P}[X=k]\cdot\mathbb{P}[Y=k] = \sum_{k=1}^n f_X(k)^2.\end{aligned}$$

Using (A)

$$\mathbb{P}[X=Y] = \sum_{k=1}^n f_X(k)^2 \geq \frac{1}{n}\Big(\sum_{k=1}^n f_X(k)\Big)^2 = \frac{1}{n}.$$

Solution to Q2.

(A) Let $\varepsilon_n \searrow 0$. Let $r \in \mathbb{R}$. The events $A_n := \{X \geq r - \varepsilon_n\}$ are decreasing in n; indeed, if $n > m$ then $\varepsilon_n < \varepsilon_m$ so $r - \varepsilon_n > r - \varepsilon_m$, which implies

$$A_n = \{X \geq r - \varepsilon_n\} \subset \{X \geq r - \varepsilon_m\} = A_m.$$

Thus $\lim A_n = \bigcap_n A_n = \{X \geq r\}$ and by continuity of probability,

$$T_X(r) = \mathbb{P}[X \geq r] = \lim_{n\to\infty} \mathbb{P}[X \geq r - \varepsilon_n] = \lim_{n\to\infty} T_X(r - \varepsilon_n).$$

Since this holds for any $\varepsilon_n \searrow 0$ we have that T_X is left continuous.

(B) If $r \geq r'$ then $\{X \geq r\} \subset \{X \geq r'\}$, so

$$T_X(r) = \mathbb{P}[X \geq r] \leq \mathbb{P}[X \geq r'] = T_X(r').$$

(C) Define $A_n = \{X \geq n\}$ and $B_n = \{X \geq -n\}$. So $(A_n)_n$ are decreasing with limit $\lim A_n = \bigcap_n A_n = \emptyset$ and $(B_n)_n$ are increasing with limit $\lim B_n = \bigcup_n B_n = \Omega$. Hence by continuity of probability,

$$0 = \mathbb{P}[\emptyset] = \lim \mathbb{P}[A_n] = \lim T_X(n),$$

$$1 = \mathbb{P}[\Omega] = \lim \mathbb{P}[B_n] = \lim T_X(-n).$$

(D) We have

$$\{a \leq X < b\} = \{X \geq a\} \setminus \{X \geq b\}$$

with $\{X \geq b\} \subset \{X \geq a\}$ because $a < b$, so

$$\mathbb{P}[a \leq X < b] = \mathbb{P}[X \geq a] - \mathbb{P}[X \geq b] = T_X(a) - T_X(b).$$

(E) The events $A_n = \{a \leq X < a + 2^{-n}\}$ for a decreasing sequence of events with limit $\lim A_n = \bigcap_n A_n = \{X = a\}$. So

$$\mathbb{P}[X = a] = \lim \mathbb{P}[A_n] = \lim \mathbb{P}[a \leq X < a + 2^{-n}] = T_X(a) - \lim T_X(a + 2^{-n}).$$

Solution to Q3.

(A) Since X is continuous, $\mathbb{P}[X = 0] = 0$. So $\mathbb{P}[Y \leq t] = \mathbb{P}[X^{-1/n} \leq t]$ for any t. If $t < 1$ then

$$\mathbb{P}[Y \leq t] = \mathbb{P}[X^{-1/n} \leq t] = 0.$$

For $t \geq 1$,

$$\mathbb{P}[Y \leq t] = \mathbb{P}[X^{1/n} \geq t^{-1}] = \mathbb{P}[X \geq t^{-n}] = 1 - t^{-n}.$$

If we define

$$f_Y(s) = \begin{cases} 0 & s \le 1 \\ ns^{-n-1} & s > 1 \end{cases}$$

we obtain that for $t \ge 1$

$$F_Y(t) = 1 - t^{-n} = \int_1^t ns^{-n-1} ds = \int_{-\infty}^t f_Y(s) ds,$$

and for $t < 1$,

$$F_Y(t) = 0 = \int_{-\infty}^t f_Y(s) ds.$$

So Y is absolutely continuous with density f_Y.

(B)

$$\mathbb{E}[Y] = \int_{-\infty}^{\infty} s f_Y(s) ds = \int_1^{\infty} ns^{-n} ds = -\tfrac{n}{n-1} s^{-(n-1)} \Big|_1^{\infty} = \tfrac{n}{n-1}.$$

(C)

$$\mathbb{E}[Y^n] = \int_{-\infty}^{\infty} s^n f_Y(s) ds = \int_1^{\infty} ns^{-n-1} s^n ds$$
$$= n \cdot \lim_{k \to \infty} \int_1^k s^{-1} ds = n \cdot \lim_{k \to \infty} \log k = \infty.$$

Solution to Q4.

(A) If X, Y are independent, then the events $\{X = 1\}, \{Y = 1\}$ are independent, so

$$\mathbb{E}[XY] = \mathbb{P}[XY = 1] = \mathbb{P}[X = 1\ ,\ Y = 1] = \mathbb{P}[X = 1] \cdot \mathbb{P}[Y = 1] = \mathbb{E}[X] \cdot \mathbb{E}[Y].$$

So $\mathrm{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X] \cdot \mathbb{E}[Y] = 0$ and X, Y are uncorrelated.

If X, Y are uncorrelated, then in order to prove that they are independent, it suffices to prove that $\mathbb{P}[X = a, Y = b] = \mathbb{P}[X = a] \cdot \mathbb{P}[Y = b]$ for all $a, b \in \mathbb{R}$. Since $R_X = R_Y = \{0, 1\}$ it suffices to show this for $a, b \in \{0, 1\}$.

Indeed, using $\mathbb{E}[XY]=\mathbb{E}[X]\cdot\mathbb{E}[Y]$,

$$\mathbb{P}[X=1,Y=1]=\mathbb{E}[XY]=\mathbb{E}[X]\cdot\mathbb{E}[Y]=\mathbb{P}[X=1]\cdot\mathbb{P}[Y=1].$$

$$\mathbb{P}[X=0,Y=1]=\mathbb{E}[(1-X)Y]=\mathbb{E}[Y]-\mathbb{E}[X]\cdot\mathbb{E}[Y]$$

$$=\mathbb{P}[Y=1]\cdot(1-\mathbb{P}[X=1])=\mathbb{P}[Y=1]\cdot\mathbb{P}[X=0].$$

$$\mathbb{P}[X=1,Y=0]=\mathbb{E}[X(1-Y)]=\mathbb{E}[X]-\mathbb{E}[X]\cdot\mathbb{E}[Y]$$

$$=\mathbb{P}[X=1]\cdot(1-\mathbb{P}[Y=1])=\mathbb{P}[X=1]\cdot\mathbb{P}[Y=0].$$

$$\mathbb{P}[X=0,Y=0]=\mathbb{E}[(1-X)(1-Y)]=1-\mathbb{E}[X]-\mathbb{E}[Y]+\mathbb{E}[X]\cdot\mathbb{E}[Y]$$

$$=(1-\mathbb{E}[X])\cdot(1-\mathbb{E}[Y])=\mathbb{P}[X=0]\cdot\mathbb{P}[Y=0].$$

(B) We want to choose p,q,r so that $\mathbb{E}[XYZ]=\mathbb{E}[X]\cdot\mathbb{E}[Y]\cdot\mathbb{E}[Z]$ but that they are not independent. Note that

$$\mathbb{E}[XYZ]=\mathbb{P}[XYZ=1]=\mathbb{P}[X=1\ ,\ Y=1\ ,\ Z=1].$$

In order to define all three random variables X,Y,Z we need to specify $f_{X,Y,Z}(x,y,z)$ for all $2^3=8$ possible vectors $(x,y,z)\in\{0,1\}^3$. If $f_{X,Y,Z}(x,y,z)=\frac{1}{8}$ for all such vectors, then one can check that X,Y,Z are independent, so this is not a good option.

However, we will take this uniform distribution on 8 vectors, and "shift" it a bit, still keeping the marginal distributions of X,Y,Z to be Ber(1/2), and keeping $f_{X,Y,Z}(1,1,1)=\frac{1}{8}$. This will ensure that

$$\mathbb{P}[X=1\ ,\ Y=1\ ,\ Z=1]=\tfrac{1}{8}=2^{-3}=\mathbb{P}[X=1]\cdot\mathbb{P}[Y=1]\cdot\mathbb{P}[Z=1],$$

so that $\mathbb{E}[XYZ]=\mathbb{E}[X]\cdot\mathbb{E}[Y]\cdot\mathbb{E}[Z]$. To ensure that $\mathbb{P}[X=1]=\mathbb{P}[Y=1]=\mathbb{P}[Z=1]=\frac{1}{2}$ we need to make sure that summing the two irrelevant coordinates in $f_{X,Y,Z}$ gives $\frac{1}{2}$.

One possibility is as follows:

$$f_{X,Y,Z}(x,y,z) = \begin{cases} \frac{1}{8} & (x,y,z) = (0,0,0) \\ \frac{1}{8} & (x,y,z) = (0,0,1) \\ \frac{3}{16} & (x,y,z) = (0,1,0) \\ \frac{1}{16} & (x,y,z) = (1,0,0) \\ \frac{1}{16} & (x,y,z) = (0,1,1) \\ \frac{1}{8} & (x,y,z) = (1,1,0) \\ \frac{3}{16} & (x,y,z) = (1,0,1) \\ \frac{1}{8} & (x,y,z) = (1,1,1) \end{cases}$$

Indeed it may be easily checked that

$$\mathbb{P}[X=1] = \mathbb{P}[Y=1] = \mathbb{P}[Z=1] = \tfrac{1}{2}$$

and that X, Y, Z are not independent because, for example,

$$\mathbb{P}[X=0\ ,\ Y=1\ ,\ Z=0] = \tfrac{3}{16} \neq 2^{-3} = \mathbb{P}[X=0] \cdot \mathbb{P}[Y=1] \cdot \mathbb{P}[Z=0].$$