Probability

Solutions to Exam B, Fall 2014

Solution Q1:

(A) Suppose the brothers of the soldiers are $\{1, 2, \ldots, n\}$. Let $\phi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be a function such that $\phi(j)$ is the soldier the receives the package from j.

Thus, our probability space is uniform measure on all possible $\phi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ that are $1 - 1$ and onto.

The size of the space is thus $n!$.

The probability of A_k is thus the number of ϕ such that $\phi(k) = k$ divided by $n!$. There are $(n-1)!$ many such ϕ, so

$$\mathbb{P}[A_k] = \frac{(n-1)!}{n!} = \frac{1}{n}.$$

Also, $A_k \cap A_j$ is the set of all functions ϕ such that $\phi(k) = k, \phi(j) = j$. There are $(n-2)!$ such functions, so:

$$\mathbb{P}[A_k \cap A_j] = \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}.$$

(B) Note that

$$X = \sum_{k=1}^{n} \mathbf{1}_{A_k}.$$

So

$$\mathbb{E}[X] = \sum_{k=1}^{n} \mathbb{P}[A_k] = n\frac{1}{n} = 1.$$

(C) We use the Pythagorean Theorem:

$$\mathrm{Var}[X] = \mathrm{Var}[\sum_{k=1}^{n} \mathbf{1}_{A_k}] = \sum_{k=1}^{n} \mathrm{Var}[\mathbf{1}_{A_k}] + \sum_{k \neq j} \mathrm{Cov}(\mathbf{1}_{A_k}, \mathbf{1}_{A_j}).$$

It it immediate that

$$\mathrm{Var}[\mathbf{1}_{A_k}] = \mathbb{P}[A_k] - \mathbb{P}[A_k]^2 = \tfrac{1}{n}\left(1 - \tfrac{1}{n}\right) = \tfrac{n-1}{n^2}.$$

Also, for $j \neq k$,

$$\begin{aligned}\mathrm{Cov}(\mathbf{1}_{A_k}, \mathbf{1}_{A_j}) &= \mathbb{E}[\mathbf{1}_{A_k}\mathbf{1}_{A_j}] - \mathbb{E}[\mathbf{1}_{A_k}] \cdot \mathbb{E}[\mathbf{1}_{A_j}] \\ &= \mathbb{P}[A_k \cap A_j] - \mathbb{P}[A_k] \cdot \mathbb{P}[A_j] = \tfrac{1}{n(n-1)} - \tfrac{1}{n^2}.\end{aligned}$$

Summing everything,

$$\mathrm{Var}[X] = \tfrac{n-1}{n} + 1 - \tfrac{n(n-1)}{n^2} = 1.$$

Solution Q2:

(A) Since $X \geq 1, Y \geq 0$ we have that $0 < \frac{X}{X+Y} \leq 1$. So, if $t \leq 0$ then $F_Z(t) = 0$ and if $t \geq 1$ then $F_Z(t) = 1$.

Let $0 < t < 1$. Then, using the law of total probability and independence,

$$\begin{aligned}\mathbb{P}[Z \leq t] &= \mathbb{P}[X \leq t(X+Y)] = \mathbb{P}[Y \geq \tfrac{1-t}{t} X] \\ &= \sum_{k=1}^{\infty} \mathbb{P}[X = k, Y \geq \tfrac{1-t}{t} k] = \sum_{k=1}^{\infty} \mathbb{P}[X = k]\, \mathbb{P}[Y \geq \tfrac{1-t}{t} k].\end{aligned}$$

Since Y is continuous,

$$\begin{aligned}\mathbb{P}[Z \leq t] &= \sum_{k=1}^{\infty} (1-p)^{k-1} p \cdot e^{-\lambda \frac{1-t}{t} k} \\ &= p \cdot e^{-\lambda \frac{1-t}{t}} \cdot \frac{1}{1 - (1-p) e^{-\lambda \frac{1-t}{t}}} \\ &= \frac{p}{e^{-\lambda} \cdot e^{\lambda/t} + p - 1}.\end{aligned}$$

We conclude that

$$F_Z(t) = \begin{cases} 0 & t \leq 0 \\ \phi(t) & 0 < t < 1 \\ 1 & t \geq 1 \end{cases}$$

where

$$\phi(t) = \frac{p}{e^{-\lambda} e^{\lambda/t} + p - 1} \qquad t \in (0,1).$$

(B) Note that ϕ is differentiable on $(0,1)$. Also note that $\phi(0) = 0$ and $\phi(1) = 1$. If we define

$$f_Z(t) = \begin{cases} 0 & t \notin (0,1) \\ \phi'(t) & t \in (0,1) \end{cases}$$

then we have by the fundamental theorem of calculus, for any $t \in (0,1)$,

$$\int_{-\infty}^{t} f_Z(s) ds = \int_{0}^{t} \phi'(s) ds = \phi(t) - \phi(0) = \phi(t) = F_Z(t).$$

For $t \geq 1$,

$$\int_{-\infty}^{t} f_Z(s) ds = \int_{0}^{1} \phi'(s) ds = \phi(1) - \phi(0) = 1 = F_Z(t).$$

For $t < 0$,

$$\int_{-\infty}^{t} f_Z(s) ds = 0 = F_Z(t).$$

So Z is absolutely continuous with density f_Z.

Solution Q3:

(A) Let $A_n = \{|X_n - Y_n| > 2^{-n}\}$. Since $\sum_n \mathbb{P}[A_n] < \infty$, by Borel-Cantelli (first lemma),

$$\mathbb{P}[\limsup_n A_n] = \mathbb{P}[A_n i.o.] = 0.$$

That is,

$$\mathbb{P}[\exists\ n\ :\ \forall\ k \geq n \quad A_k^c] = \mathbb{P}[\bigcup_n \bigcap_{k \geq n} A_k^c] = \mathbb{P}[\liminf A_n^c] = 1.$$

Note that if $\omega \in \bigcup_n \bigcap_{k \geq n} A_k^c$, then there exists n such that for all $k \geq n$ we have $|X_n(\omega) - Y_n(\omega)| \leq 2^{-k}$. Thus, for such ω,

$$\sum_{m=1}^{\infty} |X_m(\omega) - Y_m(\omega)| \leq \sum_{m=1}^{n-1} |X_m(\omega) - Y_m(\omega)| + \sum_{m=n}^{\infty} 2^{-m} \leq \sum_{m=1}^{n-1} |X_m(\omega) - Y_m(\omega)| + 2^{-(n-1)} < \infty.$$

So we conclude that for any $\omega \in \bigcup_n \bigcap_{k \geq n} A_k^c$ we have that the sum $\sum_{m=1}^{\infty} (X_m(\omega) - Y_m(\omega))$ converges absolutely.

Thus, the event $\liminf A_n^c$ implies the event that $\sum_m (X_m - Y_m)$ converges. So

$$\mathbb{P}[\sum_m (X_m - Y_m) \text{ converges }] \geq \mathbb{P}[\liminf A_n^c] = 1.$$

(B) Let A be the event that $\sum_m (X_m - Y_m)$ converges. In (A) we saw that $\mathbb{P}[A] = 1$. Let B be the event $\{Y_n \to Y\}$. We are given that $\mathbb{P}[B] = 1$. So $\mathbb{P}[A \cap B] = 1$.

Now, if $\omega \in A \cap B$ then the sum $\sum_m (X_m(\omega) - Y_m(\omega))$ converges. So $X_m(\omega) - Y_m(\omega) \to 0$. But also, $Y_m(\omega) \to Y(\omega)$. So we must have that $X_m(\omega) \to Y(\omega)$. Thus, $A \cap B$ implies the event $\{X_m \to Y\}$. Hence,

$$\mathbb{P}[X_m \to Y] \geq \mathbb{P}[A \cap B] = 1.$$

So $(X_n)_n$ converges a.s. to Y.