Probability

Solutions to Exam B, Fall 2013

Solution Q1:

(A) The total mass of a density function is 1, so

$$1 = \int_{-\infty}^{\infty} f_Y(s)ds = \int_0^1 Cs^2 ds = C\tfrac{1}{3},$$

implying that $C = 3$.

For $\mathbb{E}[Y]$ we have

$$\mathbb{E}[Y] = \int_{-\infty}^{\infty} s f_y(s)ds = \int_0^1 3s^3 ds = \tfrac{3}{4}.$$

For $\mathbb{E}[X]$ we use the fact that

$$f_{X,Y}(t,s) = \begin{cases} s^{-1}e^{-t/s} \cdot 3s^2 & \text{for } s \in (0,1), t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

So

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} t f_{X,Y}(t,s)dtds = \int_0^1 3s^2 \int_0^{\infty} s^{-1}te^{-t/s}dtds$$

The inner integral is exactly the expectation of $\text{Exp}(s^{-1})$, which is s. So

$$\mathbb{E}[X] = \int_0^1 3s^2 \cdot sds = \tfrac{3}{4}.$$

(B) First we compute $\mathbb{E}[XY]$ and $\mathbb{E}[Y^2]$ and $\mathbb{E}[X^2]$.

As in the previous item,

$$\mathbb{E}[XY] = \in \int ts f_{X,Y}(t,s)dtds = \int_0^1 3s^3 \int_0^{\infty} ts^{-1}e^{-t/s}dtds$$
$$= \int_0^1 3s^4 ds = \tfrac{3}{5}.$$

$$\mathbb{E}[Y^2] = \int_0^1 3s^4 ds = \tfrac{3}{5}.$$

$$\mathbb{E}[X^2] = \int\int t^2 f_{X,Y}(t,s)dtds = \int_0^1 3s^2 \int_0^\infty t^2 s^{-1} e^{-t/s} dtds,$$

the inner integral is the second moment of $\mathrm{Exp}(s^{-1})$. If $Z \sim \mathrm{Exp}(s^{-1})$ then

$$\mathbb{E}[Z^2] = \mathrm{Var}[Z] + (\mathbb{E}[Z])^2 = 2s^2,$$

so plugging this into the inner integral for every $s \in (0,1)$,

$$\mathbb{E}[X^2] = \int_0^1 6s^4 ds = \tfrac{6}{5}.$$

Finally, we combine all the above to get

$$\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\cdot\mathbb{E}[Y] = \tfrac{3}{5} - \tfrac{3}{4}\cdot\tfrac{3}{4} = \tfrac{3}{80}.$$

$$\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\,\mathrm{Cov}(X,Y) = \tfrac{6}{5} - \tfrac{9}{16} + \tfrac{3}{5} - \tfrac{9}{16} + 2\cdot\tfrac{3}{80} = \tfrac{60}{80} = \tfrac{3}{4}.$$

$$\begin{aligned}\mathrm{Cov}(X-Y,Y) &= \mathrm{Cov}(X,Y) - \mathrm{Cov}(Y,Y) = \mathrm{Cov}(X,Y) - \mathrm{Var}[Y] \\ &= \tfrac{3}{80} - \tfrac{3}{5} + \tfrac{9}{16} = 0.\end{aligned}$$

(C) Take $X \sim \mathrm{Exp}(\lambda)$. We prove the claim by induction on n.

For $n = 0$ the claim is immediate because $\mathbb{E}[X^0] = 1$.

Assume the claim for n. We compute for $n+1$: Using integration by parts, with the functions $u(t) = t^{n+1}$ and $v(t) = \lambda e^{-\lambda t}$, we obtain

$$\begin{aligned}\mathbb{E}[X^{n+1}] &= \int_0^\infty t^{n+1}\lambda e^{-\lambda t}dt = -t^{n+1}e^{-\lambda t}\Big|_0^\infty + \int_0^\infty (n+1)t^n e^{-\lambda t}dt \\ &= 0 + (n+1)\lambda^{-1}\int_0^\infty t^n \lambda e^{-\lambda t}dt = (n+1)\lambda^{-1}\,\mathbb{E}[X^n] = (n+1)!\lambda^{-(n+1)}.\end{aligned}$$

So we have shown by induction that

$$\mathbb{E}[X^n] = n!\lambda^{-n}.$$

Solution Q2:

(A) First we compute the distribution function $F_Z(t)$. For any $t \in \mathbb{R}$, by the law of totally probability (with the partition into disjoint events $\{Y=0\}, \{Y=1\}$)

$$F_Z(t) = \mathbb{P}[Z \le t] = \mathbb{P}[-X \le t, Y = 0] + \mathbb{P}[X \le t, Y = 1]$$
$$= \mathbb{P}[X \ge -t] \cdot \mathbb{P}[Y = 0] + \mathbb{P}[X \le t] \cdot \mathbb{P}[Y = 1] = \tfrac{1}{2}(1 - F_X(-t)) + \tfrac{1}{2}F_X(t),$$

where we have used that X, Y are independent, and that X is continuous.

Since $X \sim \text{Exp}(\lambda)$, we have that

$$F_Z(t) = \begin{cases} \frac{1+1-e^{-\lambda t}}{2} = 1 - \frac{1}{2}e^{-\lambda t} & \text{for } t \ge 0 \\ \frac{1}{2}e^{\lambda t} & \text{for } t < 0. \end{cases}$$

Now, we can choose $f_Z(t) = \frac{1}{2}\lambda e^{-\lambda|t|}$, and check that for $t \le 0$,

$$\int_{-\infty}^{t} f_Z(s)ds = \int_{-\infty}^{t} \tfrac{1}{2}\lambda e^{\lambda s} ds = \tfrac{1}{2}e^{\lambda s}\Big|_{-\infty}^{t} = \tfrac{1}{2}e^{\lambda t},$$

and for $t \ge 0$,

$$\int_{-\infty}^{t} f_Z(s)ds = \int_{-\infty}^{0} \tfrac{1}{2}\lambda e^{\lambda s} ds + \int_{0}^{t} \tfrac{1}{2}\lambda e^{-\lambda s} ds = \tfrac{1}{2} - \tfrac{1}{2}e^{-\lambda s}\Big|_{0}^{t} = 1 - \tfrac{1}{2}e^{-\lambda t}.$$

Since we get that

$$F_Z(t) = \int_{-\infty}^{t} f_Z(s)ds$$

for all $t \in \mathbb{R}$, we have that Z is absolutely continuous with density f_Z given above.

(B) The long way is to calculate $\int t^2 f_Z(t)dt$ directly.

However, note that $Z^2 = (2Y-1)^2 X^2$ and since $2Y - 1 \in \{-1, 1\}$ we have that $Z^2 = X^2$. Since $X^2 \sim \text{Exp}(\lambda)$,

$$\mathbb{E}[X^2] = \text{Var}[X] + (\mathbb{E}[X])^2 = \frac{2}{\lambda^2}.$$

If we were to go the long way, it would not be so terrible, as:

$$\mathbb{E}[Z^2] = \int_{-\infty}^{\infty} t^2 f_Z(t)dt = \int_{-\infty}^{0} t^2 \tfrac{1}{2}\lambda e^{\lambda t} dt + \int_{0}^{\infty} t^2 \tfrac{1}{2}\lambda e^{-\lambda t} dt$$
$$= 2\int_{0}^{\infty} \tfrac{1}{2}\lambda e^{-\lambda t} dt = \mathbb{E}[X^2].$$

Even an explicit integration by parts here is not so terrible.

(C) We use linearity of expectation and the fact that X, Y are independent, so any functions of X and of Y are independent. Specifically, $(2Y-1)$ and X^2 are independent and $(2Y-1)$ and X are independent. Thus,

$$\mathbb{E}[ZX] = \mathbb{E}[(2Y-1)X^2] = \mathbb{E}[(2Y-1)] \cdot \mathbb{E}[X^2],$$

$$\mathbb{E}[Z] = \mathbb{E}[(2Y-1)X] = \mathbb{E}[(2Y-1)] \cdot \mathbb{E}[X].$$

(This already shows that $\mathrm{Cov}(Z, X) = \mathbb{E}[(2Y-1)] \cdot \mathrm{Var}[X]$.) Now, since

$$\mathbb{E}[(2Y-1)] = 2\,\mathbb{E}[Y] - 1 = 0$$

we get that $\mathbb{E}[ZX] = \mathbb{E}[Z] = 0$ and

$$\mathrm{Cov}(Z, X) = \mathbb{E}[ZX] - \mathbb{E}[Z] \cdot \mathbb{E}[X] = 0.$$

(So X, Z are uncorrelated.)

We claim that X, Z are *not* independent. To show this, it suffices to find t, s such that

$$\mathbb{P}[X \le t, Z \le s] \neq \mathbb{P}[X \le t] \cdot \mathbb{P}[Z \le s].$$

Let us choose $t = s = 1$. Then,

$$\mathbb{P}[X \le 1, Z \le 1] = \mathbb{P}[X \le 1, -X \le 1, Y = 0] + \mathbb{P}[X \le 1, Y = 1]$$
$$= \tfrac{1}{2}\,\mathbb{P}[-1 \le X \le 1] + \tfrac{1}{2}\,\mathbb{P}[X \le 1] = 1 - e^{-\lambda}.$$

However, $\mathbb{P}[X \leq 1] = 1 - e^{-\lambda}$ and (using the distribution function computed above)

$$\mathbb{P}[Z \leq 1] = F_Z(1) = 1 - \tfrac{1}{2}e^{-\lambda}.$$

Multiplying we get

$$\mathbb{P}[X \leq 1] \cdot \mathbb{P}[Z \leq 1] = 1 - e^{-\lambda} - \tfrac{1}{2}e^{-\lambda} + \tfrac{1}{2}e^{-2\lambda} \neq 1 - e^{-\lambda} = \mathbb{P}[X \leq 1, Z \leq 1],$$

because $e^{-2\lambda} \neq e^{-\lambda}$ for any $\lambda > 0$.

Solution Q3:

(A) For every n let $Z_n = X_n - Y_n$. So $(Z_n)_n$ are independent, all have finite variance

$$\sigma^2 := \mathrm{Var}[Z_n] = \mathrm{Var}[X_n] + \mathrm{Var}[Y_n] = \tfrac{1-p}{p^2} + \tfrac{1}{p^2},$$

and $\mathbb{E}[Z_n] = \frac{1}{p} - \frac{1}{p} = 0$.

By the Central Limit Theorem,

$$\tfrac{1}{\sqrt{n}\sigma} \sum_{k=1}^{n} Z_n \xrightarrow{\mathcal{D}} N(0,1).$$

In other words,

$$\lim_{n\to\infty} \mathbb{P}[\sum_{k=1}^{n} X_k > \sum_{k=1}^{n} Y_k] = \lim_{n\to\infty} \mathbb{P}[\sum_{k=1}^{n} Z_k > 0]$$

$$= \lim_{n\to\infty} \mathbb{P}[\tfrac{1}{\sqrt{n}\sigma} \sum_{k=1}^{n} Z_k > 0] = \mathbb{P}[N(0,1) > 0].$$

Since $s \mapsto e^{-s^2/2}$ is an even function,

$$\mathbb{P}[N(0,1) > 0] = \int_0^{\infty} \tfrac{1}{\sqrt{2\pi}} e^{-s^2/2} ds = \tfrac{1}{2} \int_{-\infty}^{\infty} \tfrac{1}{\sqrt{2\pi}} e^{-s^2/2} ds = \tfrac{1}{2}.$$

Which completes the proof.

(B) Y_1 is absolutely continuous, so

$$\mathbb{E}[e^{\alpha Y_1}] = \int_{-\infty}^{\infty} e^{\alpha t} f_Y(t)dt = \int_0^{\infty} e^{\alpha t} p e^{-pt} dt.$$

If $\alpha - p \geq 0$ this is $\mathbb{E}[e^{\alpha Y_1}] = \infty$. For $\alpha < p$ this becomes

$$\mathbb{E}[e^{\alpha Y_1}] = \tfrac{p}{\alpha - p} e^{(\alpha-p)t}\Big|_0^{\infty} = \tfrac{p}{p-\alpha}.$$

(C) Since X_1, Y_1 are independent, we have that so are $e^{\alpha Y_1}, e^{\alpha X_1}$. So $\mathbb{E}[e^{\alpha(X_1+Y_1)}] = \mathbb{E}[e^{\alpha X_1}] \cdot \mathbb{E}[e^{\alpha Y_1}]$.

We have already seen that if $\alpha \geq p$ then $\mathbb{E}[e^{\alpha Y_1}] = \infty$, so since $e^{\alpha X_1} \geq 0$, we have that if $\alpha \geq p$ then $\mathbb{E}[e^{\alpha(X_1+Y_1)}] = \infty$.

We turn to the case $\alpha < p$. Then, since X_1 is discrete with range $\{1, 2, \ldots, \}$,

$$\mathbb{E}[e^{\alpha X_1}] = \sum_{k=1}^{\infty} e^{\alpha k} p(1-p)^{k-1} = \tfrac{p}{1-p} \sum_{k=1}^{\infty} \left((1-p)e^{\alpha}\right)^k.$$

If $e^{\alpha} \geq \frac{1}{1-p}$ then this is $\mathbb{E}[e^{\alpha X_1}] = \infty$. If $e^{\alpha}(1-p) < 1$ then this becomes

$$\mathbb{E}[e^{\alpha X_1}] = \frac{p}{1-p} \cdot \frac{(1-p)e^{\alpha}}{1-(1-p)e^{\alpha}} = \frac{pe^{\alpha}}{1-(1-p)e^{\alpha}}.$$

To sum up,

$$\mathbb{E}[e^{\alpha(X_1+Y_1)}] = \begin{cases} \frac{p^2 e^{\alpha}}{(p-\alpha)(1-(1-p)e^{\alpha})} & \text{if } \alpha < p \text{ and } e^{\alpha} < \frac{1}{1-p}, \\ \infty & \text{otherwise .} \end{cases}$$

Solution Q4:

(A)

$$\frac{\binom{k-1}{n} + \binom{k-1}{n-1}}{\binom{k}{n}} = \frac{k-n}{k} + \frac{n}{k} = 1$$

(B) We prove this by induction on k. The base case is $k = n + 1$. Then the claim is just

$$\binom{n-1}{n-1} = \binom{n}{n}$$

which is obviously true.

Assume the claim holds for some $k \geq n+1$. For $k+1$ we compute using the induction hypothesis, and using (A),

$$\sum_{m=n}^{k} \binom{m-1}{n-1} = \sum_{m=n}^{k-1} \binom{m-1}{n-1} + \binom{k-1}{n-1} = \binom{k-1}{n} + \binom{k-1}{n-1} = \binom{k}{n},$$

which proves the induction step.

(C) We prove the claim by induction on n. The base case $n = 1$ is just $\mathbb{P}[X_1 = k] = p(1-p)^{k-1}$ for integer $k \geq 1$ and 0 otherwise, which is obvious since $X_1 \sim \text{Geo}(p)$.

Assume the claim is true for n. Let $S = S_n$ and $X = X_{n+1}$. We need to show the induction step, which is to show that

$$\mathbb{P}[S + X = k] = \begin{cases} \binom{k-1}{n} p^{n+1}(1-p)^{k-n-1} & \text{for an integer } k \geq n+1 \\ 0 & \text{otherwise} \end{cases}$$

Indeed, by the induction hypothesis, S has range $\{n, n+1, n+2, \ldots, \}$. Since S, X are independent, the law of total probability (or the discrete convolution identity shown in class) gives

$$\mathbb{P}[S + X = k] = \sum_{m=n}^{\infty} \mathbb{P}[S = m, X = k - m] = \sum_{m=n}^{\infty} Pr[S = m]\, \mathbb{P}[X = k - m].$$

If k is not an integer this is 0. Since the range of X is the positive integers, we get for any integer $k \leq n$, $\mathbb{P}[S + X = k] = 0$ and for any integer $k \geq n + 1$, again by the induction hypothesis,

$$\mathbb{P}[S+X = k] = \sum_{m=n}^{k-1} \binom{m-1}{n-1} p^n (1-p)^{m-n} \cdot p(1-p)^{k-m-1} = p^{n+1}(1-p)^{k-(n+1)} \sum_{m=n}^{k-1} \binom{m-1}{n-1}.$$

That is, using (B),

$$\mathbb{P}[S_{n+1} = k] = \binom{k-1}{n} p^{n+1}(1-p)^{k-(n+1)},$$

for all integers $k \geq n+1$ and $\mathbb{P}[S_{n+1} = k] = 0$ otherwise. This proves the induction step.

(D) Let $Y_n = X_n - \mathbb{E}[X_n] = X_n - \frac{1}{p}$. So $\mathbb{E}[Y_n] = 0$ and $(Y_n)_n$ are independent with finite variance. The law of large numbers tells us that

$$\tfrac{1}{n}\sum_{k=1}^{n} Y_k \xrightarrow{\text{a.s.}} 0.$$

For any ω, the sequence $\frac{1}{n}S_n(\omega) \to C$ if and only if

$$\frac{1}{n}\sum_{k=1}^{n} Y_k(\omega) = \tfrac{1}{n}S_n(\omega) - \tfrac{1}{p} \to C - \tfrac{1}{p}.$$

Thus, by the law of large numbers $\frac{1}{n}S_n \xrightarrow{\text{a.s.}} \frac{1}{p}$.