Solution Q1:

(A) The total mass of a density function is 1, so

\[
1 = \int_{-\infty}^{\infty} f_Y(s)ds = \int_{0}^{1} Cs^2 ds = C \frac{1}{3},
\]

implying that \( C = 3 \).

For \( \mathbb{E}[Y] \) we have

\[
\mathbb{E}[Y] = \int_{-\infty}^{\infty} s f_Y(s)ds = \int_{0}^{1} 3s^3 ds = \frac{3}{4}.
\]

For \( \mathbb{E}[X] \) we use the fact that

\[
f_{X,Y}(t, s) = \begin{cases} 
  s^{-1}e^{-t/s} \cdot 3s^2 & \text{for } s \in (0, 1), t \geq 0 \\
  0 & \text{otherwise}
\end{cases}
\]

So

\[
\mathbb{E}[X] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} t f_{X,Y}(t, s)dtds = \int_{0}^{1} 3s^2 \int_{0}^{\infty} s^{-1}te^{-t/s}dtds
\]

The inner integral is exactly the expectation of \( \text{Exp}(s^{-1}) \), which is \( s \). So

\[
\mathbb{E}[X] = \int_{0}^{1} 3s^2 \cdot sds = \frac{3}{4}.
\]

(B) First we compute \( \mathbb{E}[XY] \) and \( \mathbb{E}[Y^2] \) and \( \mathbb{E}[X^2] \).

As in the previous item,

\[
\mathbb{E}[XY] = \int t s f_{X,Y}(t, s)dtds = \int_{0}^{1} 3s^3 \int_{0}^{\infty} ts^{-1}e^{-t/s}dtds = \int_{0}^{1} 3s^4 ds = \frac{3}{5}.
\]
\[ \mathbb{E}[Y^2] = \int_0^1 3s^4 \, ds = \frac{3}{5}. \]

\[ \mathbb{E}[X^2] = \int \int t^2 f_{X,Y}(t,s) \, dt \, ds = \int_0^1 3s^2 \int_0^\infty t^2 s^{-1} e^{-t/s} \, dt \, ds, \]

the inner integral is the second moment of \( \text{Exp}(s^{-1}) \). If \( Z \sim \text{Exp}(s^{-1}) \) then

\[ \mathbb{E}[Z^2] = \text{Var}[Z] + (\mathbb{E}[Z])^2 = 2s^2, \]

so plugging this into the inner integral for every \( s \in (0, 1) \),

\[ \mathbb{E}[X^2] = \int_0^1 6s^4 \, ds = \frac{6}{5}. \]

Finally, we combine all the above to get

\[ \text{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X] \cdot \mathbb{E}[Y] = \frac{3}{5} - \frac{3}{4} \cdot \frac{3}{4} = \frac{3}{80}. \]

\[ \text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y] + 2 \text{Cov}(X,Y) = \frac{6}{5} - \frac{9}{16} + \frac{3}{5} - \frac{9}{16} + 2 \cdot \frac{3}{80} = \frac{60}{80} = \frac{3}{4}. \]

\[ \text{Cov}(X-Y,Y) = \text{Cov}(X,Y) - \text{Cov}(Y,Y) = \text{Cov}(X,Y) - \text{Var}[Y] \]

\[ = \frac{3}{80} - \frac{3}{5} + \frac{9}{16} = 0. \]

(C) Take \( X \sim \text{Exp}(\lambda) \). We prove the claim by induction on \( n \).

For \( n = 0 \) the claim is immediate because \( \mathbb{E}[X^0] = 1 \).

Assume the claim for \( n \). We compute for \( n + 1 \): Using integration by parts, with the functions \( u(t) = t^{n+1} \) and \( v(t) = \lambda e^{-\lambda t} \), we obtain

\[ \mathbb{E}[X^{n+1}] = \int_0^\infty t^{n+1} \lambda e^{-\lambda t} \, dt = -t^{n+1} e^{-\lambda t} \bigg|_0^\infty + \int_0^\infty (n+1)t^n e^{-\lambda t} \, dt \]

\[ = 0 + (n+1)\lambda^{-1} \int_0^\infty t^n \lambda e^{-\lambda t} \, dt = (n+1)\lambda^{-1} \mathbb{E}[X^n] = (n+1)! \lambda^{-(n+1)}. \]

So we have shown by induction that

\[ \mathbb{E}[X^n] = n! \lambda^{-n}. \]
Solution Q2:

(A) First we compute the distribution function $F_Z(t)$. For any $t \in \mathbb{R}$, by the law of totally probability (with the partition into disjoint events $\{Y = 0\}$, $\{Y = 1\}$)

$$F_Z(t) = \mathbb{P}[Z \leq t] = \mathbb{P}[-X \leq t, Y = 0] + \mathbb{P}[X \leq t, Y = 1]$$

$$= \mathbb{P}[X \geq -t] \cdot \mathbb{P}[Y = 0] + \mathbb{P}[X \leq t] \cdot \mathbb{P}[Y = 1] = \frac{1}{2}(1 - F_X(-t)) + \frac{1}{2}F_X(t),$$

where we have used that $X, Y$ are independent, and that $X$ is continuous.

Since $X \sim \text{Exp}(\lambda)$, we have that

$$F_Z(t) = \begin{cases} 
\frac{1+1-e^{-\lambda t}}{2} = 1 - \frac{1}{2}e^{-\lambda t} & \text{for } t \geq 0 \\
\frac{1}{2}e^{\lambda t} & \text{for } t < 0.
\end{cases}$$

Now, we can choose $f_Z(t) = \frac{1}{2}\lambda e^{-\lambda|t|}$, and check that for $t \leq 0$,

$$\int_{-\infty}^{t} f_Z(s)ds = \int_{-\infty}^{t} \frac{1}{2}\lambda e^{\lambda s}ds = \frac{1}{2} e^{\lambda |t|}|_{-\infty}^{t} = \frac{1}{2}e^{\lambda t};$$

and for $t \geq 0$,

$$\int_{-\infty}^{t} f_Z(s)ds = \int_{-\infty}^{0} \frac{1}{2}\lambda e^{\lambda s}ds + \int_{0}^{t} \frac{1}{2}\lambda e^{-\lambda s}ds = \frac{1}{2} - \frac{1}{2}e^{-\lambda s}|_{0}^{t} = 1 - \frac{1}{2}e^{-\lambda t}.$$

Since we get that

$$F_Z(t) = \int_{-\infty}^{t} f_Z(s)ds$$

for all $t \in \mathbb{R}$, we have that $Z$ is absolutely continuous with density $f_Z$ given above.

(B) The long way is to calculate $\int t^2 f_Z(t)dt$ directly.

However, note that $Z^2 = (2Y - 1)^2X^2$ and since $2Y - 1 \in \{-1, 1\}$ we have that $Z^2 = X^2$. Since $X^2 \sim \text{Exp}(\lambda)$,

$$\mathbb{E}[X^2] = \text{Var}[X] + (\mathbb{E}[X])^2 = \frac{2}{\lambda^2}.$$
If we were to go the long way, it would not be so terrible, as:

\[
\mathbb{E}[Z^2] = \int_{-\infty}^{\infty} t^2 f_Z(t) dt = \int_{-\infty}^{0} t^2 \frac{1}{2} \lambda e^{\lambda t} dt + \int_{0}^{\infty} t^2 \frac{1}{2} \lambda e^{-\lambda t} dt
\]

\[
= 2 \int_{0}^{\infty} \frac{1}{2} \lambda e^{-\lambda t} dt = \mathbb{E}[X^2].
\]

Even an explicit integration by parts here is not so terrible.

(C) We use linearity of expectation and the fact that \(X, Y\) are independent, so any functions of \(X\) and of \(Y\) are independent. Specifically, \((2Y - 1)\) and \(X^2\) are independent and \((2Y - 1)\) and \(X\) are independent. Thus,

\[
\mathbb{E}[ZX] = \mathbb{E}[(2Y - 1)X^2] = \mathbb{E}[(2Y - 1)] \cdot \mathbb{E}[X^2],
\]

\[
\mathbb{E}[Z] = \mathbb{E}[(2Y - 1)X] = \mathbb{E}[(2Y - 1)] \cdot \mathbb{E}[X].
\]

(This already shows that \(\text{Cov}(Z, X) = \mathbb{E}[(2Y - 1)] \cdot \text{Var}[X].\) Now, since

\[
\mathbb{E}[(2Y - 1)] = 2 \mathbb{E}[Y] - 1 = 0
\]

we get that \(\mathbb{E}[ZX] = \mathbb{E}[Z] = 0\) and

\[
\text{Cov}(Z, X) = \mathbb{E}[ZX] - \mathbb{E}[Z] \cdot \mathbb{E}[X] = 0.
\]

(So \(X, Z\) are uncorrelated.)

We claim that \(X, Z\) are not independent. To show this, it suffices to find \(t, s\) such that

\[
\mathbb{P}[X \leq t, Z \leq s] \neq \mathbb{P}[X \leq t] \cdot \mathbb{P}[Z \leq s].
\]

Let us choose \(t = s = 1\). Then,

\[
\mathbb{P}[X \leq 1, Z \leq 1] = \mathbb{P}[X \leq 1, -X \leq 1, Y = 0] + \mathbb{P}[X \leq 1, Y = 1]
\]

\[
= \frac{1}{2} \mathbb{P}[-1 \leq X \leq 1] + \frac{1}{2} \mathbb{P}[X \leq 1] = 1 - e^{-\lambda}.
\]
However, $\mathbb{P}[X \leq 1] = 1 - e^{-\lambda}$ and (using the distribution function computed above)

$$\mathbb{P}[Z \leq 1] = F_Z(1) = 1 - \frac{1}{2}e^{-\lambda}.$$  

Multiplying we get

$$\mathbb{P}[X \leq 1] \cdot \mathbb{P}[Z \leq 1] = 1 - e^{-\lambda} - \frac{1}{2}e^{-\lambda} + \frac{1}{2}e^{-2\lambda} \neq 1 - e^{-\lambda} = \mathbb{P}[X \leq 1, Z \leq 1],$$

because $e^{-2\lambda} \neq e^{-\lambda}$ for any $\lambda > 0$.

Solution Q3:

(A) For every $n$ let $Z_n = X_n - Y_n$. So $(Z_n)_n$ are independent, all have finite variance

$$\sigma^2 := \text{Var}[Z_n] = \text{Var}[X_n] + \text{Var}[Y_n] = \frac{1-p}{p^2} + \frac{1}{p^2},$$

and $\mathbb{E}[Z_n] = \frac{1}{p} - \frac{1}{p} = 0$.

By the Central Limit Theorem,

$$\frac{1}{\sqrt{n}\sigma} \sum_{k=1}^{n} Z_n \xrightarrow{D} N(0,1).$$

In other words,

$$\lim_{n \to \infty} \mathbb{P}[\sum_{k=1}^{n} X_k > \sum_{k=1}^{n} Y_k] = \lim_{n \to \infty} \mathbb{P}[\sum_{k=1}^{n} Z_k > 0]$$

$$= \lim_{n \to \infty} \mathbb{P}[\frac{1}{\sqrt{n}\sigma} \sum_{k=1}^{n} Z_k > 0] = \mathbb{P}[N(0,1) > 0].$$

Since $s \mapsto e^{-s^2/2}$ is an even function,

$$\mathbb{P}[N(0,1) > 0] = \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-s^2/2} ds = \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-s^2/2} ds = \frac{1}{2},$$

Which completes the proof.
(B) $Y_1$ is absolutely continuous, so
\[
\mathbb{E}[e^{\alpha Y_1}] = \int_{-\infty}^{\infty} e^{\alpha t} f_Y(t) \, dt = \int_{0}^{\infty} e^{\alpha t} p e^{-pt} \, dt.
\]
If $\alpha - p \geq 0$ this is $\mathbb{E}[e^{\alpha Y_1}] = \infty$. For $\alpha < p$ this becomes
\[
\mathbb{E}[e^{\alpha Y_1}] = \left. \frac{p e^{(\alpha - p)t}}{\alpha - p} \right|_{0}^{\infty} = \frac{p}{\alpha - p}.
\]

(C) Since $X_1, Y_1$ are independent, we have that so are $e^{\alpha Y_1}, e^{\alpha X_1}$. So $\mathbb{E}[e^{\alpha (X_1 + Y_1)}] = \mathbb{E}[e^{\alpha X_1}] \cdot \mathbb{E}[e^{\alpha Y_1}]$.

We have already seen that if $\alpha \geq p$ then $\mathbb{E}[e^{\alpha Y_1}] = \infty$, so since $e^{\alpha X_1} \geq 0$, we have that if $\alpha \geq p$ then $\mathbb{E}[e^{\alpha (X_1 + Y_1)}] = \infty$.

We turn to the case $\alpha < p$. Then, since $X_1$ is discrete with range $\{1, 2, \ldots, \}$,
\[
\mathbb{E}[e^{\alpha X_1}] = \sum_{k=1}^{\infty} e^{\alpha k} p (1 - p)^{k-1} = \frac{p}{1-p} \sum_{k=1}^{\infty} ((1 - p)e^{\alpha})^k.
\]
If $e^{\alpha} \geq \frac{1}{1-p}$ then this is $\mathbb{E}[e^{\alpha X_1}] = \infty$. If $e^{\alpha}(1 - p) < 1$ then this becomes
\[
\mathbb{E}[e^{\alpha X_1}] = \frac{p}{1-p} \cdot \frac{(1 - p)e^{\alpha}}{1 - (1 - p)e^{\alpha}} = \frac{pe^{\alpha}}{1 - (1 - p)e^{\alpha}}.
\]
To sum up,
\[
\mathbb{E}[e^{\alpha (X_1+Y_1)}] = \begin{cases} 
\frac{p^2 e^{\alpha}}{(\alpha - p)(1 - (1 - p)e^{\alpha})} & \text{if } \alpha < p \text{ and } e^{\alpha} < \frac{1}{1-p}, \\
\infty & \text{otherwise}.
\end{cases}
\]

Solution Q4:

(A)
\[
\frac{\binom{k-1}{n} + \binom{k-1}{n-1}}{\binom{k}{n}} = \frac{k - n}{k} + \frac{n}{k} = 1
\]
(B) We prove this by induction on $k$. The base case is $k = n + 1$. Then the claim is just
\[
\binom{n - 1}{n - 1} = \binom{n}{n}
\]
which is obviously true.

Assume the claim holds for some $k \geq n + 1$. For $k + 1$ we compute using the induction hypothesis, and using (A),
\[
\sum_{m=n}^{k} \binom{m-1}{n-1} = \sum_{m=n}^{k-1} \binom{m-1}{n-1} + \binom{k-1}{n-1} = \binom{k}{n} + \binom{k-1}{n-1} = \binom{k}{n},
\]
which proves the induction step.

(C) We prove the claim by induction on $n$. The base case $n = 1$ is just $\Pr[X_1 = k] = p(1-p)^{k-1}$ for integer $k \geq 1$ and 0 otherwise, which is obvious since $X_1 \sim \text{Geo}(p)$.

Assume the claim is true for $n$. Let $S = S_n$ and $X = X_{n+1}$. We need to show the induction step, which is to show that
\[
\Pr[S + X = k] = \begin{cases} \binom{k-1}{n-1} p^{n+1} (1-p)^{k-1-n} & \text{for an integer } k \geq n + 1 \\ 0 & \text{otherwise} \end{cases}
\]

Indeed, by the induction hypothesis, $S$ has range \{n, n + 1, n + 2, \ldots, \}. Since $S, X$ are independent, the law of total probability (or the discrete convolution identity shown in class) gives
\[
\Pr[S + X = k] = \sum_{m=n}^{\infty} \Pr[S = m, X = k - m] = \sum_{m=n}^{\infty} \Pr[S = m] \Pr[X = k - m].
\]

If $k$ is not an integer this is 0. Since the range of $X$ is the positive integers, we get for any integer $k \leq n$, $\Pr[S + X = k] = 0$ and for any integer $k \geq n + 1$, again by the induction hypothesis,
\[
\Pr[S + X = k] = \sum_{m=n}^{k-1} \binom{m-1}{n-1} p^n (1-p)^{m-n} p(1-p)^{k-m-1} = p^{n+1}(1-p)^{k-(n+1)} \sum_{m=n}^{k-1} \binom{m-1}{n-1}.
\]
That is, using (B),
\[ P[S_{n+1} = k] = \binom{k-1}{n} p^{n+1}(1 - p)^{k-(n+1)}, \]
for all integers \( k \geq n + 1 \) and \( P[S_{n+1} = k] = 0 \) otherwise. This proves the induction step.

(D) Let \( Y_n = X_n - E[X_n] = X_n - \frac{1}{p} \). So \( E[Y_n] = 0 \) and \( (Y_n)_n \) are independent with finite variance. The law of large numbers tells us that
\[ \frac{1}{n} \sum_{k=1}^{n} Y_k \xrightarrow{a.s.} 0. \]

For any \( \omega \), the sequence \( \frac{1}{n} S_n(\omega) \rightarrow C \) if and only if
\[ \frac{1}{n} \sum_{k=1}^{n} Y_k(\omega) = \frac{1}{n} S_n(\omega) - \frac{1}{p} \rightarrow C - \frac{1}{p}. \]

Thus, by the law of large numbers \( \frac{1}{n} S_n \xrightarrow{a.s.} \frac{1}{p} \).