Probability

Solutions to Exam C, Fall 2014

Solution Q1:

(A) If g is any measurable function then

$$\mathbb{E}[g(U)] = \int_{-\infty}^{\infty} g(t) f_U(t) dt = \int_{-\pi/2}^{\pi/2} \tfrac{1}{\pi} g(t) dt.$$

So we want a measurable function $g : [-\frac{\pi}{2}, \frac{\pi}{2}] \to \mathbb{R}$ such that $\int_{-\pi/2}^{\pi/2} g(t)dt = \infty$.

One such function could be $g(x) = \frac{1}{x^2}$ for $x \neq 0$ and $g(0) = 0$. This is indeed a measurable function: for any $0 < t \in \mathbb{R}$,

$$g^{-1}[t, \infty) = \left\{ x \neq 0 \ : \ x^2 \leq \tfrac{1}{t} \right\} = [-\tfrac{1}{\sqrt{t}} 0) \cup (0, \tfrac{1}{\sqrt{t}}],$$

and for $t \leq 0$ we have that $g^{-1}[t, \infty) = \mathbb{R}$.

Now, using a change of variables,

$$\mathbb{E}[g(U)] = \tfrac{1}{\pi} \int_{-\pi/2}^{0} \tfrac{1}{t^2} dt + \tfrac{1}{\pi} \int_{0}^{\pi/2} \tfrac{1}{t^2} dt = \tfrac{2}{\pi} \int_{0}^{\pi/2} \tfrac{1}{t^2} dt$$
$$= \tfrac{2}{\pi} \cdot \left(-\tfrac{1}{t}\right) \Big|_{0}^{\pi/2} = \infty.$$

Actually, if we want to formally justify this integration, we could consider the sequence $X_n = g_n(U)$ where $g_n(x) = g(x) \mathbf{1}_{\{|x| > n^{-1}\}}$. These are all measurable as a measurable function times an indicator. Also, $g_n \leq g_{n+1}$ so $(X_n)_n$ is a monotone sequence that converges to $g(U)$. Also, as above,

$$\mathbb{E}[X_n] = \tfrac{2}{\pi} \int_{n^{-1}}^{\pi/2} \tfrac{1}{t^2} dt = \tfrac{2}{\pi} \cdot \left(n - \tfrac{2}{\pi}\right).$$

By monotone convergence,

$$\mathbb{E}[g(U)] = \lim_{n \to \infty} \mathbb{E}[X_n] = \infty.$$

(B) A simple possibility is to take $g(x) = x + 1$. This is a continuous function, so also measurable. Then,

$$\mathbb{E}[g(U)] = \mathbb{E}[U] + 1 = 1.$$

Another possibility is to take $g(x) = \frac{12}{\pi^2}x^2$ which is measurable since it is continuous. Since $\mathbb{E}[U] = 0$ we have that $\mathbb{E}[U^2] = \mathrm{Var}[U] = \frac{\pi^2}{12}$. So

$$\mathbb{E}[g(U)] = \tfrac{12}{\pi^2}\,\mathbb{E}[U^2] = 1.$$

(C) Define $g(x) = \frac{1}{x^3}$ if $x \neq 0$ and $g(0) = 0$. This is a measurable function since: For $t > 0$,

$$g^{-1}[t, \infty) = \left\{x > 0 \ : \ x^3 \leq \tfrac{1}{t}\right\} = (0, t^{-1/3}].$$

For $t = 0$,

$$g^{-1}[0, \infty) = \{x \ : \ x \geq 0\} = [0, \infty).$$

For $t < 0$,

$$\begin{aligned} g^{-1}[t, \infty) = g^{-1}[t, 0) \biguplus g^{-1}[0, \infty) &= \left\{x < 0 \ : \ |x|^3 \geq \tfrac{1}{|t|}\right\} \biguplus [0, \infty) \\ &= \left\{-x \ : \ x \geq \tfrac{1}{|t|^{1/3}}\right\} \biguplus [0, \infty) = (-\infty, -\tfrac{1}{|t|^{1/3}}] \biguplus [0, \infty). \end{aligned}$$

Let $X = g(U)$. Then,

$$X^+ = \max\{g(U), 0\} = \begin{cases} g(U) & U \geq 0 \\ 0 & U \leq 0 \end{cases}$$

Similarly,

$$X^- = \max\{-g(U), 0\} = \begin{cases} -g(U) & U \leq 0 \\ 0 & U \geq 0. \end{cases}$$

Thus, for $g^+(x) = g(x)\mathbf{1}_{\{x \geq 0\}}$ and $g^-(x) = -g(x)\mathbf{1}_{\{x \leq 0\}}$, we have that $X^+ = g^+(U)$ and $X^- = g^-(U)$.

Thus,

$$\mathbb{E}[X^+] = \int_{-\infty}^{\infty} g^+(t) f_U(t) dt = \tfrac{1}{\pi} \int_0^{\pi/2} t^{-3} dt$$
$$= \tfrac{1}{2\pi} \cdot \left(-t^{-2}\right) \Big|_0^{\pi/2} = \infty.$$

$$\mathbb{E}[X^-] = \int_{-\infty}^{\infty} g^-(t) f_U(t) ft = -\tfrac{1}{\pi} \int_{-\pi/2}^{0} t^{-3} dt$$
$$= \tfrac{1}{2\pi} \cdot t^{-2} \Big|_{-\pi/2}^{0} = \infty.$$

So $\mathbb{E}[X^+] = \mathbb{E}[X^-] = \infty$ and thus $\mathbb{E}[X]$ is not defined.

Solution Q2:

(A) For every n, we have that $S_n = \sum_{k=1}^n X_k$ is a function of $X_1, \ldots, X_n$. Since N is independent of $X_1, \ldots, X_n$ this implies that N, S_n are independent.

(B) Since N, S_n are independent, also $S_n, \mathbf{1}_{\{N=n\}}$ are independent, as functions of independent random variables. Thus, they are uncorrelated and

$$\mathbb{E}[S_n \mathbf{1}_{\{N=n\}}] = \mathbb{E}[S_n]\, \mathbb{E}[\mathbf{1}_{\{N=n\}}] = \sum_{k=1}^{n} \mathbb{E}[X_k] \cdot \mathbb{P}[N = n] = \mu n\, \mathbb{P}[N = n].$$

(C) Using the law of total probability, since the range of N is in the positive integers,

$$1 = \sum_{n=1}^{\infty} \mathbf{1}_{\{N=n\}}.$$

For every n define $K_n = S_n \mathbf{1}_{\{N=n\}}$. We have seen that $\mathbb{E}[K_n] = \mu n\, \mathbb{P}[N = n]$. Note that

$$S_N = \sum_{n=1}^{\infty} S_N \mathbf{1}_{\{N=n\}} = \sum_{n=1}^{\infty} K_n.$$

Set

$$R_n = \sum_{j=1}^{n} K_j.$$

Since all X_k are non-negative, also S_k are non-negative, and thus K_j are all non-negative. Hence $R_n \leq R_{n+1}$ and also $R_n \nearrow S_N$.

By monotone convergence and linearity of expectation we now obtain that

$$\begin{aligned}\mathbb{E}[S_N] &= \lim_{n\to\infty} \mathbb{E}[R_n] = \lim_{n\to\infty} \sum_{j=1}^{n} \mathbb{E}[K_j] \\ &= \lim_{n\to\infty} \sum_{j=1}^{n} \mu j \, \mathbb{P}[N=j] = \mu \cdot \sum_{j=1}^{\infty} j \, \mathbb{P}[N=j] \\ &= \mu \cdot \mathbb{E}[N].\end{aligned}$$

Solution Q3:

(A) Fix $\varepsilon > 0$. Let $A_n = \{|X_n - Y_n| > \varepsilon\}$. Let $\omega \in \Omega$ be such that $Z(\omega) > \varepsilon$. By definition of $\limsup$ this implies that for every n there exists $k \geq n$ such that $|X_n(\omega) - Y_n(\omega)| > \varepsilon$. That is, $\omega \in \bigcap_n \bigcup_{k\geq n} A_k = \limsup_n A_n$. So we have shown that

$$\{Z > \varepsilon\} \subseteq \limsup_{n\to\infty} A_n.$$

Now, we are given that $\sum_n \mathbb{P}[A_n] < \infty$. The first Borel-Cantelli lemma states that in this case $\mathbb{P}[\limsup_n A_n] = 0$. Hence $\mathbb{P}[Z > \varepsilon] \leq \mathbb{P}[\limsup_n A_n] = 0$.

Since $\varepsilon > 0$ was arbitrary, we are done.

(B) Let A be the event $\{Y_n \to Y\}$. We are given that $\mathbb{P}[A] = 1$.

Let B be the event $\{|X_n - Y_n| \to 0\}$. In part (A) we have seen that for any $k > 0$,

$$\mathbb{P}[\limsup_{n\to\infty} |X_n - Y_n| > k^{-1}] = 0.$$

Thus,

$$\mathbb{P}[\limsup_{n\to\infty} |X_n - Y_n| > 0] \leq \sum_{k=1}^{\infty} \mathbb{P}[\limsup_{n\to\infty} |X_n - Y_n| > k^{-1}] = 0.$$

So with probability 1 we have that $\limsup_n |X_n - Y_n| = 0$ which is to say that $\mathbb{P}[B] = 1$.

Now, if $\omega \in B \cap A$ then

$$\limsup_n |X_n(\omega) - Y(\omega)| \leq \limsup_n |X_n(\omega) - Y_n(\omega)| + \limsup_n |Y_n(\omega) - Y(\omega)| = 0.$$

So the event $A \cap B$ implies the event $\{X_n \to Y\}$. Thus,

$$\mathbb{P}[X_n \to Y] \geq \mathbb{P}[A \cap B] = 1,$$

and $(X_n)_n$ converges to Y a.s.