Probability

Solutions to Exam C, Fall 2013

Solution Q1:

(A) We need to show that for any continuity point of F_L, we have $F_{X_n}(t) \to F_L(t)$. (Since L is discrete, with range $\{0, 1, \dots, \}$ the continuity points of F_L are $\mathbb{R} \setminus \{0, 1, \dots, \}$.)

For any $t < 0$, because X_n, L take only non-negative values,

$$F_{X_n}(t) = \mathbb{P}[X_n \le t] = 0 = \mathbb{P}[L \le t].$$

Let $t > 0$. If $n > t$ then,

$$F_{X_n}(t) = \sum_{k=0}^{\lfloor t \rfloor} \mathbb{P}[X_n = k],$$

so by the assumption,

$$\lim_{n\to\infty} F_{X_n}(t) = \sum_{k=0}^{\lfloor t \rfloor} \mathbb{P}[L = k] = \mathbb{P}[L \le t].$$

For $t = 0$ we have that because the range of X_n is $\{0, 1, \dots, n\}$,

$$F_{X_n}(0) = \mathbb{P}[X_n \le 0] = \mathbb{P}[X_n = 0] \to \mathbb{P}[L = 0] = \mathbb{P}[L \le 0] = F_L(0).$$

Thus, $F_{X_n}(t) \to F_L(t)$ for all t, and specifically, $X_n \xrightarrow{\mathcal{D}} L$.

(B) Note that the range of B_n is $\{0, 1, \dots, n\}$ and the range of P is $\{0, 1, \dots\}$ as in (A).

Let $p = \frac{\lambda}{n}$. For any non-negative integer k we have that if $n \ge k$ then

$$\mathbb{P}[B_n = k] = \binom{n}{k} p^k (1-p)^{n-k} = \frac{n!}{(n-k)!n^k} \cdot \frac{\lambda^k}{k!} \cdot \left(1 - \tfrac{\lambda}{n}\right)^n \cdot \left(1 - \tfrac{\lambda}{n}\right)^{-k}$$
$$\to e^{-\lambda} \cdot \frac{\lambda^k}{k!} = \mathbb{P}[P = k].$$

By (A) this implies that $B_n \xrightarrow{\mathcal{D}} P$.

Solution Q2:

(A) Since X^{2k} is non-negative, by Markov's inequality,

$$\mathbb{P}[X \geq 1+\varepsilon] = \mathbb{P}[X^{2k} \geq (1+\varepsilon)^{2k}] \leq \mathbb{E}[X^{2k}](1+\varepsilon)^{-2k} \leq M(1+\varepsilon)^{-2k}.$$

Taking $k \to \infty$ we get 0 on the right-hand side.

(B) We have

$$\{X > 1\} = \bigcup_n \left\{X > 1 + n^{-1}\right\}.$$

Thus, by Boole's inequality (union bound)

$$\mathbb{P}[X > 1] \leq \sum_n \mathbb{P}[X > 1 + n^{-1}] = 0,$$

by (A).

(C) Take $M = 1$. Since

$$\left\{Y^{2k} > 1\right\} \subseteq \{|Y| > 1\} = \{-1 \leq Y \leq 1\}^c,$$

we have that

$$\mathbb{P}[Y^{2k} > 1] = 0.$$

Thus,

$$\mathbb{E}[Y^{2k}] = \mathbb{E}[Y^{2k}\mathbf{1}_{\left\{Y^{2k} \leq 1\right\}}] \leq 1.$$

Solution Q3:

(A) For Z we have: If $r \notin [-1, 1]$,

$$f_Z(r) = \int\int f_{X,Y,Z}(t, s, r)dtds = 0.$$

For $r \in [-1, 1]$,

$$f_Z(r) = \int\int f_{X,Y,Z}(t, s, r)dtds = \int_{-1}^{1}\int_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} \tfrac{1}{2\pi}dtds \int_{-1}^{1} \tfrac{1}{\pi}\sqrt{1-s^2}ds = \tfrac{1}{2}.$$

So

$$f_Z(r) = \begin{cases} \frac{1}{2} & r \in [-1,1] \\ 0 & r \notin [-1,1] \end{cases}$$

As for X, we have: If $t \notin [-1,1]$ then $t^2 > 1$ so

$$f_X(t) = \int \int f_{X,Y,Z}(t,s,r) ds dr = 0.$$

If $t \in [-1,1]$ then

$$f_X(t) = \int_{-1}^{1} \int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} \tfrac{1}{2\pi} ds dr = \frac{2 \cdot \sqrt{1-t^2}}{\pi}.$$

So

$$f_X(t) = \begin{cases} \frac{2}{\pi} \cdot \sqrt{1-t^2} & t \in [-1,1] \\ 0 & t \notin [-1,1] \end{cases}$$

(B) By (A) we have that $Z \sim U[-1,1]$ so $\mathrm{Var}[Z] = \frac{2^2}{12} = \frac{1}{3}$. This can also be easily calculated directly, since

$$\mathbb{E}[Z] = \int_{-1}^{1} \tfrac{1}{2} r dr = 0,$$

and so

$$\mathrm{Var}[Z] = \mathbb{E}[Z^2] = \int_{-1}^{1} \tfrac{1}{2} r^2 dr = \tfrac{1}{2} \cdot \tfrac{r^3}{3}\Big|_{-1}^{1} = \tfrac{1}{3}.$$

(C) First we calculate $f_{X,Y}$. If $t^2 + s^2 > 1$ then

$$f_{X,Y}(t,s) = \int f_{X,Y,Z}(t,s,r) dr = 0.$$

For $t^2 + s^2 \leq 1$,

$$f_{X,Y}(t,s) = \int_{-1}^{1} \tfrac{1}{2\pi} dr = \tfrac{1}{\pi}.$$

Thus,

$$\begin{aligned}\mathbb{E}[XY] &= \int\int ts f_{X,Y}(t,s)dtds = \int_{-1}^{1}\int_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} \tfrac{1}{\pi} tsdtds \\ &= \int_{-1}^{1} \tfrac{s}{\pi}\left(\int_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} t\ dt\right) ds = 0.\end{aligned}$$

Also,

$$\begin{aligned}\mathbb{E}[Y] &= \int\int s f_{X,Y}(t,s)dtds = \int_{-1}^{1} \tfrac{s}{\pi}\left(\int_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} dt\right) ds \\ &= \int_{-1}^{1} \tfrac{s}{\pi} \cdot 2\sqrt{1-s^2}\ ds = 0,\end{aligned}$$

because the function in the integral is an odd function, and the integral is symmetric around 0.

$\mathbb{E}[X]$ can be computed similarly, although we may also use (A) to compute

$$\mathbb{E}[X] = \int t f_X(t)dt = \tfrac{2}{\pi}\int_{-1}^{1} t \cdot \sqrt{1-t^2}\ dt = 0,$$

again because the function is an odd function and the integral is symmetric around 0.

Altogether,

$$\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\cdot\mathbb{E}[Y] = 0.$$

Solution Q4:

(A) It is immediate that Z is discrete with range in $\{0, 1, \dots, \}$. For any non-negative integer k, using the independence of X, Y,

$$\begin{aligned}\mathbb{P}[Z = k] &= \sum_{n=0}^{\infty} \mathbb{P}[X = n, Z = k] = \sum_{n=0}^{\infty} \mathbb{P}[X = n, Y = k - n] \\ &= \sum_{n=0}^{k} \mathbb{P}[X = k] \cdot \mathbb{P}[Y = k - n] = \sum_{n=0}^{k} e^{-\alpha} \tfrac{\alpha^k}{k!} \cdot e^{-\beta} \tfrac{\beta^{k-n}}{(k-n)!} \\ &= e^{-(\alpha+\beta)} \tfrac{1}{k!} \cdot \sum_{n=0}^{k} \binom{k}{n} \alpha^k \beta^{k-n} = e^{-(\alpha+\beta)} \frac{(\alpha+\beta)^k}{k!}.\end{aligned}$$

This is exactly the density of $\mathrm{Poi}(\alpha + \beta)$.

(B) For any $t \in \mathbb{R}$, by independence of x and Y,

$$\mathbb{P}[Z > t] = \mathbb{P}[X > t, Y > t] = \mathbb{P}[X > t] \cdot \mathbb{P}[Y > t] = (1 - F_X(t)) \cdot (1 - F_Y(t)).$$

Thus,

$$\mathbb{P}[Z > t] = \begin{cases} 1 & t < 0 \\ e^{-\alpha t} \cdot e^{-\beta t} & t \geq 0 \end{cases}$$

That is,

$$F_Z(t) = 1 - \mathbb{P}[Z > t] = \begin{cases} 0 & t < 0 \\ 1 - e^{-(\alpha+\beta)t} & t \geq 0 \end{cases}$$

This is exactly the distribution function of $\mathrm{Exp}(\alpha + \beta)$.