Probability

Solutions to Exam A, Fall 2015

Solution Q1:

(A) φ is always defined because it is the expectation of a non-negative random variable, since $e^{tX} \geq 0$ always. It is always positive, because if it was $\varphi_X(t) = 0$, then this being the expectation of a non-negative random variable, we would have $e^{tX} = 0$ a.s. But that could only happen if $X = -\infty$ a.s., which is contradictory to the definition of a random variable.

Now, if X, Y are independent, then also e^{tX}, e^{tY} are independent, as functions of independent random variables. So e^{tX}, e^{tY} are uncorrelated, which is to say that

$$\varphi_{X+Y}(t) = \mathbb{E}[e^{tX} \cdot e^{tY}] = \mathbb{E}[e^{tX}] \cdot \mathbb{E}[e^{tY}] = \varphi_X(t) \cdot \varphi_Y(t).$$

Also, for any constant c,

$$\varphi_{cX}(t) = \mathbb{E}[e^{ctX}] = \varphi_X(ct).$$

(B) We have that $Y_n \leq Y_{n+1}$ for all n, because $Y_{n+1} - Y_n = \frac{t^{n+1} X^{n+1}}{(n+1)!} \geq 0$. Since $Y_n(\omega) \to e^{tX(\omega)}$ for any ω, we have that $Y_n \nearrow X$. By the Monotone Convergence Theorem, $\mathbb{E}[Y_n] \to \mathbb{E}[e^{tX}] = \varphi_X(t)$.

Now, for any n, by linearity of expectation we have that

$$\mathbb{E}[Y_n] = \sum_{k=0}^{n} \frac{t^k \, \mathbb{E}[X^k]}{k!}.$$

Taking $n \to \infty$ on both sides of this equation gives

$$\varphi_X(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \, \mathbb{E}[X^k].$$

(C) Recall that $X = X_1 + \cdots + X_n$ for $X_1, \ldots, X_n$ all independent and $X_j \sim \mathrm{Ber}(p)$. Thus, using (A) repeatedly,

$$\varphi_X(t) = \varphi_{X_1+\cdots+X_n}(t) = \varphi_{X_1+\cdots+X_{n-1}}(t) \cdot \varphi_{X_n}(t) = \cdots = \varphi_{X_1}(t) \cdots \varphi_{X_n}(t).$$

For any j we have that

$$\varphi_{X_j}(t) = e^{t0}(1-p) + e^t p = 1 - p(1 - e^t).$$

So

$$\varphi_X(t) = \big(1 - p(e^t - 1)\big)^n .$$

Solution Q2:

(A) For any $\varepsilon > 0$ we have that the event $\{|X - Y| > \varepsilon\}$ implies the event $\{|X - X_n| > \varepsilon/2\} \bigcup \{|Y - X_n| > \varepsilon/2\}$. (This is because $|X - Y| \leq |X - X_n| + |Y - X_n|$.) So for any $k > 0$ we have that

$$\mathbb{P}[|X - Y| > k^{-1}] \leq \mathbb{P}[|X - X_n| > \tfrac{1}{2k}] + \mathbb{P}[|Y - Y_n| > \tfrac{1}{2k}] \to 0$$

as $n \to \infty$. Thus, $\mathbb{P}[|X - Y| > k^{-1}] = 0$ for all k. However, $\{X \neq Y\} \subset \bigcup_k \{|X - Y| > k^{-1}\}$, so Boole's inequality tells us that

$$\mathbb{P}[X \neq Y] \leq \sum_k \mathbb{P}[|X - Y| > k^{-1}] = 0.$$

(B) Since

$$F_X(t) = \begin{cases} 1 & t \geq c \\ 0 & t < c \end{cases}$$

we know that for any $t \neq c$,

$$\mathbb{P}[X_n \leq t] \to \mathbb{P}[X \leq t] = F_X(t).$$

Thus, for any $\varepsilon > 0$, we have that

$$\begin{aligned}\mathbb{P}[|X_n - X| > \varepsilon] &= \mathbb{P}[|X_n - c| > \varepsilon, X = c] + \mathbb{P}[|X_n - X| > \varepsilon, X \neq c] \\ &\leq \mathbb{P}[|X_n - c| > \varepsilon] = \mathbb{P}[X_n > c + \varepsilon] + \mathbb{P}[X_n < c - \varepsilon] \\ &\leq 1 - \mathbb{P}[X_n \leq c + \varepsilon] + \mathbb{P}[X_n \leq c - \varepsilon] \to 1 - F_X(c + \varepsilon) + F_X(c - \varepsilon) = 0.\end{aligned}$$

So $(X_n)_n$ converges to X in probability.

Solution Q3:

(A) For any $t \in [-1, 1]$,

$$f_X(t) = \int_{-\infty}^{\infty} f_{X,Y}(t, s)ds = \int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} c ds = 2c\sqrt{1 - t^2}.$$

If $t \notin [-1, 1]$ then $f_{X,Y}(t, s) = 0$ for any s so $f_X(t) = 0$.

Two ways to find c: One, the support of $f_{X,Y}$ is just the unit disc, so $\int\int f_{X,Y}(t, s)dtds = c\pi$ and $c = \pi^{-1}$. The second is to integrate f_X above. So

$$f_X(t) = \begin{cases} \frac{2}{\pi} \cdot \sqrt{1 - t^2} & t \in [-1, 1] \\ 0 & \text{otherwise} \end{cases}$$

Now,

$$\mathbb{E}[X] = \int t f_X(t)dt = \int_{-1}^{1} t\sqrt{1 - t^2}dt = 0,$$

because $t \mapsto t\sqrt{1 - t^2}$ is an odd function on $[-1, 1]$.

(B) For any $t \notin [-1, 1]$, we have that $f_{X,Y}(t, s) = 0$ for all s, so $f_X(t) = 0$. Similarly, if $s \notin [-1, 1]$ then $f_Y(s) = 0$.

For $t \in [-1, 1]$ we have

$$f_X(t) = \int_{-\infty}^{\infty} f_{X,Y}(t, s)ds = \int_{|t|-1}^{1-|t|} c ds = 2c(1 - |t|).$$

If we integrate this we get

$$1 = \int_{-\infty}^{\infty} f_X(t)dt = \int_{-1}^{1} 2c(1-|t|)dt = 2c\int_{-1}^{0}(1+t)dt + 2c\int_{0}^{1}(1-t)dt$$
$$= 2c \cdot \left((t + \tfrac{1}{2}t^2)\Big|_{-1}^{0} + (t - \tfrac{1}{2}t^2)\Big|_{0}^{1} \right) = 2c.$$

So $c = \frac{1}{2}$, and

$$f_X(t) = \begin{cases} 1 - |t| & t \in [-1,1] \\ 0 & \text{otherwise} \end{cases}$$

Similarly, for $s \in [-1,1]$,

$$f_Y(s) = \int_{|s|-1}^{1-|s|} c dt = 2c(1-|s|) = 1 - |s|,$$

and actually $f_Y = f_X$.

Since $f_Y = f_X$ we have that

$$\mathbb{E}[Y^2] = \mathbb{E}[X^2] = \int_{-\infty}^{\infty} t^2 f_X(t)dt = \int_{-1}^{1} t^2(1-|t|)dt$$
$$= \int_{-1}^{0} t^2(1+t)dt + \int_{0}^{1} t^2(1-t)dt = (\tfrac{1}{3}t^3 + \tfrac{1}{4}t^4)\Big|_{-1}^{0} + (\tfrac{1}{3}t^3 - \tfrac{1}{4}t^4)\Big|_{0}^{1}$$
$$= -(-\tfrac{1}{3}) - \tfrac{1}{4} + \tfrac{1}{3} - \tfrac{1}{4} = \tfrac{1}{6}.$$

(C) By linearity,

$$\mathrm{Cov}(A,B) = \alpha\,\mathrm{Cov}(X-Y,B) = \alpha^2\,\mathrm{Cov}(X-Y,X+Y)$$
$$= \alpha^2 \cdot (\mathrm{Cov}(X,X) - \mathrm{Cov}(Y,X) + \mathrm{Cov}(X,Y) - \mathrm{Cov}(Y,Y))$$
$$= \alpha^2 \cdot (\mathrm{Var}[X] - \mathrm{Var}[Y]).$$

Since $f_X = f_Y$ we have that $\mathrm{Var}[X] = \mathrm{Var}[Y]$ and so $\mathrm{Cov}(A,B) = 0$.