Probability

Solutions to Exam B, Fall 2015

Solution Q1:

(A) $S_m - S_n = X_{n+1} + \cdots + X_m$. For any t, s we have that $\{S_m - S_n \leq t\} \in \sigma(X_{n+1}, \ldots, X_m)$ and $\{S_n \leq s\} \in \sigma(X_1, \ldots, X_n)$. These σ-algebras are independent, so $\{S_m - S_n \leq t\}$ and $\{S_n \leq s\}$ are independent events. So for all t, s,

$$\mathbb{P}[S_m - S_n \leq t \ , \ S_n \leq s] = \mathbb{P}[S_m - S_n \leq t] \cdot \mathbb{P}[S_n \leq s].$$

This implies that $S_m - S_n$ and S_n are independent.

(B) The random variable $S_n \cdot \mathbf{1}_{A_n}$ is a function of $X_1, \ldots, X_n$. The random variable $S_m - S_n$ is a function of $X_{n+1}, \ldots, X_m$. Since $(X_1, \ldots, X_n)$ is independent of $(X_{n+1}, \ldots, X_m)$, we get that also $S_m - S_n$ is independent of $S_n \cdot \mathbf{1}_{A_n}$. So these random variables are also uncorrelated and

$$\mathbb{E}[(S_m - S_n) \cdot S_n \cdot \mathbf{1}_{A_n}] = \mathbb{E}[S_m - S_n] \cdot \mathbb{E}[S_n \cdot \mathbf{1}_{A_n}].$$

Since $\mathbb{E}[S_m - S_n] = \mathbb{E}[X_{n+1}] + \cdots + \mathbb{E}[X_m] = 0$ we have that $\mathbb{E}[(S_m - S_n) \cdot S_n \cdot \mathbf{1}_{A_n}] = 0$.

(C) Write

$$\begin{aligned}
\mathbb{E}[S_m^2 \cdot \mathbf{1}_{A_n}] &= \mathbb{E}[(S_m - S_n + S_n)^2 \cdot \mathbf{1}_{A_n}] \\
&= \mathbb{E}[(S_m - S_n)^2 \cdot \mathbf{1}_{A_n}] + \mathbb{E}[S_n^2 \cdot \mathbf{1}_{A_n}] + 2\,\mathbb{E}[(S_m - S_n) \cdot S_n \cdot \mathbf{1}_{A_n}] \\
&= \mathbb{E}[(S_m - S_n)^2 \cdot \mathbf{1}_{A_n}] + \mathbb{E}[S_n^2 \cdot \mathbf{1}_{A_n}] \\
&\geq \mathbb{E}[S_n^2 \cdot \mathbf{1}_{A_n}].
\end{aligned}$$

(D) We will show that

$$\{M_n \geq a\} = \biguplus_{k=1}^{n} A_k.$$

So $\mathbf{1}_{\{M_n \geq a\}}(\omega) = 1$ if and only if there is exactly one $1 \leq k \leq n$ such that $\mathbf{1}_{A_k}(\omega) = 1$. This implies that

$$\mathbf{1}_{\{M_n \geq a\}} = \sum_{k=1}^{n} \mathbf{1}_{A_k}.$$

If $M_n \geq a$, then, there exists $j \leq n$ such that $|S_j| \geq a$. Thus, there exists a minimal such j; that is, there exists k for which $|S_j| < a$ if $j < k$ and $|S_k| \geq a$. So we have shown that $\{M_n \geq a\} \subset \bigcup_{k=1}^{n} A_k$.

On the other hand, if A_k occurs then $|S_k| \geq a$, so also $M_n \geq a$. That is, $A_k \subset \{M_n \geq a\}$ for all $k \leq n$, which implies that $\bigcup_{k=1}^{n} A_k \subset \{M_n \geq a\}$.

The two inclusions prove that

$$\{M_n \geq a\} = \bigcup_{k=1}^{n} A_k.$$

So we are left with showing that the union is disjoint.

If $k > j$ we have

$$A_k \cap A_j \subset \{|S_j| < a\} \cap \{|S_j| \geq a\} = \emptyset.$$

So $(A_k)_k$ are pairwise disjoint.

(E) Since $S_n^2 \geq S_n^2 \mathbf{1}_{\{M_n \geq a\}}$, by linearity,

$$\mathbb{E}[S_n^2] \geq \mathbb{E}[S_n^2 \mathbf{1}_{\{M_n \geq a\}}] = \sum_{k=1}^{n} \mathbb{E}[S_n^2 \mathbf{1}_{A_k}] \geq \sum_{k=1}^{n} \mathbb{E}[S_k^2 \mathbf{1}_{A_k}],$$

where the last inequality is from (C).

Markov's inequality gives that for the non-negative random variable $S_n^2 \mathbf{1}_{A_n}$,

$$\mathbb{P}[S_n^2 \mathbf{1}_{A_n} \geq a^2] \leq \mathbb{E}[S_n^2 \mathbf{1}_{A_n}] \cdot \tfrac{1}{a^2}.$$

Also, if $M_n \geq a$ then there exists $k \leq n$ such that $\mathbf{1}_{A_k} = 1$ and $S_k^2 \geq a^2$. So,

$$\begin{aligned}
\mathbb{P}[M_n \geq a] &\leq \mathbb{P}[\bigcup_{k=1}^n \{S_k^2 \mathbf{1}_{A_k} \geq a^2\}] \\
&\leq \sum_{k=1}^n \mathbb{P}[S_k^2 \mathbf{1}_{A_k} \geq a^2] \leq \frac{1}{a^2} \cdot \sum_{k=1}^n \mathbb{E}[S_k^2 \mathbf{1}_{A_k}] \\
&\leq \frac{1}{a^2} \cdot \mathbb{E}[S_n^2].
\end{aligned}$$

Finally, since $\mathbb{E}[S_n] = \sum_{k=1}^n \mathbb{E}[X_k] = 0$ we have that $\mathbb{E}[S_n^2] = \mathrm{Var}[S_n]$ and so

$$\mathbb{P}[M_n \geq a] \leq \frac{1}{a^2} \cdot \mathrm{Var}[S_n].$$

Solution Q2:

(A) Since X is absolutely continuous,

$$\begin{aligned}
\mathbb{E}[(X^+)^2] &= \int_{-\infty}^{\infty} (t^+)^2 f_X(t) dt = \int_0^\infty t^2 f_X(t) dt \\
&= \int_0^\infty \int_0^t 2s ds f_X(t) dt = \int_0^\infty \int_0^t 2s f_X(t) ds dt \\
&= \int_0^\infty \int_s^\infty 2s f_X(t) dt ds = \int_0^\infty 2s\, \mathbb{P}[X > s] ds.
\end{aligned}$$

(B) We have

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{k=0}^\infty k\, \mathbb{P}[X = k] = \sum_{k=1}^\infty \sum_{m=0}^{k-1} \mathbb{P}[X = k] \\
&= \sum_{m=0}^\infty \sum_{k=m+1}^\infty \mathbb{P}[X = k] = \sum_{m=0}^\infty \mathbb{P}[X \geq m+1] \\
&= \sum_{m=0}^\infty \mathbb{P}[X > m].
\end{aligned}$$

Solution Q3:

(A) Note that for any k we have that $n_k, n_{k+1} \geq n_k$ so

$$\mathbb{P}[|Y_{k+1} - Y_k| > 2^{-k}] = \mathbb{P}[|X_{n_{k+1}} - X_{n_k}| > 2^{-k}] < 2^{-k}.$$

For any n we have

$$\mathbb{P}[\bigcup_{k\geq n} \{|Y_k - Y_{k+1}| > 2^{-k}\}] \leq \sum_{k\geq n} \mathbb{P}[|Y_{k+1} - Y_k| > 2^{-k}] \leq \sum_{k=n}^{\infty} 2^{-k} = 2^{-n+1}.$$

For any n we have that $F \subset \bigcup_{k\geq n} \{|Y_k - Y_{k+1}| > 2^{-k}\}$, so $\mathbb{P}[F] \leq 2^{-n+1}$ for all n. Thus, $\mathbb{P}[F] = 0$.

(B) We know that $(Y_k)_k$ converges if and only if $(Y_k)_k$ is a Cauchy sequence. So we need to show that $(Y_k)_k$ is a Cauchy sequence a.s.

Now, if $\omega \notin F$ (for F as in (A)), then

$$\omega \in F^c = \bigcup_n \bigcap_{k\geq n} \{|Y_k - Y_{k+1}| \leq 2^{-k}\}.$$

That is, if $\omega \notin F$ then there exists n such that for all $k \geq n$ we have $|Y_{k+1}(\omega) - Y_k(\omega)| \leq 2^{-k}$. In this case, for any $k \geq n$ and $m \geq 0$ we have that

$$|Y_{k+m}(\omega)-Y_k(\omega)| \leq \sum_{j=0}^{m-1} |Y_{k+j+1}(\omega)-Y_{k+j}(\omega)| \leq \sum_{j=0}^{\infty} |Y_{k+i+1}(\omega)-Y_{k+i}(\omega)| \leq \sum_{j=0}^{\infty} 2^{-k-j} = 2^{-k+1}.$$

That is, for any $\varepsilon > 0$ there exists n such that for all $k, m \geq n$ we have $|Y_k(\omega) - Y_m(\omega)| \leq \varepsilon$. So for any $\omega \notin F$ we have that $(Y_k(\omega))_k$ forms a Cauchy sequence. Thus,

$$\mathbb{P}[(Y_k)_k \text{ is a Cauchy sequence }] \geq \mathbb{P}[F^c] = 1.$$

(C) $A = \{Z = W\} = \{Z - W = 0\} = (Z-W)^{-1}(\{0\})$. Since Z, W are random variables, so is $Z - W$, and so $A = (Z - W)^{-1}(\{0\})$ is an event.

Now, for any ω we have that the limit $\lim_k X_k(\omega)$ exists if and only if $Z(\omega) = W(\omega)$. If the limit exists, then it is equal to $Z(\omega) = W(\omega)$. Thus,

$$Y(\omega) = \left\{ \begin{array}{ll} Z(\omega) & \text{if } Z(\omega) = W(\omega) \\ 0 & \text{if } Z(\omega) \neq W(\omega) \end{array} \right\} = Z(\omega)\mathbf{1}_{\{Z=W\}}(\omega).$$

That is, $Y = Z \cdot \mathbf{1}_A$. Because A is an event, $\mathbf{1}_A$ is a random variable, and thus so is $Y = Z \cdot \mathbf{1}_A$ as a product of two random variables.