Automorphism groups of trees acting locally with affine permutations.

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Abstract. We consider the structure of groups that act on a p^n-regular tree in a vertex transitive way with the local action (i.e. the action of the vertex stabilizer on the link) isomorphic to the group of affine transformations on a finite affine line.

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1 Introduction

Let T be the $q-$regular tree, and H a closed subgroup of automorphisms of T acting transitively on the vertices T. The stabilizer of a vertex acts on the link of the vertex via a finite permutation group say $G < Sym_q$ which is independent of the given vertex. We say that the local action of H is given by the finite permutation group G. Given a finite permutation group G, Burger and Mozes ([BM00a]) define a universal group $U(G) < \mathrm{Aut}(T)$ containing a conjugate of every vertex transitive group H whose local action is given by G.

Of special interest are the groups $U(G)$ where G is a two-transitive permutation group. In fact for every vertex x the stabilizer $U(G)_x$ acts transitively on the boundary ∂T iff the group G is two-transitive.

Let G be a two-transitive permutation group and let G_0 be the stabilizer of a point. Investigating the subgroup structure of $U(G)$, Burger and Mozes show that if the group G_0 is non Abelian simple (or close to being non Abelian simple) then the family of closed vertex transitive groups whose local action is described by G is very restricted:

Theorem 1.1. *(Burger-Mozes [BM00a]) Let $G < Sym_q$ be a two transitive permutation group, with G_0 non Abelian simple. Then there is a finite number of closed subgroups of $U(G)$ whose local action is given by G. All such proper subgroups are discrete.*

On the other hand if G_0 is solvable there can be many closed vertex transitive groups whose action is locally described by G. An example for this is the group $G = PGL_2(\mathbb{F}_p)$ acting on the projective line $\mathbb{P}^1\mathbb{F}_p$. Indeed, if we take K to be any complete discretely valued field with residue field $\mathbb{F}_p$ then the action of $PGL_2(K)$ on its Bruhat-Tits tree is locally described by G.

In this paper we attempt to investigate the extreme situation where the group G_0 is very far from being non-Abelian simple. We consider the group $A = \mathrm{Aff}(\mathbb{F}_q)$ of affine transformations acting on the affine line of some finite field. We use the facts that A is strictly two transitive permutation group and that A_0 is Abelian [1].

The paper is organized as follows. Section (2) gives some definitions and describes some technical tools. The most important idea introduced in this section is the notion of the local permutation map associated with an automorphism and its connection with the commensurability group of the tree. This idea was introduced by Lubotzky-Mozes-Zimmer in ([LMZ94]) and is reproduced here for the convenience of the reader. In section (3) the group $U(A)$ is introduced and some basic properties are mentioned. In section (4) the main technical tool of this paper (theorem 4.4) is developed. This theorem enables us to combinatorially analyze the orbits for the action of cyclic subgroups of the vertex stabilizer $U(A)_{x_0}$ on the tree. In sections (5) and (6) we draw some corollaries of theorem (4.4) among them:

- The group $U(A)_{x_0}$ is a pro-solvable group. We give an explicit description of its pro-Sylow subgroups.
- We show an explicit description of the commensurator of a discrete subgroup of $PGL_2(\mathbb{Q}_p)$ which is not a lattice.
- We give an example of an interesting closed vertex transitive subgroup of $U(A)$ whose local action is described by A. This is the junction group defined in (3.6). We prove that this group has virtually torsion free vertex stabilizers.
- Irreducible uniform lattices in $\mathrm{Aut}(T_1 \times T_2)$ whose action on both trees is locally described by primitive permutation groups are extensively studied by Burger, Mozes and Zimmer in ([BM00b, BM00a, BM97, BMZ]). In section (6) we give a new constructions of such lattices using the combinatorial methods developed earlier.

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2 Local permutation maps and Commensurator elements

The main technical tool used in this paper is the description of an automorphism of a tree using the local permutation that it induces on every vertex link. This description of elements in $\mathrm{Aut}(T)$ was introduced in ([LMZ94]) where it is also shown that this description of automorphisms is especially suitable for describing elements of the commensurability subgroup as these give rise to periodic local permutation maps. In this section we recall the definitions and main results concerning local permutation maps. We refer the reader to ([LMZ94]) for more details on the subject.

[1] In fact these two properties characterize the groups $\mathrm{Aff}(\mathbb{F}_q)$

We will use the following terminology

Definition 2.1. *A graph X is a 5-tuple $X = (VX, EX, i, t, ^{-})$ with VX the set of vertices, EX the set of (oriented) edges, $i, t : EX \to VX$ the initial and terminal maps, $^{-} : EX \to EX$ a fixed point free involution called the edge inversion taking $e \to \overline{e}$ and satisfying $i(\overline{e}) = t(e)$ and $t(\overline{e}) = i(e)$. A path in a graph is a sequence of edges $\alpha = (e_0, e_1, \dots, e_N)$ with $t(e_i) = i(e_{i+1}) \,\forall i = 0, \dots N-1$, we denote $i(\alpha) = i(e_0)$ and $t(\alpha) = t(e_N)$. We say that α has a backtracking if for some i we have $e_i = \overline{e_{i+1}}$. A path α is called a loop if $i(\alpha) = t(\alpha)$. A graph X is called a tree, if every closed loop without backtracking is trivial (i.e. has no edges). We define the star of a vertex $St(v) = t^{-1}(v)$ for $v \in VX$. A graph morphism $l : X \to Y$ consists of two maps $l : VX \to VY$ and $l : EX \to EY$ consistent with the structure maps $i, t, ^{-}$.*

Definition 2.2. *A* **legal edge coloring** *of a regular graph X is an edge coloring such that every edge has the same color as its opposite, and for every vertex $v \in VX$ the coloring induces a bijection between the edges in $St(v)$ and the set of colors.*

Definition 2.3. *A* **V-structure** *on a graph X is a map $\phi : VX \to G$ into a finite group G.*

Definition 2.4. *A* **C-structure** *on a graph X is a map $\psi : EX \to G$ into a finite group G, satisfying the reversibility condition $\psi_{\overline{e}} = (\psi_e)^{-1}$.*

Definition 2.5. *Let (X, x_0) be a q-regular graph with a base point, F a set of colors of cardinality q. Then a* **legal V-structure** *on (X, x_0) is defined to be a triplet (X, h, ϕ) with $h : EX \to F$ a legal edge coloring and with $\phi : VX \to Sym(F)$ a V-structure satisfying $\phi_{i(e)}(h(e)) = \phi_{t(e)}(h(e))$. Similarly a* **legal C-structure** *is a triplet (X, h, ψ) with $\psi : EX \to Sym(F)$ a C-structure satisfying the consistency condition $\psi_e(h(e)) = h(e)$.*

Definition 2.6. *If $l : X \to Y$ is a graph morphism and $\phi : VY \to G$ is a V-structure (resp $\psi : EY \to G$ is a C-structure) on Y. Then we define a V-structure (resp C-structure) $i^*\phi \overset{\text{def}}{=} \phi \circ i : EX \to G$ on X called* **the pull back of the V-structure (resp. C-structure) to X.**

Notice that for covering morphisms of graphs the pull back of a legal V-structure (resp C-structure) is still legal with respect to the pull back of the coloring.

Definition 2.7. *If ϕ is a given V-structure on a graph X then the* **differential of** ϕ *is the C-structure on the same graph defined by:*

$$d\phi_e \overset{\text{def}}{=} (\phi_{i(e)})^{-1} \circ \phi_{t(e)} \tag{2.1}$$

Notice that the differential of a legal V-structure is a legal C-structure.

Definition 2.8. *Let (X, x_0) be a pointed graph and $\psi : EX \to G$ a C-structure. We define a homomorphism called* **the homomorphism induced on the fundamental group** $\tilde{\psi} : \pi_1(X, x_0) \to G$ *defined by $(e_1, e_2, \dots e_N) \to \psi_{e_1}\psi_{e_2}\dots\psi_{e_N}$.*

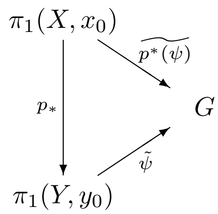
Definition 2.9. *We say that a C-structure $\psi : E(X, x_0) \to G$ on a pointed graph satisfies the* **Kirchoff loop law** *if the homomorphism $\tilde{\psi} : \pi_1(X, x_0) \to G$ is trivial.*

Proposition 2.10. *A C-structure ψ on a finite pointed connected graph X satisfies the Kirchoff loop law iff it is obtained as a differential of some V-structure ϕ on the same graph.*

Proof. Assume that ψ satisfies the Kirchoff loop law. We define a V-structure ϕ in the following way. Choose arbitrarily $\phi_{x_0} \in G$ and for every $x \in VX$ define $\phi_x \stackrel{\text{def}}{=} \psi_{e_1}\psi_{e_2}\dots\psi_{e_N}$, where $(e_1, e_2, \dots e_N)$ is a path from x_0 to x. The fact that ψ satisfies the Kirchoff loop law shows that this does not depend on the choice of the path. It is now easy to check that indeed $\psi = d\phi$. The other direction is even easier. □

The following lemma gives the connection between the above definitions, its proof is immediate.

Lemma 2.11. *Given a covering morphism of two pointed graphs $p : (X, x_0) \to (Y, y_0)$ and given a C-structure $\psi : E(Y, y_0) \to G$ on the graph Y then the following diagram is commutative.*



Lemma (2.11) gives us the following:

Corollary 2.12. *Given a C-structure $\psi : E(Y, y_0) \to G$ on a graph Y. There exists a finite regular cover $p : (X, x_0) \to (Y, y_0)$ such that the induced C-structure $p^*\psi$ on X satisfies the Kirchoff loop law*

Proof. Let $\Lambda = ker(\tilde{\psi}) \triangleleft \pi_1(Y, y_0)$ and (X, x_0) the covering of (Y, y_0) corresponding to Λ (via the Galois correspondence between subgroups of $\pi_1(Y, y_0)$ and covering spaces of Y). Let $p : (X, x_0) \to (Y, y_0)$ be the covering morphism. The fact that $p^*(\psi)$ satisfies the Kirchoff loop law now follows from the definition of Λ and from lemma (2.11). The cover is regular because Λ is a normal subgroup. □

Let $T = \mathcal{T}_q$ be the q-regular tree. We choose a base vertex $t_0 \in VT$. We let F be a finite set of cardinality q of which we think as a set of colors. And we choose, once and for all, a legal edge coloring $h : ET \to F$ of T.

Definition 2.13. *([LMZ94]) Given an automorphism $a \in \mathrm{Aut}(T)$, there is a legal V-structure (T, h, ϕ^h_a) called* **the local permutation map induced by a**, *where ϕ^h_a is a map*

$$\begin{array}{rcl} \phi^h_a : VT & \to & Sym(F) \\ v & \to & \phi^h_{a,v} \end{array} \tag{2.2}$$

defined by the equation:

$$\phi^h_{a,v} = h \circ a \circ (h|_{St(v)})^{-1} \tag{2.3}$$

If the coloring h in the above definition is understood we will omit it from the notation.

The point is that once we know the image of one vertex, the automorphism $a \in \mathrm{Aut}(T)$ is completely determined by its local permutation map. So every element $a \in Aut(T)$ can be reconstructed from the following data:

1. <u>The initial data:</u> The image of the base point $a(t_0)$.

2. The local data: The local permutation map ϕ_a.

The element a can be reconstructed from the differential of ϕ_a as well, but with some more initial data. Namely we need:

1. The image of the base point $a(t_0)$.

2. The local action on the star of the base point ϕ_{a,t_0}.

3. The differential of the local permutation map $d\phi_a$.

One of the fundamental observations in [LMZ94] is that elements in the commensurability group of T can be characterized as those admitting a periodic local permutation map. This makes the combinatorial description of commensurator elements possible.

Let $\Gamma < \mathrm{Aut}(T)$ be the lattice of color preserving automorphisms of T and $C = Comm_{\mathrm{Aut}(T)}(\Gamma) = Comm(T)$ its commensurator in the whole automorphism group. By Leighton's theorem ([Lei82, BK90]) C depends on Γ only up to conjugation. Note that if $\Lambda < \Gamma$ acts on T without inversion (i.e. Λ is torsion free) then the legal edge coloring $h : ET \to F$ factors through the covering morphism $T \to \Lambda \backslash T$.

Theorem 2.14. *(Lubotzky-Mozes-Zimmer [LMZ94]) An element $c \in \mathrm{Aut}(T)$ is in the commensurator C if and only if the function ϕ_c is periodic, i.e. iff there exists a uniform torsion free lattice $\Lambda < \Gamma$ such that the map $\phi_c : VT \to Sym(F)$ factors through the covering map $T \to \Lambda \backslash T$. This holds iff the differential $d\phi_c$ is also periodic.*

Proof. The only part not proven in ([LMZ94]) is the equivalence of the third assertion. If ϕ_c is periodic then $d\phi_c$ must also be periodic (with respect to the same lattice Λ) since the differential commutes with the projection map. Conversely if $d\phi_c$ factors through $T \to \Lambda \backslash T$ corollary (2.12) gives us that a finite index subgroup $\Lambda' < \Lambda$ such that $d\phi_c$ satisfies the Kirchoff loop law on $\Lambda' \backslash T$. Proposition (2.10) now assures us that $d\phi_c$ is the differential of some V-structure on $\Lambda' \backslash T$, the later must coincide with ϕ_c. □

Given an element $c \in C$ we choose a lattice Λ as in theorem (2.14) and we obtain a pointed finite graph $(Y, y_0) \stackrel{\text{def}}{=} (\Lambda \backslash T, \Lambda t_0)$. The edge coloring h is preserved by Λ so it induces a legal edge coloring of Y, also denoted by h. By our choice of Λ, ϕ_c induces a legal V-structure on Y, which will again be denoted by ϕ_c. Thus we have associated to every element $c \in C$ a finite graph with a legal V-structure (Y, h, ϕ_c). Note that the graph Y is not determined uniquely by c, as we can always take a subgroup of Λ thereby passing to a finite cover (X, x_0) of (Y, y_0). The V-structure and C-structure induced on X by c will be the same as the pull back of the corresponding structures on Y. So the two structures (X, h, ϕ_c) and (Y, h, ϕ_c) are equivalent in the sense of the following definition.

Definition 2.15. *Two pointed graphs (X, x_0) and (Y, y_0) with a given covering map $p : (X, x_0) \to (Y, y_0)$ and with two given legal edge colorings h_X, h_Y and V-structures ϕ_X, ϕ_Y are said to be equivalent if $p^*(h_Y) = h_X$ and $p^*(\phi_Y) = \phi_X$.*

If $p : (T, t_0) \to (Y, y_0)$ is the universal covering we can retrieve the local permutation map on T induced by c, by pulling back its projection to Y, thus reconstructing the original legal V-structure (T, h, ϕ_c) and (up to the ambiguity in choosing the image of the base point) the original automorphism c of the tree.

The above discussion can be summarized in the following theorem, asserting an equivalence of three sets of data.

Theorem 2.16. *The following three sets of data are equivalent.*

1. *An element c of the commensurator C.*
2. *A legal V-structure on a finite pointed graph, together with a choice of a base point t_0 and its image by c, ct_0, on the universal cover of the graph.*
3. *A legal C-structure on a finite pointed graph, together with a base point t_0, its image ct_0 on the universal cover of the graph, and a permutation $\phi_{c,t_0} \in Sym(F)$.*

Remark: It is easy to see that if for an element $a \in \mathrm{Aut}(T)$ we have $\phi_{a,VT} < G < Sym(F)$ then the image of the differential of the local permutation map is also contained in G. The converse is true if we assume that $\phi_{a,v} \in G$ for at least one $v \in VT$.

3 The group $U(A)$ of locally affine automorphisms

The main object of interest in this paper is the group U(A) defined below. In this chapter we introduce this group and some of its basic properties.

Definition 3.1. *(Burger-Mozes ([BM00b, BM00a])) If $G < Sym(F)$ is a permutation group on the set of colors then the* **universal group associated with** G *is defined by*

$$U(G) \stackrel{\text{def}}{=} \left\{a \in \mathrm{Aut}(T) \,\middle|\, Im\,(\phi_a^h) < G\right\} \tag{3.1}$$

We also define

$$C(G) \stackrel{\text{def}}{=} U(G) \cap C \tag{3.2}$$

Let $q = p^n$ be some fixed prime power, and $\mathbb{F} = \mathbb{F}_q$ be the corresponding finite field.

Definition 3.2. *The affine group of the field $A = \mathrm{Aff}(\mathbb{F})$ is the group of all permutations on the elements of $\mathbb{F}$ induced by affine transformations.*

$$A = \{x \to ax + b\,|a \in \mathbb{F}^*; b \in \mathbb{F}\} \tag{3.3}$$

We denote by P and M the subgroups of A isomorphic to the additive and multiplicative groups of $\mathbb{F}$ respectively. So $P = \langle P_y \in A\,|P_y(x) = x + y\ \rangle_{y\in\mathbb{F}}$ and $M = \langle M_y \in A\,|M_y(x) = xy\rangle_{y\in\mathbb{F}^*}$. Notice that A is isomorphic to the semi-direct product $M \ltimes P$, with the action $M_a \circ P_b \circ (M_a)^{-1} = P_{ab}$. We let $\chi : A \to M$ be the projection map. It is well known that the action of A on $\mathbb{F}$ is strictly two-transitive. The stabilizer of a point for the action of A is a conjugate of M and is therefore Abelian.

From now on we assume that $T = \mathcal{T}_q$ is the q-regular tree. We choose q colors corresponding to the elements of $\mathbb{F}$ and a legal coloring $h : ET \to \mathbb{F}$. As before we let $\Gamma < \mathrm{Aut}(T)$ be the lattice of color preserving automorphisms of T, and let $C = Comm_{\mathrm{Aut}(T)}(\Gamma) = Comm(T)$. $C(A) < C$ is the group of all commensurator elements that induce on every vertex link an affine permutation.

We choose an edge e_0 of the tree (say the edge in the star of t_0 which is colored 0) there is an involution $\sigma \in \Gamma$ which inverts this edge. We can now write

$$U(A) = \langle\sigma\rangle \ltimes U(A)^0 \tag{3.4}$$

where $U(A)^0$ is an index two subgroup that acts without inversion. The group $U(A)^0$ acts locally transitively and therefore decomposes as an amalgam

$$U(A)^0 = U(A)_{i(e_0)} *_{U(A)^0_{e_0}} U(A)_{t(e_0)} \tag{3.5}$$

$U(A)^0$ is a pro-solvable group.

By section (2) we can describe any element of $U(A)$ by its local permutation map or by the differential of its local permutation map together with some initial data. This gives rise to the following

Definition 3.3. *An* **affine structure** *on a q-regular graph Y is a legal C-structure (Y, h, ψ), with h taking values in the field $\mathbb{F}$ and ψ taking values in $A < \mathrm{Sym}(\mathbb{F})$.*

If (Y, h, ψ) is such an affine structure, we can associate a new C-structure to it taking values in the group M, namely $(Y, h, \chi \circ \psi)$. If $\alpha : M \to \mathbb{F}^*$ is the isomorphism of M with $\mathbb{F}^*$ then we denote $\overline{\psi} \stackrel{\mathrm{def}}{=} \alpha \circ \chi \circ \psi$ so that $(Y, h, \overline{\psi})$ is a C-structure of Y into the multiplicative group of $\mathbb{F}$. Note that there is no loss of information by passing to $\overline{\psi}$, in fact, using the assumption $\psi_e \in Stab_A(h(e))$, we can reconstruct ψ from $\overline{\psi}$. This is summarized in the following proposition.

Proposition 3.4. *Let $a \in A$ be a permutation such that $a(0) = h(e)$, then the following equation holds*

$$\psi_e = a \circ (\chi \circ \psi_e) \circ a^{-1} \tag{3.6}$$

Proof. First notice that the right hand side of equation (3.6) does not depend on the choice of a as different choices of a correspond to multiplying a on the right by elements of $A_0 = M$ and the group M is Abelian. Thus the proposition can be proved by substituting a concrete transformation for a. If we choose $a = P_{h(e)}$ and we let $a^{-1} \circ \psi_e \circ a = M_y$ we obtain

$$\begin{aligned} \psi_e &= P_{h(e)} \circ M_y \circ P^{-1}_{h(e)} \\ &= P_{h(e)} \circ \left(M_y \circ P^{-1}_{h(e)} \circ M_y^{-1} \right) \circ M_y \\ &= P_{h(e) - y * h(e)} \circ M_y \end{aligned} \tag{3.7}$$

Which shows that $\chi(\psi_e) = M_y$ and completes the proof. □

Note

- If we write $\psi_e(x) = c_1 x + c_0$ then $\overline{\psi_e} = c_1$.
- Proposition (3.4) and the remarks in the previous section imply that an element $a \in U(A)$ can be reconstructed from $\overline{d\phi_a}$ together with some initial data - the image of the star of a vertex.

Definition 3.5. *We say that an affine structure (Y, h, ψ) satisfies the* **Kirchoff junction law** *if for every vertex $v \in VY$ we have*

$$\prod_{e \in St(v)} \overline{\psi}_e = 1 \tag{3.8}$$

It is obvious that if an affine structure satisfies the junction law then so does the induced structure on every covering graph.

Definition 3.6. *We define*

$$J = \{c \in U(A) \,|\, \text{c satisfies the junction law}\} \tag{3.9}$$

and we call J the **junction group of** $U(A)$.

It is still left to show that this is indeed a group. We leave the proof of the following proposition to the reader.

Proposition 3.7. *The junction group defined above is a closed non discrete group with $\Gamma < J$. J acts vertex transitively on the tree and its local action is given by A.*

4 The orbit structure of elements in $U(A)$

In this section we give the main result of this paper. We use two properties special to the group A - It acts strictly two-transitively with an Abelian stabilizer. These enable us to give a nice description for the orbits of the action of cyclic subgroups on the edges of the tree - ET.

We fix an element $c \in U(A)_{t_0}$ and we wish to study the orbits for the action of $\langle c \rangle$ on the edges of the tree ET. We denote the spheres around t_0 by $S(n) \stackrel{\text{def}}{=} \{v \in VT \,|d(v,t_0) = n\}$ and the balls by $B(n) \stackrel{\text{def}}{=} \{x \in T \,|d(x,t_0) \leq n\}$. We also denote by $R(m,n) \stackrel{\text{def}}{=} B(n) \setminus \overset{o}{B}(m) = \{x \in T \,|m \leq d(x,t_0) \leq n\}$ for every $n > m \geq 0$. The orbits for the action of c on ET lie within the sets of the form $R(n,n+1)$.

Definition 4.1. *Let $H < Stab_{\mathrm{Aut}(T)}(t_0)$ be a group, $O = \{f_0, f_1, \ldots, f_{N-1}\} \subset ER(n-1,n)$ an orbit for the action of H on ET. Then we say that* **the orbit O splits into M pieces at the $n-$th sphere** *if there are exactly M orbits for the action of H on $ER(n,n+1)$ that intersect with O non trivially (on $S(n)$). We call these orbits the* **direct descendents of** O. *The* **descendents of** O *are defined as the orbits of H which are obtained from O by an iterated passage to direct descendents. The* **Shadow of O in** $R(m-1,m)$ *for some $m > n$ is the union of all orbits of H in $ER(m-1,m)$ which are descendents of O.*

Lemma 4.2. *Let H, O be as in definition (4.1). For every $i \in \mathbb{Z}/N\mathbb{Z}$, let $v_i = t(f_i) \in S(n)$, $W_i \stackrel{\text{def}}{=} St(v_i)$ and Orb_i be the set of orbits for the action of $H_{v_i} = Stab_H(v_i)$ on W_i. Let $\mathcal{O} \stackrel{\text{def}}{=} \{O = O_0, O_1, \ldots O_M\}$ be a set containing O and all of its direct descendents. Then for every $i \in \mathbb{Z}/N\mathbb{Z}$ there is a bijective correspondence:*

$$\begin{array}{ccc} \mathcal{O} & \rightarrow & Orb_i \\ O_j & \rightarrow & O_j \cap W_i \end{array} \tag{4.1}$$

preserving the ratio of the sizes of the orbits.

Proof. Since H acts transitively on O, the action of H_{v_i} on W_i and the action of H_{v_j} on W_j are conjugate. Thus we restrict ourselves to the study of the action of H_{v_0} on W_0. It is obvious that the orbits of H_{v_0} are subsets of the orbits of H. In the other direction assume that $d_1, d_2 \in W_0$ are two edges which are in the same H orbit, so that $d_2 = h(d_1)$. The fact that both d_i are in W_0 implies that $h \in H_{v_0}$. The remark from the beginning of the proof now shows that the size of $O_j \cap W_i$ does not depend on i which finishes the proof of the lemma. ∎

Figure 1: Definition of the mapping ζ_n

For every $n \geq 2$, we define a map $\zeta_n : ER(n-1,n) \to EB(n-1)$ to be the map associating to each edge f the unique edge $\zeta_n(f)$ such that:

- $h(f) = h(\zeta_n(f))$, i.e both edges are of the same color.
- The distance $d(f, \zeta_n(f))$ is minimal with respect to the previous property.

This is illustrated in figure (1) . We consider the following subgroup of $\mathrm{Aut}(T)$:

$$\begin{aligned} G^{(n)} &\stackrel{\text{def}}{=} \left\{ g \in \mathrm{Stab}_{Aut(T)}(t_0) \,|h \circ g(f) = h \circ g \circ \zeta_n(f) \qquad \forall f \in R(n-1,n) \right\} \\ &= \left\{ g \in \mathrm{Stab}_{Aut(T)}(t_0) \,|\zeta_n \circ g(f) = g \circ \zeta_n(f) \qquad \forall f \in R(n-1,n) \right\} \end{aligned} \tag{4.2}$$

The question whether an element $g \in \mathrm{Stab}_{Aut(T)}(t_0)$ is in $G^{(n)}$ depends only on its restriction to B_n. We will therefore, by abuse of notation, refer to $G^{(n)}$ also as a subgroup of $\mathrm{Aut}(B_n)$.

We notice that if $g \in Aut(B_n)$, then g admits a unique extension to $G^{(n+1)}$ denoted $\tilde{g}$. More precisely we have:

$$\tilde{g}(f) = \begin{cases} g(f) & \text{if} f \in B_n \\ (\zeta_{n+1})^{-1} \circ g \circ \zeta_{n+1}(f) & \text{if} f \in R(n,n+1) \end{cases} \tag{4.3}$$

Note that if g acts on the tree in a local affine way then so does its extension $\tilde{g}$.

Lemma 4.3. *The group $G^{(n+1)}(A) \stackrel{\text{def}}{=} \{g \in G^{(n+1)} | \phi_{g,VB(n)} < A\}$ acts transitively on $S(n)$ but its action splits into $q-1$ orbits at $S(n)$.*

Proof. As A is a two-transitive permutation group, the group $U(A)$ acts transitively on $S(n)$ for every n (see [BM00a]). Now since each element of $U(A)|_{Aut(B(n))}$ can be extended to an element of $G^{(n+1)}(A)$, it follows that the later group still acts transitively on $S(n)$. By lemma (4.2) it will be enough to show that if an element $g \in G^{(n+1)}(A)$ fixes a vertex $v \in S(n)$ then it fixes $St(v)$ pointwise. Since the action of A on $\mathbb{F}$ is strictly two-transitive, it is enough to show that g fixes two edges of $St(v)$. Let w be the vertex on $S(n-1)$ adjacent to v. It is clear that g fixes two edges in $\mathrm{St}(w)$, namely the two edges on the geodesic from t_0 to v. By definition of $G^{(n+1)}$ we have $\phi_{g,v} = \phi_{g,w}$ so that g must also fix two edges in $\mathrm{St}(v)$. ☐

Theorem 4.4. *Let $c \in U(A)_{t_0}$, and $n \geq 3$. Let $O = \{f_0, f_1, \ldots, f_{N-1}\} \in ET$ be an orbit for the action of c on $ER(n-1,n)$ (assume that all edges are directed from $S(n-1)$ to $S(n)$). We define an element $\xi = \xi(O) \in \mathbb{F}^*$*

$$\xi = \xi(O) \stackrel{\text{def}}{=} \prod_{j=0}^{N-1} \overline{d\phi_{c,f_j}} \tag{4.4}$$

Then O splits into $(q-1)/ord(\xi)$ orbits of equal sizes at the $n-th$ sphere.

Proof. Let $v_j = t(f_j) \in S(n)$, and $W_j = St(v_j)$. Without loss of generality we may assume that $c(f_j) = f_{j+1} \quad \forall j \in \mathbb{Z}/N\mathbb{Z}$.

Let us associate with each one of the q-orbits for the action of $G^{(n+1)}$ on $ER(n-1, n+1)$ (c.f. lemma (4.3)) an element of the field $\mathbb{F}$ by taking $O \to h(O \cap W_0)$. We now define a new edge coloring taking each edge in $R(n-1, n+1)$ to the color associated with its orbit:

$$\begin{aligned} \overline{h} : ER(n-1, n+1) &\to \mathbb{F} \quad (4.5) \\ e &\to h\left(G^{(n+1)}(A)(e) \cap W_0\right) \quad (4.6) \end{aligned}$$

We have $h|_{W_0} = \overline{h}|_{W_0}$. We may assume without loss of generality that the notation is such that $\overline{h}|_{R(n-1,n)} = 0$. Now we have two colorings of $R(n-1, n+1)$ and c induces two corresponding local permutation maps ϕ_c^h and $\phi_c^{\overline{h}}$ on the vertices of $S(n)$. By definition if $d \in G^{(n+1)}(A)$ and if $d(v_0) = v_j$ for some $j \in \mathbb{Z}/N\mathbb{Z}$ then:

$$\begin{aligned} \overline{h}|_{W_j} &= \overline{h}|_{W_0} \circ d^{-1}|_{W_j} \\ &= h|_{W_0} \circ d^{-1}|_{W_j} \\ &= \phi^h_{d^{-1}, t(f_j)} \circ h \quad (4.7) \end{aligned}$$

By lemma (4.2) applied to $H = \langle c \rangle$, it will be enough to show that c^N acts on W_0 with $(q-1)/ord(\xi)+1$ orbits: one singleton and the rest of equal sizes. The action of $\phi^{\overline{h}}_{c^N, v_0}$ on $\mathbb{F}$ is isomorphic to the action of c^N on W_0. We decompose the former permutation as a product:

$$\phi^{\overline{h}}_{c^N, v_0} = \phi^{\overline{h}}_{c, v_{N-1}} \circ \phi^{\overline{h}}_{c, v_{N-2}} \circ \ldots \circ \phi^{\overline{h}}_{c, v_0} \quad (4.8)$$

We define the element $d = \widetilde{c|_{B_m}}$ to be the unique extension of $c|_{B_m}$ to $G^{(n+1)}$. Using equation (4.7) we obtain the following formula for each one of the factors in equation (4.8)

$$\begin{aligned} \phi^{\overline{h}}_{c, v_j} &= \overline{h} \circ c \circ (\overline{h}|_{W_j})^{-1} \\ &= (\phi^h_{d^{j+1}, v_0})^{-1} \circ h \circ c \circ (h|_{W_j})^{-1} \circ \phi^h_{d^j, v_0} \\ &= (\phi^h_{d^{j+1}, v_0})^{-1} \circ \phi^h_{c, v_j} \circ \phi^h_{d^j, v_0} \\ &= \left(\phi^h_{d^j, v_0}\right)^{-1} \left[(\phi^h_{d, v_j})^{-1} \circ \phi^h_{c, v_j}\right] \circ \phi^h_{d^j, v_0} \\ &= \left(\phi^h_{d^j, v_0}\right)^{-1} \left[d\phi_{f_j}\right] \circ \phi^h_{d^j, v_0} \\ &= M_{\xi_{f_j}} \quad (4.9) \end{aligned}$$

By definition the local permutation that d induces on $St(v_j) = St(t(f_j))$ is identical to the permutation that c induces on $St(i(f_j))$. This explains the equality before the last one in equation (4.9). The last equality follows from proposition (3.4).

Substituting equation (4.9) into equation (4.8) we get a formula for the action of c^N on W_0

$$\begin{aligned} \phi^{\overline{h}}_{c^N, v_0}(x) &= \phi^{\overline{h}}_{c, v_{N-1}} \circ \phi^{\overline{h}}_{c, v_{N-2}} \circ \ldots \circ \phi^{\overline{h}}_{c, v_0}(x) \\ &= M_{\xi_{f_{N-1}}} \circ M_{\xi_{f_{N-2}}} \circ \ldots M_{\xi_{f_0}} \\ &= M_{\xi} \quad (4.10) \end{aligned}$$

So the action is isomorphic to multiplication by ξ on $\mathbb{F}$. Multiplication by ξ acts on $\mathbb{F}$ with one orbit of size one and $(q-1)/ord(\xi)$ orbits of size $ord(\xi)$, which completes the proof. □

5 Applications of the orbit structure theorem

In this section we give some corollaries of the orbit structure theorem (4.4): We calculate the pro-Sylow groups of $U(A)_{t_0}$, we discuss the structure of the junction group and we construct some examples of commensurator elements with large orbits. These allow us later to construct explicit examples of irreducible lattices in products of two trees.

5.1 The Pro-Sylow groups of $U(A)_{t_0}$

The group $U(A)_{t_0}$ is pro-solvable and thus decomposes as a product of its pro-Sylow subgroups. Furthermore it is easy to determine exactly which primes occur by writing $U(A)_{t_0}$ as an inverse limit. Here we wish to give an explicit description of the pro-Sylow subgroups of $U(A)_{t_0}$ in terms of local permutation maps. We first deal with the normal subgroup $K = Stab_{U(A)}(St(t_0)) \triangleleft U(A)_{t_0}$, the point stabilizer of $St(t_0)$.

Theorem 5.1. *Let $L < \mathbb{F}^*$ be a subgroup and define:*

$$\hat{L} = \{a \in K | \overline{d\phi_{a,e}} \in L \qquad \forall e \in ET\} \tag{5.1}$$

1. *$\hat{L}$ is a closed subgroup of $U(A)_{t_0}$.*
2. *If l is a prime and L is an $l-$subgroup of $\mathbb{F}^*$ then $\hat{L}$ is a pro-l subgroup of K.*
3. *If L is an $l-Sylow$ subgroup of $\mathbb{F}^*$ then $\hat{L}$ is a pro-l-Sylow subgroup of K.*

Proof. It is easy to verify that the following cocycle equation holds for every edge $e \in ET$:

$$\overline{d\phi_{ab,e}} = \overline{d\phi_{a,be}}\,\overline{d\phi_{b,e}} \tag{5.2}$$

which implies that $\hat{L}$ is a group whenever L is. If $a \notin \hat{L}$ then there exists a specific edge $e \in ET$ such that $\overline{d\phi_{a,e}} \notin L$. This later condition is open, so $\hat{L}$ must be closed, proving (1). Assume now that L is an $l-$group. Since $\hat{L}$ is closed it is a pro-finite group, it follows that:

$$\hat{L} = \underset{\leftarrow}{Lim}\, \hat{L}|_{B(i)} \tag{5.3}$$

So we only have to check that $\hat{L}|_{B(i)}$ is an $l-$group for every $i \in \mathbb{N}$. To prove this it is enough to show that any element has an order which is a power of l. We pick an element $a \in \hat{L}$ and show that when it acts on the sphere $S(i)$ it acts with orbits all of whose cardinalities are powers of l. This is true for $i = 1$ because by assumption all orbits there are singletons. We proceed by induction on i. We have $\overline{d\phi_{a,e}} \in L \quad \forall e \in ET$. The induction hypothesis is that any a orbit, O on $ER(i-1,i)$ consists of l^r points for some r. By theorem (4.4) any such orbit splits into orbits of size $l^r ord(\xi)$ on the $i-$th sphere, where $\xi = \Pi_{f \in O} \overline{d\phi_{a,f}}$. By our assumptions $d\phi_{a,f} \in L \quad \forall f \in O$ which implies $\xi \in L$, thus ξ must have an order which is a power of l. This completes the proof of (2). To prove the last assertion we express $\mathbb{F}^*$ as a (direct) product of Sylow subgroups $\mathbb{F}^* = L_1 \times L_2 \times \ldots \times L_m$

and show that K is the product of the corresponding $pro-l_i-Sylow$ subgroups: $K = \hat{L_1}\hat{L_2}\ldots\hat{L_m}$. Let $k \in K$ be given. We have to find $k_i \in \hat{L_i}$ such that $k = k_m k_{m-1} \ldots k_1$. Recall that by section (2) and by proposition (3.4) it is enough to specify the differential of the local permutation map of each one of these k_i's in order to define them. We write $\overline{d\phi_{k,e}} = l_{k,e,m} l_{k,e,m-1} \ldots l_{k,e,1}$, where $l_{k,e,i} \in L_i$. Now define k_1 to be the element satisfying $\overline{d\phi_{k_1,e}} = l_{k,e,1}$ then define recursively k_i by the equation $\overline{d\phi_{k_i,k_{i-1}\circ k_{i-2}\circ\ldots\circ k_1(e)}} = l_{k,e,i}$ The cocycle equation (5.2) proves that $k = k_m k_{m-1} \ldots k_1$. This completes the proof. □

Let us recall that $U(A)_{t_0} = K \rtimes A$. For any subgroup $B < A$ and any prime l we can find a pro-l-Sylow subgroup of $K \rtimes B$ which factors as a product of a pro-l-Sylow subgroup of K and an l-Sylow subgroup of B this can be applied to the group $U(A)_{t_0}$ itself and also to the stabilizer of an edge $U(A)_{e_0}$ to give the following corollaries:

Corollary 5.2.

- *The group $U(A)_{e_0}$ admits pro-l-Sylow subgroups iff l divides $|\mathbb{F}^*| = (q-1)$.*
- *The group $U(A)_{v_0}$ admits pro-l-Sylow subgroups iff l divides $|A| = q(q-1)$. Furthermore it admits an infinite pro-l-Sylow subgroups iff l divides $|\mathbb{F}^*| = (q-1)$.*

5.2 An application to $PGL_2(L)$

There is a strong connection between the finite groups $A = \mathrm{Aff}(\mathbb{F})$ and $PGL_2(\mathbb{F})$. The former being the stabilizer of a point for the action of the later on the projective line over $\mathbb{F}$. The group $U(PGL_2(\mathbb{F}))$ contains many closed, vertex transitive subgroups whose local action is given by $PGL_2(\mathbb{F})$. An example is $G = PGL_2(L)$ (where L is a non-archimedian local field with residue field $\mathbb{F}$), acting on its Bruhat-Tits tree Y. In this section we make use of the connection between A and $PGL_2(\mathbb{F})$ in order to calculate the commensurator in G of a discrete subgroup of $PGL_2(L)$ which is not a lattice.

Let $y_0 \in VY$ be a base vertex. We can "color" in a natural way the star of y_0 with a set of colors corresponding to the projective line $\mathbb{P}^1\mathbb{F} = \mathbb{F} \cup \{\infty\}$ in such a way that the action of G_{y_0} on this star factors through the action of $PGL_2(\mathbb{F})$ on $\mathbb{P}^1\mathbb{F}$. Denote this coloring by $St(y_0) = \{e_\eta\}_{\eta\in\mathbb{P}^1\mathbb{F}}$. We now wish to extend this coloring to a legal coloring of EY in such a way that $G < U(PGL_2(\mathbb{F}))$. Choose a set of involutions $g_\eta \in G$ inverting the edges e_η. The group $\Lambda = \langle g_\eta \rangle_{\eta\in\mathbb{P}^1\mathbb{F}}$ will be a lattice acting simply transitively on the oriented edges of Y. Thus Λ defines an edge coloring of Y - coloring an edge e with the same color as the edge $\Lambda e \cap St(y_0)$.

We might hope to obtain an interesting subgroup of $U(A)$ in the following way. Let $T \subset Y$ be the connected component of y_0 in the forest which is obtained by deleting all the edges in Y colored by ∞. T is a $|\mathbb{F}|-$regular tree, it inherits from Y a legal coloring by a set of colors corresponding to the elements of $\mathbb{F}$. Let $H < G$ be the subgroup of G which fixes T as a set. If $|\mathbb{F}| > 2$ the map $H \to Aut(T)$ is injective since an element of G is determined by its action on any three points of ∂Y. For every vertex $v \in VT$ and any element of H the local permutation map $\phi_{h,v}$ fixes ∞, we can therefore view H as a subgroup of $U(A)$.

The group H contains the group $\Gamma = \langle g_\eta \rangle_{\eta\in\mathbb{F}}$ which is a lattice in $\mathrm{Aut}(T)$. In fact, H even contains the commensurator of Γ in G:

Lemma 5.3. *$Comm_G(\Gamma) < H$.*

Proof. $Comm_G(\Gamma)$ is generated as a group by Γ and a vertex stabilizer $Comm_G(\Gamma)_{y_0}$. Since $\Gamma < H$, it is enough to prove that $Comm_G(\Gamma)_{y_0} < H$. Let $c \in Comm_G(\Gamma)_{y_0}$, pick $\Sigma < \Gamma$ of finite index such that $c\Sigma c^{-1} < \Gamma$. We observe that:

$$c\Sigma y_0 = c\Sigma c^{-1} c y_0 \subset \Gamma c y_0 = \Gamma y_0 = T \tag{5.4}$$

from this and from the fact that the convex hull of Σy_0 is T it follows that $cT \subset T$. □

We now calculate H explicitly thus showing that the commensurator is small:

Proposition 5.4. *$[H : \Gamma] < \infty$.*

Proof. H contains Λ so it must act transitively on the vertices of the tree T so $[H : \Gamma] = [H_{y_0} : \Gamma_{y_0}] = |H_{y_0}|$. The group G_{y_0} is a virtually pro-p group (where $p = char(\mathbb{F})$) and so must be its closed subgroup H_{y_0}, but as we have just seen in corollary (5.2)the group $U(A)$ does not contain infinite pro-p groups. This implies that H_{y_0} is finite and H is discrete. □

The following follows immediately:

Corollary 5.5. *$[Comm_G(\Gamma) : \Gamma] < \infty$.*

5.3 The Junction group

The junction law and the junction group are introduced in definitions (3.5,3.6). We wish to discuss the meaning of the junction law (3.5) in the context of theorem (4.4). An immediate consequence of the junction law is that if an element $c \in J_{t_0}$ admits an orbit $O \subset R(n-1, n)$ as in theorem (4.4) and if this orbit has M descendents $O(0), O(1), \ldots O(M-1)$ in $R(m-1, m)$ for some $m > n$ then (using the notation of theorem(4.4)).

$$\xi(O) = \prod_{k=0}^{M-1} \xi(O(k)) \tag{5.5}$$

This enables us to conclude a few facts about the junction group.

Corollary 5.6. *Assume that $c \in C(A)_{t_0}$ satisfies the junction law. Suppose that c admits an orbit $O = \{f_0, f_1, \ldots, f_{N-1}\} \in R(n, n+1)$ for some $n \geq 2$ with $\xi(O)$ a generator of the cyclic group $\mathbb{F}^*$. Then the orbit O never splits, i.e. c acts transitively on the shadow of O on the sphere $S(m)$ for every $m \geq n$.*

Proof. We prove this by induction using theorem (4.4) in every step. Theorem (4.4) implies that $O = O(n)$ does not split in the n'th sphere so it has only one descendent orbit $O(n+1) \subset R(n+1, n+2)$. The junction law now implies that for this orbit again $\xi(O(n+1)) = \xi$. □

Corollary 5.7. *The junction group J has virtually torsion free vertex stabilizers. More specifically the subgroup of J which fixes pointwise the set* $\mathrm{St}(t_0)$ *is torsion free.*

Proof. We pick an element $c \in J_{t_0}$ which fixes pointwise $St(t_0)$ (Note that for such an element the proof of theorem (4.4) works also in the case $n = 2$). It is enough to prove that either $c = id$ or c admits orbits of arbitrarily large cardinality. If $c \neq id$ then there exists at least one edge $e \in ET$ with $\xi(e) \neq 1$, such an edge of minimal distance from t_0 will also be fixed under the action of c. This edge now splits into orbits of size larger then one. The junction law implies that one of these new orbits again carries a $\xi \neq 1$ which means that it will again split into larger orbits and we continue by induction. □

5.4 Calculating the orbits for a specific commensurator element.

In this section we use theorem (4.4) to give explicit examples for commensurator elements which act with few orbits on the boundary of the tree. This enables us in the next section to give explicit constructions for irreducible lattices acting on the product of two trees.

In a typical example we describe an element $c \in C(A)$ by giving its local permutation map ϕ_c, the differential of the local permutation map $d\phi_c$ or the projection of the later on the multiplicative group of the field $\overline{d\phi_c}$, each with the necessary initial data. Since by assumption c is an element of the commensurator all this information can be given by $V-structures$ or $C-structures$ on finite graphs.

We fix once and for all a $q-$regular tree with a base point (T, t_0) and a legal coloring $h : ET \to \mathbb{F}$ (where $\mathbb{F}$ is the finite field of q elements). We identify T with the universal cover of all our pointed finite graphs. For convenience we will work only with elements $c \in C(A)_{t_0}$ so that we always have $ct_0 = t_0$, such an element is described by one of the following (graphic) notations (these notations are illustrated for a specific commensurator element in figure (2)).

- A pointed graph (X, x_0) with the legal coloring $h : EX \to \mathbb{F}$ drawn on the edges, and the local permutation map $\phi_c : VX \to A$ drawn on the vertices.
- A pointed graph (X, x_0), with the legal edge coloring $h : EX \to \mathbb{F}$ and the differential of the local permutation map, $d\phi_c : EX \to A$ drawn on every edge as a pair $(h, d\phi_c)$. Together with ϕ_{c,t_0}.
- A pointed graph (X, x_0), with the legal edge coloring $h : EX \to \mathbb{F}$ and the multiplicative part of the differential of the local permutation map, $\overline{d\phi_c} = \alpha \circ \chi \circ d\phi_c : EX \to A$, drawn on every edge as a pair $(h, \overline{d\phi_c})$. Together with ϕ_{c,t_0}.

Proposition 5.8. *Let c be the element of $C(A)_{t_0}$ defined by any of the three equivalent descriptions in figure (2). If the following conditions hold* [2]

1. *σ fixes $1 \in \mathbb{F}$ and acts as a $(q-1)$-cycle on the rest of the elements of the field.*
2. *The element $y \overset{\text{def}}{=} \alpha \circ \chi(\sigma^{-1} \circ \tau)$ is a generator of the cyclic group $\mathbb{F}^*$.*

Then $\overline{\langle c \rangle}$, the closure of the cyclic group generated by c, fixes the unique edge $\tilde{e}$ in $\mathrm{St}(t_0)$ with $h(\tilde{e}) = 1$, and acts transitively on the boundary of each one of the two connected components of $T \backslash \tilde{e}$.

Proof. The proof uses induction. As the two halves of the tree around $\tilde{e}$ are symmetric, we will prove the theorem only for the half containing t_0. When using the notations $S(m), B(m), R(m, n)$ we will understand (for the purpose of this proof only) the intersection of the respective sets with the relevant half tree.

It is obvious that the assumption of the theorem about σ implies that c fixes $\tilde{e}$ and acts transitively on $\mathrm{St}(t_0)\backslash\tilde{e}$, i.e. c acts transitively on $S(1)$, thus all the edges in $R(0, 1)$ form one c-orbit denoted O_1. We can easily verify that $\xi(O_1) = y \in \mathbb{F}^*$ is a generator of the multiplicative group, where $\xi(O_1)$ is defined as in theorem (4.4). Theorem (4.4) (which holds in this situation also for $n = 2$) now implies that c acts transitively on the edges of $R(1, 2)$. We denote this orbit by O_2 and continue by induction. The only thing we need for the induction step is the following combinatorial lemma.

[2]The values specified in figure (2) are examples for a choice of elements which satisfy the conditions of the theorem

Figure 2: Three equivalent definitions for the element c

Figure 3: Labeled oriented graph

Lemma 5.9. *Let* $\xi_n \stackrel{\text{def}}{=} \prod_{f \in R(n-1,n)} \overline{d\phi_{c,f}} \in \mathbb{F}^*$*, then* ξ_n *is a generator of the cyclic group* $\mathbb{F}^*$*.*

proof. The tree T is the universal covering of the graph in figure (3). We give a label and a sign to each directed edge of the graph in the following way. Every edge on the left (resp. right) side of the graph is getting the sign $-$ (resp. $+$). Each edge labeled 0 and directed from left to right (resp. right to left) gets the sign $+$ (resp. $-$). The directed edges of T acquire labels and signs via the covering morphism. The edge $\tilde{e}$ covers the edge labeled 1 on the left and is therefore labeled $(1, -)$. We define $J(n, j) \stackrel{\text{def}}{=} \{$ number of edges labeled $''j''$, directed from $S(n-1)$ to $S(n)$ and counted with their signs }. We obtain the following:

$$\begin{array}{rcll} J(1,0) & = & 1 & \\ J(1,1) & = & 0 & \\ J(1,j) & = & -1 & \forall j \in \mathbb{F}\backslash\{0,1\} \\ J(n+1,0) & = & -J(n,1)-(q-2)J(n,z) & \\ J(n+1,1) & = & J(n,0)+(q-2)J(n,z) & \\ J(n+1,j) & = & J(n,0)+J(n,1)+(q-3)J(n,z) & \forall j \in \mathbb{F}\backslash\{0,1\} \end{array} \tag{5.6}$$

Here we used twice the fact that there is a symmetry between all labels different from 0 and 1. We use z in the above equations to denote any element of the field which is neither 0 nor 1, we could have replaced this by any other element. Using equations (5.6) it is easy to prove by induction the following formula

$$\begin{array}{rcl} J(n+1,0) & = & J(n+1,1)-J(n+1,z) \\ & = & J(n,z)-J(n,1) \\ & = & -J(n,0)=\ldots=(-1)^n \pmod{q-1} \end{array} \tag{5.7}$$

Thus $\xi_n = y^{J(n,0)}$ is always congruent to y or to y^{-1} which proves the lemma and the proposition. □

6 Examples of lattices in products of two trees

Recently much work was done on uniform lattices in the automorphism group of the product of two trees. In a series of papers, Burger, Mozes and Zimmer show that these lattices have many properties reminiscent of lattices in higher rank Lie groups ([BM00b, BM97, BMZ]). As in the case of Lie groups a notion of an irreducible lattice is necessary.

In the setting of semisimple Lie groups there is a dichotomy: A lattice in the product of two simple Lie groups $\Lambda < G_2 \times G_2$ has projections which are either discrete (a reducible lattice) or dense (an irreducible lattice). In products of two trees this dichotomy no longer holds, in fact it turns out that a uniform lattice $\Gamma < \mathrm{Aut}(\Delta)$ can never have dense projections (see [BM00b]). Still we would like to think of an "irreducible" lattice as such a lattice which has "large" projections. A family of definitions was given by Burger, Mozes and Zimmer all of which require, in a stronger or weaker sense, that the projections of the lattice be big.

Definition 6.1. *Let T_1, T_2 be two regular trees, $\Delta \stackrel{\text{def}}{=} T_1 \times T_2$ their product (a two dimensional square complex), $pr_i : \mathrm{Aut}(\Delta) \to \mathrm{Aut}(T_i)\ _{i\in\{1,2\}}$ the projections, $\Gamma < \mathrm{Aut}(\Delta)$ a uniform lattice and $H_i \stackrel{\text{def}}{=} \overline{pr_i(\Gamma)}$ the closures of the projections of the lattice. We say that Γ* **is an irreducible lattice** *if both H_i are non discrete.*

Remark 6.2. *By a lemma of Burger, Mozes and Zimmer, Γ is irreducible iff any one of the H_i's is non discrete (see [BM00b]).*

Definition 6.3. *A closed subgroup* $H < Aut(T)$ *of the automorphism group of a tree is called* **locally primitive**, *if for every vertex* x *of* T, $Stab_H(x)$ *acts as a primitive permutation group on the link* $Lk(x)$.

Definition 6.4. *A closed group* $H < Aut(T)$ *in the automorphism group of a tree is called* **locally infinitely transitive**, *if for every vertex* x *of* T, $Stab_H(x)$ *acts as a transitive permutation group on the sphere* $S(x, m)$ *for every* $m \in \mathbb{N}$.

Stronger notions of "irreducibility" for Γ are obtained by requiring the projection closures H_i to be non discrete and locally primitive, or locally infinitely transitive. Many theorems have been proved ([BM00b]) in this setting showing many similarities, but also many differences, between the theory of "irreducible" lattices in the product of two regular trees (in each of the above settings) and the theory of irreducible lattices in semisimple Lie groups.

In this paper we will give a method for construction of such lattices using the techniques discussed in the earlier sections.

It turns out that there is a strong connection between the commensurator of a lattice $\Gamma < Aut(T_1)$ and embedding of Γ into lattices in products of trees. As an indication we bring the following facts.

Proposition 6.5. *Let* $\Gamma < \mathrm{Aut}(\Delta)$ *be a lattice in the product of two trees (using the notations of definition (6.1)). Considering only the action of* Γ *on* T_2 *we obtain - as a quotient - a finite graph of groups* $\Gamma \backslash\backslash T_2$ *where all the vertex and edge groups are commensurable, uniform lattices in* $\mathrm{Aut}(T_1)$.

Theorem 6.6. *Let* $(G(X), x_0)$ *be a finite graph of groups with all the groups* $\{G_\sigma\}_{\sigma \in VX}$ *being commensurable uniform lattices in* $\mathrm{Aut}(T_1)$ *where* T_1 *is some uniform tree. Let* $L \stackrel{\text{def}}{=} \pi_1(G(X), x_0)$ *and let* $T_2 \stackrel{\text{def}}{=} (\widetilde{G(X), x_0})$, *the Bass-Serre tree corresponding to this graph of groups. Let* $\Delta \stackrel{\text{def}}{=} T_1 \times T_2$. *Then*

1. *There is a natural action of* L *on* Δ *making* L *into a uniform lattice in* $\mathrm{Aut}(\Delta)$.
2. $pr_1(L) = \langle G_\sigma | \sigma \in VX \rangle < \mathrm{Aut}(T_1)$ *and thus* L *is irreducible iff* $\langle G_\sigma | \sigma \in VX \rangle < \mathrm{Aut}(T_1)$ *is non discrete.*
3. $pr_2(L) < \mathrm{Aut}(T_2)$ *is given by the fundamental group of the effective quotient the graph of groups* $(G(X), x_0)$. *Thus* L *is reducible iff the effective quotient of* $G(X)$ *is a graph of finite groups. The lattice* L *acts locally transitively on* T_2 *iff* X *is a single edge (i.e.* L *is an amalgamated product* $\Sigma *_{\Sigma \cap \Sigma} \Sigma'$*). In such a case* L *acts locally primitively on* T_2 *iff* $\Sigma \cap \Sigma'$ *is a maximal subgroup in both* Σ *and* Σ'.

Proof. By the universal property of the fundamental group of a graph of groups there is a well defined map $L \to \mathrm{Aut}(T_1)$ by which L acts on T_1. The action of L on T_2 is the regular action of the fundamental group on the Bass-Serre tree. We let the action $L \rightsquigarrow \Delta$ be the diagonal one $l(x_1, x_2) = (l(x_1), l(x_2))$. In order to prove (1) we must show that L acts on Δ with discrete stabilizers and with a compact fundamental domain. If $x = (x_1, x_2)$ is a vertex of Δ then $Stab_L(x) = Stab_{Stab_L(x_2)}(x_1)$. This later group is finite because $Stab_L(x_2)$ acts on T_1 as a discrete group (a uniform lattice which is conjugate to one of the $\{G_\sigma\}_{\sigma \in VX}$). To show that L is cocompact let $Y \stackrel{\text{def}}{=} \{y_\sigma\}_{\sigma \in VX} \subset T_2$ be a section $VX \to T_2$ with $L_{y_\sigma} = G_\sigma$. Applying some element of L to $x = (x_1, x_2)$ we may assume that $x_2 = y_\tau \in Y$. Now applying an element of G_τ we can move x_1 into some compact fundamental domain for the action of G_τ on T_1, without affecting the second coordinate. Thus we have proved (1).

Assertion (2) follows from remark (6.2).

As for (3), the equivalence of L being reducible with $G(X)$ having an effective quotient which is a graph of finite groups again follows from remark (6.2). The assertion about local transitivity is obvious. As for the local primitivity we wish to check the action of $Stab_L(x_2)$ on $Lk(x_2)$ for an arbitrary vertex $x_2 \in VT_2$. By conjugation though we may assume that x_2 is the vertex stabilized by Σ (or Σ') and that the stabilizer of one of the edges in the link is $\Sigma \cap \Sigma'$. The elements acting trivially on the link are exactly those in the (normal) subgroup stabilizing the one-sphere around x_2. Dividing by this group we are reduced to the well known fact that a group action is primitive iff it is transitive and the point stabilizers are maximal subgroups. □

It follows from theorem (6.6) that we can generate irreducible lattices by finding two commensurable uniform lattices in $\Sigma, \Sigma' < \mathrm{Aut}(T_1)$ which generate a large group of $\mathrm{Aut}(T_1)$. In this section we give an example of how this can be achieved, using the methods developed in the previous sections. Namely, we construct a family of irreducible lattices in $\mathrm{Aut}(T_1 \times T_2)$ where T_1 and T_2 are two regular trees. These lattices will act locally primitively on one of the trees and locally infinitely transitively on the other.

The strategy is as follows, using theorem (4.4) we can construct an element $c \in C$ of the commensurator such that $\overline{\langle c \rangle}$ acts on the boundary of T_1 with a finite number of orbits. We decompose c as a product of two commensurator elements of finite order $c = ba$. A periodic element of the commensurator is also contained in a lattice (if it commensurates Γ it also normalizes a subgroup of finite index) thus we construct two commensurable uniform lattices Γ_a, Γ_b containing a, b respectively. By theorem (6.6) we have an action of $L \stackrel{\text{def}}{=} \Gamma_a *_{\Gamma_a \cap \Gamma_b} \Gamma_b$ on a product of two trees. The projection $pr_1(L) = \langle \Gamma_a, \Gamma_b \rangle$ contains both c and a uniform lattice and it is easy to verify that it must act locally infinitely transitively on T_2.

In ([LMZ94]) the following criterion is proven for an element of the commensurator to be periodic.

Theorem 6.7. *(Lubotzky-Mozes-Zimmer [LMZ94]) Let $c \in C$ be a commensurator element on the regular tree T. Then c is of finite order iff its local permutation map factors through a covering of some finite graph $T \to X$ in such that the two colored graphs: $(X, f \to h(f))$ and $(X, f \to \phi_{a,t(f)}(h(f)) = (X, f \to \phi_{a,i(f)}(h(f)))$ are isomorphic.*

As a specific example we start with the commensurator element constructed in section (5.4). We take $l = q - 1 = 2^r - 1$ to be a Mersenne prime, $\mathbb{F} = \mathbb{F}_q = \langle 0, 1, z, z^2, \ldots, z^{l-1} \rangle$ the corresponding finite field, T_1 the $q-$regular tree, $h : T_1 \to \mathbb{F}_q$ a legal coloring, $\Gamma < \mathrm{Aut}(T_1)$ the lattice of color preserving automorphisms, $\Gamma_0 < \Gamma$ the subgroup (of index 2) acting without inversions. We can think of Γ_0 as the fundamental group $\pi_1(X, x_0)$ of the graph X in figure (4) and we take $\Lambda < \Gamma_0$ to be the index two subgroup $\Lambda = \pi_1(R, r_0)$. As the cyclic group of the field is of prime order (an l-group) the generator z of the cyclic group has a square root-$\sqrt{z}$. We consider the element $c \in C$ constructed in section (5.4) which is defined by figure (2). Figure (5) gives a decomposition of c as a product of two periodic elements $c = b \circ a$. The element that b describes an automorphism of the graph R of order l. As follows from [LMZ94] this implies that b is periodic of order l. As for the element a, it does not describe an automorphism of the graph R but a can be described as an automorphism of the graph X as illustrated in figure (6). Here the action of a on the edges of X is just like the action of the permutation $\sigma \circ \delta$. A short calculation (using the fact that every element

Figure 4: The labeling of T

Figure 5: Definition of some commensurator elements

Figure 6: Description of a as a re-coloring of X

of A is of order 2 or l) shows that a is an l-cycle for every $l > 3$ and that in the case $l = 3$ it is an element of order 2.

Let us notice that $\langle \Gamma, a \rangle < N_{\mathrm{Aut}(T_1)}(\Gamma)$ and that $\langle \Gamma, b \rangle < N_{\mathrm{Aut}(T_1)}(\Lambda)$. Therefore both these groups are lattices in $\mathrm{Aut}(T_1)$. Let $L \stackrel{\text{def}}{=} \langle \Gamma, a \rangle *_{\Gamma} \langle \Gamma, b \rangle$, and let T_2 be the corresponding Bass-Serre tree. Let $\Delta \stackrel{\text{def}}{=} T_1 \times T_2$.

Proposition 6.8. *With the above notations, the following hold:*

1. *T_2 is an l-regular tree for $l > 3$, and it is the barycentric subdivision of the 3-regular tree if $l = 3$.*

2. *L is an irreducible lattice in* $\mathrm{Aut}(\Delta)$.

3. *L acts locally infinitely transitively on T_1 and locally primitively on T_2.*

Proof. In order to prove (1) we must show that: $[\langle \Gamma, a \rangle : \Gamma] = [\langle \Gamma, b \rangle : \Gamma] = l$ for $l > 3$, and that $[\langle \Gamma, a \rangle : \Gamma] = 2$ in the case of $l = 3$. Since $\langle \Gamma, a \rangle$ normalizes Γ, it maps into the automorphism group of the graph X. As Γ maps into $\mathrm{Aut}(X)$ trivially and the image of $\langle \Gamma, a \rangle$ is a cyclic group of order l (resp. 2 if $l = 3$). We get the desired indices by pulling back to $\langle \Gamma, a \rangle$.

In a similar way, the group $\langle \Gamma, b \rangle$ maps into the automorphism group of the graph R. Explicitly this group is $\mathrm{Aut}(R) \cong (S^{(l)} \times S^{(l)}) \rtimes \mathbb{Z}/2\mathbb{Z}$ (the action here is by switching the two components). The image of Γ is the factor of order two, and the image of $\langle b \rangle$ is the group $\left\langle (\sqrt{z}^{-1}, \sqrt{z}) \right\rangle \in (S^{(l)} \times S^{(l)})$, which is a cyclic group of order l, normalized by the image of Γ. Pulling back to $\langle \Gamma, b \rangle$ we get $[\langle \Gamma, b \rangle : \Gamma] = [2l : 2] = l$.

For the proof of (3). By theorem (6.6) the action of L on T_1 is the action of $\langle \Gamma, a, b \rangle$. Using the fact (proposition 5.8) that $c = b \circ a$ acts infinitely transitively on half spheres, and that Γ acts on the vertices of T_1 with two orbits we get the desired conclusion.

As L acts locally transitively on T_2, and since T_2 is regular of prime valence, the action of L must be locally primitive.

For the proof of (2) $pr_1(L) < \mathrm{Aut}(T_1)$ is non discrete and hence the lattice is irreducible by remark (6.2). □

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