

AUTOMATA AND SQUARE COMPLEXES.

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ABSTRACT. We introduce a new geometric tool for analyzing groups of finite automata. To each finite automaton we associate a square complex. The square complex is covered by a product of two trees iff the automaton is bi-reversible. Using this method we give examples of free groups and of Kazhdan groups which are generated by the different states of one finite (bi-reversible) automaton. We also reprove the theorem of Macedońska, Nekrashevych, Sushchansky, on the connection between bi-reversible automata and the commensurator of a regular tree.

1. INTRODUCTION

If $X = \{x_1, x_2, \dots, x_d\}$ is a finite set we denote by X^* the free monoid generated by X and by F_X the free group generated by X .

Definition 1.1. *A finite synchronous automaton is a quadruple of the form $A = \langle X, Q, \pi, \lambda \rangle$, where:*

- X is a finite set, referred to as the alphabet.
- Q a finite set, referred to as the states.
- $\pi : X \times Q \rightarrow Q$ is called the transition function.
- $\lambda : X \times Q \rightarrow X$ is called the output function.

An initial automaton is just an automaton with a choice of a state (A, q_0) ; $q_0 \in Q$.

The term finite means that the set of states is finite, and synchronous means that the output function gives one alphabet letter rather than a general word in X^* . We will not treat these more general automata, all our automata will be finite and synchronous.

An initial automaton (A, q_0) defines a map $\mathfrak{T}(A, q_0) : X^* \rightarrow X^*$ by the recursive definition:

$$\begin{aligned}\mathfrak{T}(A, q_0)(\emptyset) &= \emptyset \\ \mathfrak{T}(A, q_0)(xw) &= \lambda(x, q_0)\mathfrak{T}(A, \pi(x, q_0))w \quad \forall x \in X, w \in X^*.\end{aligned}$$

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This map from X^* to itself does not preserve the monoid structure, it is merely a word length preserving transformation of X^* which is compatible with truncations. Equivalently $\mathfrak{T}(A, q_0)$ is an automorphism of the d -regular rooted tree. If this map is invertible for all $q \in Q$ we call the automaton A invertible.

Definition 1.2. *Let A be an automaton (resp. an invertible automaton) We denote by $S(A)$ (resp. $G(A)$) the semigroup (resp. group) of mappings of X^* generated by all different choices of initial automata $\{\mathfrak{T}(A, q)\}_{q \in Q}$.*

Recent years have seen some renewed interest in finite automata and the groups and semigroups that they generate. Groups with interesting properties (e.g. infinite torsion groups, groups with intermediate growth) were constructed from very simple automata. On the other hand well known groups (e.g. the lamplighter group) were realized as groups of type $G(A)$, giving rise to new understanding of their properties [GZ01, GLSŻ00]. We refer the readers to the beautiful paper [GNS00] and the references therein for a survey on this subject.

Another theory that has seen many recent advances is the theory of finite square complexes and in particular the theory of lattices acting on products of trees, [BM97, BMZ, Wis96, NR97].

Our purpose in this paper is to introduce a new geometric tool to the theory of finite automata. With each finite automaton we associate a square complex, conversely certain finite square complexes give rise to finite automata.

Section 2 is dedicated to describing the correspondence between square complexes and automata and its basic properties. Some ideas and constructions, well known in the theory of finite automata, become particularly appealing in this new geometric setting. The idea of associating with every automaton its *dual automaton* by interchanging the alphabet with the set of states is essentially due to Aleshin [Ale83]. If the automata is invertible we can also talk about the inverse and its dual and so on. In general, up to 8 different automata can be obtained by applying the inverse and dual operations to a given one. From the geometric point of view the dual and inverse operations correspond to certain geometric operations on square complexes, namely changing the vertical and horizontal directions or inverting the vertical orientation respectively. An automaton is called bi-reversible if all eight associated automata are well defined. Geometrically an automaton is bi-reversible iff the universal covering of the associated square complex is a product of two trees. We conclude section 2 with a discussion of the beautiful paper [MNS00] in which Macedońska Nekrashevych and

Sushchansky draw the connection between bi-reversible automata and the commensurator of a regular (un-rooted) tree. Similar ideas were developed independently in [LMZ94] where commensurator elements are interpreted in terms of re-colorings of finite regular graphs.

An ongoing project is that of understanding the groups that can be generated by finite automata. Somewhat surprisingly, for example, it is not so easy to generate a free group. In [Ale83] Aleshin sketches an example of two initial automata that generate a free group, but his argument is not complete. Interesting groups like free groups [On98], free products of finite groups [On00] or the group $GL_n(R) \rtimes R^n$ where $R = \mathbb{Z}_{\{2\}}$ is the ring of rational numbers with odd denominators [BS98] can all be generated by finite automatic transformations. Let us point out that the later example contains subgroups such as $SL_n(\mathbb{Z})$ which have Kazhdan property (T). In all these examples the groups are not groups of the form $G(A)$, because they are generated by initial automata coming from a few different finite automata. Sidki suggested [Sid00] one specific automaton A (one of the automata appearing in [Ale83]) as a candidate for generating a free group. As far as we know, despite convincing experimental evidence, the question whether $G(A)$ is free remains open.

Section 3 is dedicated to the construction of new examples of automata. We construct two families of bi-reversible automata, one giving rise to free groups and the other to Kazhdan groups. For the free groups we construct automata A such that $G(A)$ is free and the different initial automata associated with A form a symmetric set of free generators. We do this by first constructing the associated square complexes; but our method is group theoretic rather than combinatorial. We construct torsion free lattices in $\text{Aut}(T_h) \times \text{Aut}(T_v)$ acting transitively on the vertices of the product of two regular trees $T_h \times T_v$.

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2. AUTOMATA AND SQUARE COMPLEXES.

In this section we describe the square complexes associated with finite automata and their basic properties. We begin by collecting basic facts and terminology concerning automata and square complexes.

2.1. Automata. Finite synchronous automata were defined in the introduction. We define here invertible automata and their inverses

Definition 2.1. A synchronous automaton $A = (X, Q, \lambda, \pi)$ is called invertible if $\lambda(\cdot, q) : X \rightarrow X$ is a permutation for every $q \in Q$. If A is invertible we associate with it an inverse automaton $A^{-1} = \langle X, \bar{Q}, \bar{\lambda}, \bar{\pi} \rangle$ where \bar{Q} is a set isomorphic to Q under the isomorphism $q \rightarrow \bar{q}$ and the output and transition functions are given by the formulas $\bar{\lambda}(\cdot, \bar{q}) = \lambda(\cdot, q)^{-1} \quad \forall q \in Q$ and $\bar{\pi}(x, \bar{q}) = \pi(\bar{\lambda}(x, q), q)$.

Remark: It is easy to check that the maps $\mathfrak{T}(A, q) : X^* \rightarrow X^*$ are invertible for all $q \in Q$ iff the automaton A is invertible in the sense of definition 2.1. Moreover the inverse transformation is given by the inverse automaton $(\mathfrak{T}(A, q))^{-1} = \mathfrak{T}(A^{-1}, \bar{q}) \quad \forall q \in Q$. Therefore our definition of invertible automata coincides with the one given in the introduction.

Definition 2.2. A morphism of synchronous automata $\Phi : \langle X, Q, \pi, \lambda \rangle \rightarrow \langle X', Q', \pi', \lambda' \rangle$ is a pair of maps $\Phi_X : X \rightarrow X'$ and $\Phi_Q : Q \rightarrow Q'$ satisfying $\lambda'(\Phi_Q(q), \Phi_X(x)) = \Phi_X(\lambda(q, x))$ and $\pi'(\Phi_Q(q), \Phi_X(x)) = \Phi_Q(\pi(q, x))$.

Definition 2.3. Let $A = \langle X, Q, \lambda, \pi \rangle$ be an automaton. The dual automaton is obtained by exchanging between the output function and the transition function, the states of the dual automaton will be the alphabet of the original and vice versa. Namely if $A = \langle X, Q, \lambda, \pi \rangle$ then $A^* = \langle Q, X, \pi, \lambda \rangle$.

2.2. Square Complexes. We recall some of the terminology pertaining to square complexes; all our notation is taken from [BM00]. A graph is a set of vertices V and a set of edges E with boundary maps $o, t : E \rightarrow V$, called the *origin and terminus maps*, and a fixed point free action $D_2 \cong \mathbb{Z}/2\mathbb{Z} \curvearrowright E$ called *edge inversion* which is compatible with the boundary maps: $o\bar{e} = te$, $t\bar{e} = oe$. We denote by Circ_4 the ‘‘circle of length four graph’’ with a set of vertices $\{1, 2, 3, 4\}$ and set of edges $\{[i, j] : i - j = \pm 1 \pmod{4}\}$.

A square complex Y is given by a graph $Y^{(1)}$ (the one-skeleton), a set of squares S , a boundary map associating with each square $s \in S$ a circle of length four: $\partial s : \text{Circ}_4 \rightarrow Y^{(1)}$ and a fixed point free action of the dihedral group of order 8 on the squares $D_4 \curvearrowright S$ which is compatible with the natural action of the same group on the set of circles of length four. In other words: in the definition of a graph (resp. of a square complex) we are keeping track of the two directed edges (resp. eight directed squares) that correspond to each geometric edge (resp. face) and the group D_2 (resp. D_4) acts on these orientations.

We call a square complex VH if every vertex link is a bipartite graph (allowing for graphs with multiple edges) and if there is a partition of

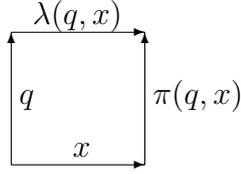


FIGURE 1. The square (x, q)

the set of edges into vertical and horizontal $E = V \amalg H$ which agrees with the bipartition of the graph on every link. Note that our definition here is a little more general than that of [BM00]: we allow for VH-complexes even if the links are not complete bipartite graphs. A VH-complex will be called a VH-T-complex if every link is a complete bipartite graph. An orientation on E is a choice of one directed representative for every geometric edge. A square complex will be called *directed* if there is a choice of orientation on E such that opposite edges of each square are of the same orientation. Namely if $s \in S$ is a square then $\partial s([1, 2])$ and $\partial s([4, 3])$ have the same orientation as well as $\partial s([2, 3])$ and $\partial s([1, 4])$.

2.3. The square complex associated with an automaton.

Definition 2.4. Let $A = \langle X, Q, \lambda, \pi \rangle$ be a finite synchronous automaton. The square complex associated with A denoted by $\Sigma(A)$ is a directed VH-square complex (V, E, S, ∂) such that.

- V contains only one vertex O .
- The 1-skeleton is $E = X^- \amalg X^+ \amalg Q^- \amalg Q^+$. Where X^\pm and Q^\pm are two disjoint copies of X and Q respectively. The \pm distinction describes the choice of orientation on E , namely $\overline{x^+} = x^- \forall x \in X$ and similarly for Q . The horizontal edges are $H = X^\pm$ and the vertical are $V = Q^\pm$.
- The geometric squares are pairs $X \times Q$. With the boundary given by (see figure 1):

$$\partial(x, q) = x^+ \pi(q, x)^+ \lambda(q, x)^- q^-.$$

The eight directed squares corresponding to each geometric square are obtained by applying formally all possible symmetries of the square.

We denote by $\Pi(A) = \pi_1(\Sigma(A), O)$ and $\widetilde{\Sigma(A)}$ the fundamental group and the universal cover of the VH-square complex (V, E, S, ∂) .

Remark: A morphism of automata defines a morphism of labelled square complexes $\Sigma(\Phi) : \Sigma(A) \rightarrow \Sigma(A')$. The map $\Sigma(\Phi)$ in turn gives rise to maps on the fundamental group and universal covering. We introduce the following short notation for these maps $\tilde{\Phi} = \widetilde{\Sigma(\Phi)} : \widetilde{\Sigma(A)} \rightarrow \widetilde{\Sigma(A')}$ and $\pi_1(\Phi) \stackrel{\text{def}}{=} \pi_1(\Sigma(\Phi)) : \Pi(A) \rightarrow \Pi(A')$.

2.4. Basic properties. Let A be an automaton $\Sigma(A)$ the associated square complex. The group D_4 acts giving rise to different choices of orientation and VH-structure on the same geometric complex. In particular the complex associated with the dual automaton $\Sigma(A^*)$ corresponds to the square complex obtained by exchanging vertical and horizontal directions in $\Sigma(A)$. The other six possibilities do not necessarily come from automata, in fact they come from the following six automata iff the latter are well defined: A^{-1} , $(A^{-1})^*$, $(A^*)^{-1}$, $((A^*)^{-1})^*$, $((A^{-1})^*)^{-1}$, $((((A^{-1})^*)^{-1})^*)^{-1} = (((A^*)^{-1})^*)^{-1}$

Definition 2.5. (See, [MNS00]) *An automaton is called reversible if its dual is invertible. It is called bi-reversible if all eight automata above are well defined.*

Remarks:

- It is enough to check that A and A^{-1} are reversible in order to deduce that A is bi-reversible, i.e. it is enough that A, A^* and $(A^{-1})^*$ all be invertible.
- It is possible for an automaton to be invertible and reversible but not bi-reversible. An example of such an automaton is depicted in figure 2, it was shown in [GŻ01] (see also [GNS00]) that this automaton defines the lamplighter group.

Proposition 2.6. *Let A be an automaton then A is bi-reversible iff $\Sigma(A)$ is VH-T-complex (i.e. iff $\widetilde{\Sigma(A)}$ is a product of trees).*

Proof. The link of the vertex $\text{Lk}(O)$ is a bipartite graph where the vertices are naturally partitioned into two sets X^\pm and Q^\pm . The orientation further divides each side into two disjoint sets. By the definition of $\Sigma(A)$, a pair of vertices of type (x^+, q^+) are always connected by a unique edge in the link (i.e. there is a unique square incident with the two edges near their origin). The existence of such a unique edge for pairs of type (x^+, q^-) (resp. (x^-, q^+) , resp. (x^-, q^-)) is equivalent to the invertibility of A (resp. A^* , resp. $(A^{-1})^*$). Thus $\text{Lk}(O)$ is a complete bipartite graph if and only if the condition of the theorem holds. It is well known that the universal covering of a VH-complex is a product of trees iff every vertex link is a complete bipartite graph. \square

Remark: The condition of the theorem is also necessary and sufficient for \widetilde{A} to be a CAT(0) space. Clearly a product of trees is CAT(0). Conversely, if the condition does not hold then, by counting considerations, there exists a pair of edges (x, q) connected by more than one square. So there is a circle of length π (two edges of length $\pi/2$ each) in the link $\text{Lk}(O)$ and $\widetilde{\Sigma(A)}$ can not be CAT(0).

Let $w \in X^*$, we can associate with it a path in the horizontal 1-skeleton of the complex $\Sigma(A)$. Such a path will follow always the positive orientation of the edges. A general path in the horizontal 1-skeleton is described by a word in F_X . In a similar fashion we can describe paths in the vertical 1-skeleton by words in Q^* or in F_Q . A positive path in the 1-skeleton of $\Sigma(A)$ is described by a word in $(X \amalg Q)^*$ and a general path by a word in $F_{X \amalg Q}$.

Definition 2.7. *An immersion of square complexes is a cellular map which induces an injection on the links.*

The fact that every pair of type (x^+, q^+) is incident with a unique square is a direct consequence of the definition of the complex $\Sigma(A)$. Using the notion of an immersion of square complexes this can be extended to longer words in X^* and Q^* . If one or more of the automata $A, A^*, (A^{-1})^*$ are invertible then we can talk about words in F_X and F_Q as well. The proof of the following theorem is a direct consequence of the definitions.

Proposition 2.8. *Let A be an automaton.*

- *Let $\underline{x} \in X^*$ and $\underline{q} \in Q^*$, there is a unique immersion of a tessellated rectangle into $\Sigma(A)$ whose lower edge describes the path \underline{x} and whose left side describes the path \underline{q} .*
- *If A is invertible then the previous statement holds for every $\underline{x} \in X^*$ and $\underline{q} \in F_Q$.*
- *If A is reversible then one can take $\underline{x} \in F_X$ and $\underline{q} \in Q^*$.*
- *If A is bi-reversible then one can take $\underline{x} \in F_X$ and $\underline{q} \in F_Q$. Note that this is not possible if one only assumes that both A, A^* are invertible.*

The upper and right sides of the tessellated rectangle from the proposition describe elements of these monoids (or groups, as the case may be: of the same type as the lower or left sides) we will denote these by $\lambda(\underline{q}, \underline{x})$ and $\pi(\underline{q}, \underline{x})$ respectively. This general notation agrees with our previous notation in a few special cases:

- *If $\underline{x} \in X, \underline{q} \in Q$ this agrees with the definition of $(\Sigma(A))$.*
- *If $\underline{q} \in Q$ and $\underline{x} \in X^*$ then $\lambda(\underline{q}, \underline{x}) = \mathfrak{T}(A, \underline{q})\underline{x}$.*

- If $\underline{q} = q_1 q_2 \dots q_l$ then $\lambda(\underline{q}, \underline{x}) = \mathfrak{T}(A, q_1) \circ \mathfrak{T}(A, q_2) \circ \dots \circ \mathfrak{T}(A, q_l) \underline{x}$. Thus the map:

$$\begin{aligned} Q^* \times X^* &\rightarrow X^* \\ (\underline{q}, \underline{x}) &\rightarrow \lambda(\underline{q}, \underline{x}) \end{aligned}$$

gives an action of Q^* on X^* . This action is not necessarily effective, in fact, by definition of $S(A)$, it factors via the action of $S(A)$ on X^* .

- If A is invertible (resp. reversible, resp. bi-reversible) we obtain more general actions of the form $F_Q \times X^* \rightarrow X^*$ (resp. $Q^* \times F_X \rightarrow F_X$ resp. $F_Q \times F_X \rightarrow F_X$). Where we remember that, by abuse of notation, the action on F_X is not an action on the group but rather an action on the Cayley graph $\text{Cay}(F_X, X)$ which fixes the base vertex e . In the invertible case the action $F_Q \times X^* \rightarrow X^*$ factors via an effective action of the group $G(A)$ on X^* .

2.5. Non-finiteness properties. Let A_0 be the automaton on a one letter alphabet and with one state. $\Sigma(A_0)$ is just a torus obtained from gluing parallel sides of a square. We can identify $\Pi(A_0) \rightsquigarrow \widetilde{\Sigma(A_0)}$ with the regular action of \mathbb{Z}^2 on \mathbb{R}^2 . Clearly there exists a unique morphism ι from any automaton A to A_0 . This gives rise to *height* functions on the universal cover and fundamental groups of A . By abuse of notation we call all of these functions $\text{ht} = (\text{ht}^1, \text{ht}^2)$

$$(2.1) \quad \begin{array}{ccc} \Pi(A) & \xrightarrow{\text{ht}} & \mathbb{Z}^2 \\ \downarrow & & \downarrow \\ \widetilde{A} & \xrightarrow{\text{ht}} & \mathbb{R}^2 \end{array}$$

The following statement is a direct consequence of the existence of the height function:

Lemma 2.9. *Any path $w \in \Pi(A)$ homotopic to the identity satisfies $\text{ht}(w) = (0, 0)$. If w is represented by a word in $F_{X \amalg Q}$ this means that in both horizontal and vertical directions the positively and negatively oriented edges should be balanced.*

Corollary 2.10. *$\Pi(A)$ is infinite, in fact, every element $w \in \Pi(A)$ with $\text{ht}(w) \neq (0, 0)$ is of infinite order.*

2.6. Automata associated with square complexes. As we have seen every square complex of the form $\Sigma(A)$ satisfies the minimal link condition defined below.

Definition 2.11. *A directed square complex is said to satisfy the minimal link condition if every pair of positively oriented edges, one vertical and one horizontal edges starting at the same vertex are incident to a unique square.*

Square complexes of the form $\Sigma = \Sigma(A)$ are all directed VH-complexes with one vertex O satisfying the minimal link condition. Conversely given such a square complex Σ we can reconstruct a finite automaton $A = A(\Sigma) = (X, Q, \pi, \lambda)$

- X is the set of horizontal positively oriented edges.
- Q is the set of vertical positively oriented edges.
- $\lambda(q, x)$ and $\pi(q, x)$ are defined to be the unique edges opposite to x and q respectively, in the unique square incident to both x and q near their origin (see figure 1).

If Σ is a VH-T-complex then we can associate with it an automaton $A^T(\Sigma) = (X, Q, \pi, \lambda)$ even without assuming that the complex is directed, by setting:

- X is the set of directed horizontal edges.
- Q is the set of directed vertical edges.
- $\lambda(q, x)$ and $\pi(q, x)$ are defined to be the unique edges opposite to x and q in the unique square incident to x and q near their origin.

If we apply this construction to $\Sigma(A)$ where A is a bi-reversible automaton then we will end up with a new automaton whose sets of states and alphabet consist of the original sets and their formal opposites.

Not every bi-reversible automaton is of the form $A^T(\Sigma)$. When it is possible, however, there are big advantages for working with non directed square complexes. If Σ is a directed square complex with universal covering $T_h \times T_v$ then the action of $\pi_1(\Sigma, O)$ on each one of the factor trees can not be very transitive because of the restriction that edge directions should be preserved. This is a serious drawback from the point of view of the theory of groups acting on products of trees. For example in section (3) we construct bi-reversible automata from non-directed VH-T-square complexes, using their strong transitivity properties in an essential way.

It is possible to formulate intermediate constructions when the link condition is defined in two ‘‘corners’’ of the square complex (i.e. when $A(\Sigma)$ is invertible or when its dual is invertible). Instead of trying to formulate a general statement we give an example. Consider the automaton A depicted in figure 3. This automaton is invertible but neither A nor A^{-1} are reversible. It is proved in [GNS00] that $G(A)$ is

FIGURE 2. A self-dual automaton defining the Lamplighter group.

FIGURE 3. An automaton generating an infinite 3 group.

the Gupta-Sidki group which is an infinite 3-group. Since A has three states and acts on an alphabet of three letters $\Sigma(A)$ is a complex of nine squares. If we notice however that $\mathfrak{T}(A, a) = \mathfrak{T}(A, c)^{-1}$ we can draw a square complex Ξ with six squares in such a way that $A^T(\Xi)$ has four states $a, c = a^-, b, b^-$ and the same alphabet, in such a way that $\mathfrak{T}(A^T(\Xi), a) = \mathfrak{T}(A, a)$, $\mathfrak{T}(A^T(\Xi), a^-) = \mathfrak{T}(A, c)$, $\mathfrak{T}(A^T(\Xi), b) = \mathfrak{T}(A, b)$, $\mathfrak{T}(A^T(\Xi), b^-) = \mathfrak{T}(A, b)^{-1}$.

2.7. Bi-Reversible automata. From now on we will restrict our attention to VH-T-complexes Σ , and to bi-reversible automata. As we have seen there are two cases here, the directed and non directed. We will treat both. In the bi-reversible setting $\tilde{\Sigma} = T_h \times T_v$ is a product of trees. The group $\pi_1(\Sigma, O)$ acts on $\tilde{\Sigma}$, preserving the VH-structure, and therefore acts on each one of the factors.

Let $\tilde{O} = (O_h, O_v) \in T_h \times T_v$ be a base vertex. The horizontal and vertical one skeletons $\Sigma^{(1),h}, \Sigma^{(1),v}$ of the complex Σ are just bouquets of circles labelled by the set X and Q respectively. Since both $\Sigma^{(1),h}$ and Σ are CAT(0) spaces then the embedding $\Sigma^{(1),h} \hookrightarrow \Sigma$ gives rise to an embedding of the actions of the fundamental groups on the universal

FIGURE 4. The action of $\Pi(A)$ on T_h

coverings see [BH99]:

$$\begin{array}{ccc} F_X & \longrightarrow & \pi_1(\Sigma, O) \\ \downarrow & & \downarrow \\ T_h & \longrightarrow & T_h \times T_v \end{array}$$

And similarly in the vertical direction. We will henceforth identify the trees $T_h \cong T_h \times \{O_v\}$ and $T_v \cong \{O_h\} \times T_v$ and the corresponding groups with the setwise stabilizers of these trees $F_X \cong \pi_1(\Sigma, O)_{O_v}$ and $F_Q \cong \pi_1(\Sigma, O)_{O_h}$.

The universal covering map to Σ gives rise to an edge coloring of T_h by the elements of X^\pm . This gives a natural identification of T_h with the Cayley graph of F_X where O_h represents the identity element of the group. The monoid X^* spans an $|X|$ -regular rooted tree $T_X \subset T_h$.

Consider now the action of $F_Q \subset \Pi(A)$ on T_h . Let $q_1 q_2 \dots q_l = \underline{q} \in F_Q$ and consider the vertical path $\gamma_{\underline{q}}$ in $\Sigma(A)$ represented by $\underline{q}^{-1} = q_l^{-1} \dots q_2^{-1} q_1^{-1}$. This path represents an element of $\pi_1(\Sigma, O)$ and it acts on $T_h \times T_v$ by a deck transformation. $\gamma\tilde{O}$ is just the endpoint of the unique lifting $\tilde{\gamma}$ of the path γ starting at \tilde{O} . Let $\underline{x} = x_1 x_2 \dots x_m$ represent a horizontal path in Σ and \tilde{x} its unique lifting to $T_h \times T_v$ starting at \tilde{O} . The path $\gamma\tilde{x}$ is the unique lifting of \underline{x} to a path starting at $\gamma\tilde{O}$. Finally projecting back to $T_h \cong T_h \times \{O_v\}$ we find that $\text{pr}_h(\gamma\tilde{x})$ is the unique lifting of the path $\lambda(\underline{q}, \underline{x})$ to a path starting at \tilde{O} (see figure 4). Summarizing everything we obtain the following

Proposition 2.12. *Let $A = (X, q, \pi, \lambda)$ be a bi-reversible automaton, $\Sigma = \Sigma(A)$ the associated square complex and $T_h \times T_v$ its universal cover and $\tilde{O} = (O_h, O_v)$ a base vertex. The group F_Q acts in two different ways on $\text{Cay}(F_X, X)$:*

- *Via the (left) action $(\underline{q}, \underline{x}) \rightarrow \lambda(\underline{q}, \underline{x})$ defined after proposition 2.8.*

- Via the identification $\pi_1(\Sigma, O)_{O_h} \cong F_Q$ and the (right) action of $\pi_1(\Sigma, O)$ on $T_h \times T_v$ which projects to an action on $T_h \cong \text{Cay}(F_X, X)$.

These two actions become isomorphic if we precompose the second action with the inverse map of the group in order to turn it into a left action.

The rooted tree X^* is invariant under these actions and any $g \in G(A)$ fixing pointwise X^* fixes also the whole tree. The action of the group F_Q on $T_h \cong \text{Cay}(F_X, X)$ is not necessarily effective but it factors through an effective action of the group $G(A)$.

Proof. We have, in fact, proved everything with the exception of the last injectivity statement. Assume that $\underline{q} \in Q^*$ is such that $\lambda(\underline{q}, \underline{x}) = \underline{x} \ \forall \underline{x} \in X^*$. We first show that $\lambda(\underline{q}, \underline{x}^{-1}) = \underline{x}^{-1} \ \forall \underline{x} \in X^*$. Indeed consider the action of \underline{q} on the periodic bi infinite word $w = \cdots \underline{xxxx} \cdots$. In geometric terms we consider an immersion of an infinite strip whose lower side describes the word w and some vertical segment describes the word \underline{q} . Since the possible number of vertical lines is finite and since the lower side and one vertical segment determine the immersion on the whole strip, the immersion is bound to be periodic. But by assumption the upper left side of the strip describes the word $\cdots \underline{xxxx}$ so the upper right hand side must describe the word $\underline{xxxx} \cdots$. The situation is completely symmetric of course: if we assume that \underline{q} acts trivially on “negative words” we can conclude in a similar manner that it acts trivially on positive words as well.

Clearly if the word \underline{q} acts trivially on X^* so does the word $\pi(\underline{q}, \underline{x})$ for any $\underline{x} \in X^*$. Furthermore by the previous paragraph \underline{q} will act trivially also on the set of negative words so that $\pi(\underline{q}, \underline{x}^{-1})$ will also act trivially on all negative words and therefore $\pi(\underline{q}, \underline{x}^{-1})$ will act trivially on X^* as well for any $\underline{x} \in X^*$.

Now decomposing a general word $w \in F_X$ into a sequence of negative and positive words we can prove by induction that $\lambda(\underline{q}, w) = w$ and this completes the proof. \square

The non directed situation is a little different: Let Σ be a VH-T-complex with one vertex, universal covering $T_h \times T_v$ and $A = A^T(\Sigma) = (X, Q, \pi, \lambda)$. Pick an orientation on X . This gives a partition of X into two sets Y, \bar{Y} . One can think of T_h as the Cayley graph of F_Y . Let us

define a homomorphism of monoids and of trees.

$$\begin{aligned}\phi : X^* &\rightarrow F_Y \\ y &\mapsto y \\ \bar{y} &\mapsto y^{-1}\end{aligned}$$

Proposition 2.13. *The action $G(A) \curvearrowright X^*$ factors, via ϕ , to an action on F_Y . Furthermore the map $\Phi : G(A) \rightarrow \text{Aut}(T_Y)$ is injective (i.e. any element acting trivially on F_Y will act trivially also on X^*). The action of $G(A) \curvearrowright F_Y$ is isomorphic to the action $F_Q \curvearrowright T_h$ as above.*

Proof. Paths in the tree $\text{Cay}(F_Y, Y)$ starting at the base vertex are in one to one correspondence with the set of paths without backtracking in the rooted tree corresponding to X^* starting at the root. This gives a bijection between the automorphisms of $\text{Cay}(F_Y, Y)$ preserving the base point, and between the automorphisms of the rooted tree $\text{Cay}(X^*, X)$ which preserve backtracking, i.e. automorphisms that take words of the type wvv^{-1} to words of type $w'v'v'^{-1}$. The definition of A implies $G(A)$ preserves “backtracking”. Our discussion earlier proves that this correspondence of paths is equivariant under the $G(A)$ action. \square

Corollary 2.14. *In both the directed and the non directed cases group $G(A)$ is isomorphic to the projection of F_Q to $\text{Aut}(T_h)$.*

2.8. Bi-Reversible automata and the commensurator of a regular tree. Let X be a finite set, $T = \text{Cay}(F_X, X)$ the right Cayley graph of F_X with respect to the free set of generators which is a $2|X|$ -regular tree. The action of F_X on T gives rise to an embedding of F_X in $\text{Aut}(T)$ as a uniform lattice. We define the commensurator

$$\begin{aligned}C &= \text{Comm}_{\text{Aut}(T)}(F_X) \\ &= \{g \in \text{Aut}(T) \mid (gF_Xg^{-1} \cap F_X) \text{ is of finite index in both } F_X \text{ and } gF_Xg^{-1}\}.\end{aligned}$$

Definition 2.15. *We call an automorphism of T directed if it preserves the orientation on the edges which is given by the choice of the set of generators. We denote the group of directed automorphisms of T by $\text{Aut}^{\text{Dir}}(T)$.*

Let $O \in T$ be a base vertex (e.g. corresponding to the identity element of F_X) and C_O the stabilizer of this point. Let $C_O^{\text{Dir}} = C_O \cap \text{Aut}^{\text{Dir}}(T)$.

Note: We have defined the commensurator for this specific lattice for convenience but, in fact, a famous theorem of Leighton [?],[?] shows that up to conjugation in $\text{Aut}(T)$ the commensurator is independent

of the uniform lattice. This is not true anymore for the directed commensurator.

We re-prove here a theorem from [MNS00] using the new terminology.

Theorem 2.16. (*Macedońska, Nekrashevych and Sushchansky*)

All bi-reversible automatic transformations over the alphabet X form a group $\text{Bat}(X)$. This group is isomorphic to the group C_O^{Dir} defined above.

A non directed version of the same theorem will say:

Theorem 2.17. *All automatic transformations of the form $(A^T(\Sigma), q)$ where Σ is a VH-T-complex with one vertex form a group. This group is naturally isomorphic to a vertex stabilizer of the commensurator of a uniform lattice in $\text{Aut}(T)$, where T is the $2|X|$ -regular tree, i.e. to the group C_O defined above.*

Proof. The proofs of the two theorems are almost identical, we will treat the directed case and leave the other to the reader. It is enough to prove the isomorphism. First we define a natural homomorphism $\phi : \text{Bat}(X) \rightarrow C_O$. Let T be the Cayley graph of F_X which can also be identified as T_h : one of the factor trees of \tilde{A} where A is any finite automaton on the alphabet X . If $\mathfrak{T}(A, q) \in \text{Bat}(X)$ is an initial bi-reversible automaton then, $\mathfrak{T}(A, q)$ is an automorphism of the rooted tree $T_X \subset T$ corresponding to the monoid $X^* \subset F_X$. However $\mathfrak{T}(A, q)$ can be extended to an automorphism $\widetilde{\mathfrak{T}(A, q)}$ of T . This can be thought of as the action of $\gamma(q^{-1}) \in \Pi(A)$, the loop corresponding to q^- as an element of the fundamental group. We define:

$$(2.2) \quad \begin{aligned} \phi : \text{Bat}(X) &\rightarrow C_O \\ \phi \mathfrak{T}(A, q) &\mapsto \widetilde{\mathfrak{T}(A, q)} \end{aligned}$$

The map ϕ is well defined because a bi-reversible transformation $\mathfrak{T}(A, q)$ which is trivial on T_X will give rise to a trivial $\widetilde{\mathfrak{T}(A, q)}$.

To see that the image of ϕ is contained in the commensurator we identify $F_X \rightsquigarrow \text{Cay}(F_X, X)$ with the action $\Pi(A)_{O_v} \rightsquigarrow T_h \times \{O_v\}$. But $\gamma(q^{-1})\Pi(A)_{O_v}\gamma(q) = \Pi(A)_{\gamma(q^{-1})O_v}$ and these two groups are commensurable. Specifically the pointwise stabilizer of the geodesic $[O_2, \gamma(q^{-1})O_2]$ is the finite index intersection.

Conversely assume that we are given an element $c \in C_{O_1}^{\text{Dir}}$. By definition $(F_X \cap (cF_Xc^{-1}))$ has a finite index in both F_X and (cF_Xc^{-1}) . By standard covering theory we obtain the corresponding covering maps

of graphs:

$$\begin{array}{ccc}
 & (F_X \cap (cF_Xc^{-1})) \setminus T & \\
 p_0 \swarrow & & \searrow p_1 \\
 F_X \setminus T & & cF_Xc^{-1} \setminus T
 \end{array}$$

We take the product of the graph $Y = (F_X \cap (cF_Xc^{-1})) \setminus T$ with an interval $[0, 1]$ and glue the graph $0 \times Y$ onto $F_X \setminus T$ according to the covering map p_0 and the graph $\{1\} \times Y$ to $cF_Xc^{-1} \setminus T$ according to p_1 . Finally we identify the two graphs $F_X \setminus T$ and $cF_Xc^{-1} \setminus T$ via the map $F_Xe \rightarrow (cF_Xc^{-1})ce$. It is easy to see, using the assumption on c , that the result is a directed VH-T-complex with one vertex describing the transformation c . \square

3. EXAMPLES

3.1. A free group. Let Σ be a VH-T-complex with one vertex and universal cover $T_h \times T_v$. Let $A = A^T(\Sigma)$ be the associated automaton. By theorem 2.13 the group $G(A)$ is the image in $\text{Aut}(T_h)$ of the group F_Q —the subgroup of $\pi_1(\Sigma, O)$ generated by all the vertical loops (i.e. loops labelled by elements of Q). We recall that F_Q is a free group with the set of states as free generators. The image of F_Q in $\text{Aut}(T_h)$ will also be free if it acts faithfully on T_h . We are lead to examine the case where $\pi_1(\Sigma_O)$ acts faithfully on the factor trees of the universal cover.

Proposition 3.1. *Let Σ be a VH-T-complex with one vertex and $\pi_1(\Sigma, O) \rightsquigarrow T_h \times T_v$ the action on the universal cover. If the action of $\pi_1(\Sigma, O)$ on the factor tree T_h is effective then $G(A^T(\Sigma))$ is a free group on the set of states.*

We proceed to develop the framework that enables us to ensure that this action is indeed effective.

Definition 3.2. *We say that a group action on a tree $G \rightsquigarrow T$ is locally infinitely transitive, if every vertex stabilizer G_x acts transitively on each sphere centered at x .*

Proposition 3.3. *Let Σ be a VH-T-complex, and $\pi_1(\Sigma, O) \rightsquigarrow T_h \times T_v$ its action on the universal cover. If the action of $\pi_1(\Sigma, O)$ on T_v is locally infinitely transitive then its action on T_h is effective.*

Proof. Assume the contrary and let $K < \pi_1(\Sigma, O)$ be the kernel of the action on T_h . The group K is normal and it acts freely on T_v . The action of $\pi_1(\Sigma, O)$ on T_v factors now through an action of $K \backslash \pi_1(\Sigma, O) \rightsquigarrow K \backslash T_h$. This new action should still be infinitely transitive (because

dividing by a group acting freely does not change the stabilizers), but this is no longer possible for an action on a graph which is not a tree. This contradiction completes the proof. \square

There are many ways of obtaining lattices acting locally infinitely transitively on the projections (see [BM00, BG02]), but the easiest way is to appeal to the theory of arithmetic lattices, in fact by the weak approximation theorem the projections of an irreducible arithmetic lattice in $\mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_q)$ are dense in each of the factor groups and in particular they act in a locally infinitely transitive fashion. Combining these observations we have

Proposition 3.4. *Let $\Gamma < \mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_q)$ be a torsion free irreducible arithmetic lattice acting transitively on the vertices of the building $T_h \times T_v$, then $G^T(\Gamma \backslash (T_h \times T_v))$ is a free group.*

Examples of such lattices are described in detail in [Vig80, Moz94]. We will not go through the proof of the desired properties but will describe the corresponding square complexes in detail for the convenience of the reader.

Assume that we are given a pair of primes $p \neq l$ both congruent to $1 \pmod{4}$. With such a pair we associate a square complex:

- There is only one vertex O .
- There are exactly $(p+1)$ projective quaternions $[x] = [a + bi + cj + dk]$ satisfying the following:
 - (1) $x = 1 \pmod{2}$ i.e. a is odd and b, c, d are all even.
 - (2) $Nx = x\bar{x} = a^2 + b^2 + c^2 + d^2 = p$.

The word projective and the square bracket notation just mean that we identify x and $-x$. We denote these projective quaternions by x_1, x_2, \dots, x_{p+1} . Similarly there are $l+1$ quaternions q_1, q_2, \dots, q_{l+1} associated with the prime l .

These quaternions will stand for the directed loops in the 1 skeleton, where x, \bar{x} are the two directed edges forming a geometric edge.

- Given two projective quaternions q_i, x_j there is a unique pair $q_{k(i,j)}, x_{l(i,j)}$ satisfying the equation (note that this is an equality of projective quaternions thus equality should hold only up to sign):

$$x_j q_i = q_{k(i,j)} x_{l(i,j)} \text{ (projective equality)}$$

The set of these relations form the directed squares with boundary $x_j q_i \overline{q_{k(i,j)} x_{l(i,j)}}$. Note that each non directed square appears here eight times in all possible orientations.

3.2. Explicit free group construction. In this section we carry out the calculation for the smallest example namely when the primes are $p = 5, l = 13$. We will obtain an automaton A on a 6 letter alphabet with 14 states such that the corresponding initial automata form a symmetric set of generators for a free group on 7 generators $G(A) = F_7$. The dual automaton will have 6 states and 14 letter alphabet with $G(A^*) = F_3$.

We list the 6 projective quaternions associated with the prime $p = 5$ and the 14 associated with the prime $l = 13$. Only one of every conjugate pair appears in the table below.

$x_1 = [1 + 2i]$	$a_1 = [1 + 2i + 2j + 2k]$
$x_2 = [1 + 2j]$	$a_2 = [1 + 2i + 2j - 2k]$
$x_3 = [1 + 2k]$	$a_3 = [1 + 2i - 2j + 2k]$
	$a_4 = [1 - 2i + 2j + 2k]$
	$a_5 = [3 + 2i]$
	$a_6 = [3 + 2j]$
	$a_7 = [3 + 2k]$

So our automaton will be $A = (X, Q, \pi, \lambda)$ where $X = (x_1, x_2, x_3, \bar{x}_1, \bar{x}_2, \bar{x}_3)$ and $Q = (a_1, \dots, a_7, \bar{a}_1, \dots, \bar{a}_7)$. A short calculation will show that the transition and output functions are given by the following table:

(π, λ)	x_1	x_2	x_3	\bar{x}_1	\bar{x}_2	\bar{x}_3
a_1	$(\mathbf{a_6}, \bar{\mathbf{x_3}})$	(a_7, \bar{x}_1)	(\bar{a}_5, \bar{x}_2)	(\bar{a}_4, x_1)	(a_3, x_2)	(a_2, x_3)
a_2	(\bar{a}_7, \bar{x}_2)	(a_5, x_3)	(a_1, \bar{x}_3)	(\bar{a}_3, x_1)	(\bar{a}_4, x_2)	(a_6, \bar{x}_1)
a_3	(a_7, x_2)	(a_1, \bar{x}_2)	(\bar{a}_6, \bar{x}_1)	(\bar{a}_2, x_1)	(a_5, \bar{x}_3)	(\bar{a}_4, x_3)
a_4	(a_1, \bar{x}_1)	(\bar{a}_5, \bar{x}_3)	(a_6, x_1)	(a_7, \bar{x}_2)	(\bar{a}_2, x_2)	(\bar{a}_3, x_3)
a_5	(a_5, x_1)	(a_3, x_3)	(\bar{a}_4, \bar{x}_2)	(a_5, \bar{x}_1)	(a_2, \bar{x}_3)	(a_1, x_2)
a_6	(\bar{a}_3, \bar{x}_3)	(a_6, x_2)	(a_2, x_1)	$(\mathbf{a_1}, \mathbf{x_3})$	(a_6, \bar{x}_2)	(a_4, \bar{x}_1)
a_7	(a_4, x_2)	(\bar{a}_2, \bar{x}_1)	(a_7, x_3)	(a_3, \bar{x}_2)	(a_1, x_1)	(a_7, \bar{x}_3)
\bar{a}_1	(\bar{a}_4, \bar{x}_1)	(\bar{a}_3, \bar{x}_2)	(\bar{a}_2, \bar{x}_3)	(\bar{a}_7, x_2)	(\bar{a}_5, x_3)	$(\mathbf{\bar{a}_6}, \mathbf{\bar{x}_1})$
\bar{a}_2	(a_3, \bar{x}_1)	(a_4, \bar{x}_2)	(\bar{a}_5, x_2)	(\bar{a}_6, \bar{x}_3)	(a_7, x_1)	(\bar{a}_1, x_3)
\bar{a}_3	(a_2, \bar{x}_1)	(\bar{a}_7, x_1)	(a_4, \bar{x}_3)	(a_6, x_3)	(\bar{a}_1, x_2)	(\bar{a}_5, \bar{x}_2)
\bar{a}_4	(\bar{a}_6, x_3)	(a_2, \bar{x}_2)	(a_3, \bar{x}_3)	(\bar{a}_1, x_1)	(\bar{a}_7, \bar{x}_1)	(a_5, x_2)
\bar{a}_5	(\bar{a}_5, x_1)	(\bar{a}_1, \bar{x}_3)	(\bar{a}_3, x_2)	(\bar{a}_5, \bar{x}_1)	(a_4, x_3)	(\bar{a}_2, \bar{x}_2)
\bar{a}_6	(\bar{a}_2, x_3)	(\bar{a}_6, x_2)	$(\mathbf{\bar{a}_1}, \mathbf{\bar{x}_1})$	(\bar{a}_4, \bar{x}_3)	(\bar{a}_6, \bar{x}_2)	(a_3, x_1)
\bar{a}_7	(\bar{a}_1, \bar{x}_2)	(\bar{a}_4, x_1)	(\bar{a}_7, x_3)	(a_2, x_2)	(\bar{a}_3, \bar{x}_1)	(\bar{a}_7, \bar{x}_3)

One can read out the associated directed square complex from the table, each entry in the table corresponds to one square cell. Since our automaton was originally obtained from a non-directed square complex Σ , there is some redundancy in the data. The original, non directed, square complex is obtained by collapsing sets of four entries in the

above table to represent one single geometric square. For example the boldface entries in the above table, corresponding to the equations $[x_1 \cdot a_1] = [a_6 \cdot \bar{x}_3]$, $[\bar{x}_1 \cdot a_6] = [a_1 \cdot x_3]$, $[\bar{x}_3 \cdot \bar{a}_1] = [\bar{a}_6 \cdot x_1]$, $[x_3 \cdot \bar{a}_6] = [\bar{a}_1 \cdot \bar{x}_1]$, all represent a single geometric square in the non directed complex.

3.3. A Kazhdan group.

3.3.1. *An extension of Mumford's example.* For the construction of an automaton A such that $G(A)$ has property T we will need to construct an irreducible lattice $\Gamma < \mathrm{PGL}_3(\mathbb{Q}_p) \times \mathrm{PGL}_3(\mathbb{Q}_q)$ with the following properties:

- Γ acts transitively on the vertices of the building $\Delta_p \times \Delta_q$ of $\mathrm{PGL}_3(\mathbb{Q}_p) \times \mathrm{PGL}_3(\mathbb{Q}_q)$.
- Γ acts freely on the geometric cubes of $\Delta_p^{(1)} \times \Delta_q^{(1)}$ — the product of the one skeletons of the buildings.

Here (p, q) are just a pair of prime numbers.

We start with an example due to Mumford ([Mum79]) of a torsion free lattice acting transitively on the building of $\mathrm{PGL}_3(\mathbb{Q}_2)$. Mumford's example is based on an arithmetic lattice in $\mathrm{PGL}_3(\mathbb{Q}_2)$. For background on arithmetic lattices and on unitary groups we refer to [PR94]. A good short summary on unitary groups in three variables can be found in the first few pages of [Rog90].

Let $\zeta = e^{2\pi i/7}$, $\lambda = \zeta + \zeta^2 + \zeta^4 = \frac{-1 + \sqrt{-7}}{2}$ and $\bar{\lambda} = \zeta^3 + \zeta^5 + \zeta^6 = \frac{-1 - \sqrt{-7}}{2}$. Let $K = \mathbb{Q}(\lambda)$ and $\mathcal{O} = \mathbb{Z}[\lambda]$ its ring of integers, \mathcal{O} is an Euclidean domain. The extension $[K : \mathbb{Q}]$ is of degree 2 with a Galois involution — the complex conjugation $z \rightarrow \bar{z}$. The extension $[\mathbb{Q}(\zeta) : K]$ is Galois of degree 3, with the Galois group generated by σ , $\sigma(\zeta) = \zeta^2$. We will, most of the time, think of $\mathbb{Q}(\zeta)$ merely as a K -vector space and as such we will denote it by $V = \mathbb{Q}(\zeta)$. V contains the lattice $L = \mathbb{Z}[\zeta]$ with a basis $1, \zeta, \zeta^2$ over \mathcal{O} . We define a Hermitian form on V :

$$h(x, y) = \mathrm{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}(\lambda)}(x\bar{y}) = x\bar{y} + \sigma(x\bar{y}) + \sigma^2(x\bar{y})$$

Taking $1, \zeta, \zeta^2$ as a basis we find that $h(x, y) = xH({}^t\bar{y})$ where H is the matrix

$$H = \begin{pmatrix} 3 & \bar{\lambda} & \bar{\lambda} \\ \lambda & 3 & \bar{\lambda} \\ \lambda & \lambda & 3 \end{pmatrix}.$$

Let $A = \mathrm{End}_K(V) = M_3(K)$ be the split central simple algebra over K . The form h defines an involution of the second kind on A given by $g \rightarrow \alpha(g) = H({}^t\bar{g})H^{-1}$. Let $G = U(h)$ be the algebraic group of all linear transformations preserving the Hermitian form h . Note that,

even though A is a K -algebra, the algebraic group G is only defined over \mathbb{Q} because the definition involves the Galois involution. The \mathbb{Q} -rational points of G are given by:

$$\begin{aligned} G(\mathbb{Q}) &= \{g \in GL(V, K) : h(xg, yg) = h(x, y) \forall x, y \in V\}. \\ &= \{g \in A \mid g\alpha(g) = 1\}. \end{aligned}$$

Let p be a place of \mathbb{Q} (possibly $p = \infty$). We denote the \mathbb{Q}_p -rational points of G by

$$G_p = \{g \in M_3(K) \otimes_{\mathbb{Q}} \mathbb{Q}_p \mid gH({}^t\bar{g})H^{-1} = 1\}.$$

If p splits in K as $p = \mathfrak{p}\bar{\mathfrak{p}}$ then $K_{\mathfrak{p}} \cong \mathbb{Q}_p$ and $K \otimes_{\mathbb{Q}} \mathbb{Q}_p = K_{\mathfrak{p}} \oplus K_{\bar{\mathfrak{p}}} = \mathbb{Q}_p \oplus \mathbb{Q}_p$. The natural extension of the Galois involution, acting trivially on \mathbb{Q}_p , switches the two summands of $K \otimes_{\mathbb{Q}} \mathbb{Q}_p$. Consequently:

$$\begin{aligned} A_p &\stackrel{\text{def}}{=} A \otimes_{\mathbb{Q}} \mathbb{Q}_p \\ &= A_{\mathfrak{p}} \oplus A_{\bar{\mathfrak{p}}}. \end{aligned}$$

Up to isomorphism, α induces the anti-involution $\alpha(g, h) = ({}^t h, {}^t g)$ on $A_p = A_{\mathfrak{p}} \oplus A_{\bar{\mathfrak{p}}}$. The projection into each of the summands gives an isomorphism of G_p with $GL_3(K_{\mathfrak{p}}) \cong GL_3(\mathbb{Q}_p)$. Since the discriminant of K is $d_K = -7$, the primes that split at K are exactly 2 and the odd primes $p \neq 7$ such that $(\frac{-7}{p}) = 1$ (i.e. all primes congruent to 1, 2 or 4 (mod 7)).

Since G_{∞} is a compact group, by reduction theory if p_1, p_2, \dots, p_l are rational primes all congruent to 1, 2, 4 (mod 7) then the group

$$\begin{aligned} \Gamma_1(p_1, \dots, p_l) &= G(\mathbb{Z}[1/p_1, 1/p_2, \dots, 1/p_l]) \\ &= Q(\lambda) - \text{linear maps } \gamma : V \rightarrow V \text{ which preserve the form } h \\ &\quad \text{and map } L[1/p_1, 1/p_2, \dots, 1/p_l] \text{ to itself} \end{aligned}$$

maps into a uniform lattice $\bar{\Gamma}_1 < PGL_3(\mathbb{Q}_{p_1}) \times PGL_3(\mathbb{Q}_{p_2}) \times \dots \times PGL_3(\mathbb{Q}_{p_l})$. We wish to work with subgroups of Γ_1 defined by some congruence conditions mod 7. Let

$$\pi_7 : \Gamma_1(p_1, \dots, p_l) = G(\mathbb{Z}[1/p_1, \dots, 1/p_l]) \rightarrow G\left(\frac{\mathbb{Z}[1/p_1, \dots, 1/p_l]}{7\mathbb{Z}[1/p_1, \dots, 1/p_l]}\right) = G(\mathbb{F}_7)$$

be the reduction map mod 7. The Hermitian form \bar{h} becomes degenerate of rank 1 over \mathbb{F}_7 with a null space L_1 spanned by $\zeta - 1$ and $(\zeta - 1)^2$. The action of $\Gamma_1(p_1, \dots, p_l)$ on L_1 gives a homomorphism

$$\eta : \Gamma_1(p_1, \dots, p_l) \rightarrow GL_2(\mathbb{F}_7).$$

Let $\hat{H} < GL_2(\mathbb{F}_7)$ be a 2-Sylow subgroup $H = \hat{H} \cap SL_2(\mathbb{F}_7)$ and $\Gamma(p_1, \dots, p_l) = \eta^{-1}(\hat{H})$. Mumford proves the following:

Proposition 3.5. (Mumford [Mum79]) *Using the terminology of the above paragraph the group $\Gamma(2)$ is torsion free and acts transitively on the vertices of the building of $\mathrm{PGL}_3(\mathbb{Q}_2)$.*

Note: Mumford in fact shows that $\eta(\Gamma_1(2)) = \mathrm{SL}_2(\mathbb{F}_7)$. He defines $\Gamma(2)$ as a pullback of a 2-Sylow subgroup of $\mathrm{SL}_2(\mathbb{F}_7)$. Since $\mathrm{SL}_2(\mathbb{F}_7)$ is normal, this is consistent with our definition. In the course of our proof we will see that, in fact even in our more general setting $\eta(\Gamma_1) \subset \mathrm{SL}_2(\mathbb{F}_7)$.

Proposition 3.6. *Let p_1, \dots, p_l be primes all congruent to 1, 2 or 4 (mod 7), Δ_i the building of $\mathrm{PGL}_3(\mathbb{Q}_{p_i})$, $\Delta_i^{(1)}$ its one skeleton and $Y = \Delta_1^{(1)} \times \dots \times \Delta_l^{(1)}$. Then the group $\Gamma(p_1, \dots, p_l)$ acts freely and transitively on the vertices of Y and it acts freely on the set of geometric cubes in Y .*

Proof. We proceed in several steps:

- (1) Set $\Gamma = \Gamma(p_1, p_2, \dots, p_l)$ and $\Gamma_1 = \Gamma_1(p_1, p_2, \dots, p_l)$. Assume first that 2 is one of the given primes, say $p_1 = 2$. We denote by pr the projection on all but the first factor:

$$\mathrm{pr} : \mathrm{PGL}_3(\mathbb{Q}_{p_1}) \times \dots \times \mathrm{PGL}_3(\mathbb{Q}_{p_l}) \rightarrow \mathrm{PGL}_3(\mathbb{Q}_{p_2}) \times \dots \times \mathrm{PGL}_3(\mathbb{Q}_{p_l}).$$

- (2) $\mathrm{pr}(\Gamma_1)$ is transitive on the vertices of $\Delta_2^{(1)} \times \dots \times \Delta_l^{(1)}$. In fact, by weak approximation $\mathrm{pr}(\Gamma_1)$ intersects with $\mathrm{PSL}_3(\mathbb{Q}_{p_2}) \times \dots \times \mathrm{PSL}_3(\mathbb{Q}_{p_l})$ in a dense subgroup, thus it acts transitively on each one of the 3^{l-1} different vertex colors. It remains only to exhibit, for each $p \in \{p_2, p_3, \dots, p_l\}$, an element $\rho_p \in \Gamma_1$ with

$$\mathrm{val}_q(\det(g)) \begin{cases} \neq 0 \pmod{3} & q = p \\ = 0 \pmod{3} & p \neq q \end{cases}$$

By assumption $p = \mathfrak{P}\bar{\mathfrak{P}}$ splits in K . Since \mathcal{O} is a UFD we may assume that $\mathfrak{P} = (P) = (s + t\sqrt{-7})$ and $\bar{\mathfrak{P}} = (\bar{P}) = (s - t\sqrt{-7})$ are principal ideals. We fix the notation in such a way that in $\mathbb{Q}_p = K_{\mathfrak{P}}$ we have $P = (\text{unit}) \cdot p$ and $\bar{P} = (\text{unit})$. Define:

$$(3.1) \quad \rho_p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{2}{\bar{P}} \frac{P-\bar{P}}{\sqrt{-7}} & -\frac{\lambda^2}{\bar{P}} \frac{P-\bar{P}}{\sqrt{-7}} & \frac{P}{\bar{P}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{4t}{\bar{P}} & \frac{-2t\lambda^2}{\bar{P}} & \frac{P}{\bar{P}} \end{pmatrix}$$

- (3) **Identifying $\Gamma(2)$ in Γ .** Let $O = (O_1, O_2, \dots, O_l) \in \Delta = \Delta_1 \times \dots \times \Delta_l$ be a base vertex, we choose the notation so that $O_i \in \Delta_i$ is the vertex whose stabilizer is $\mathrm{PGL}_3(\mathbb{Z}_{p_i}) = \mathrm{PGL}_3(\mathcal{O}_{\mathfrak{P}_i})$. We claim that

$$\Gamma_1(2) = \Gamma_1 \cap [\mathrm{GL}_3(\mathbb{Q}_{p_1}) \times \mathrm{GL}_3(\mathbb{Z}_{p_2}) \times \dots \times \mathrm{GL}_3(\mathbb{Z}_{p_l})].$$

In fact, any element of the right hand side automatically has all its matrix elements in $\mathbb{Z}[\lambda][1/2, 1/\overline{P}_2, \dots, 1/\overline{P}_l]$. Since ${}^t\overline{g} = H^{-1}g^{-1}H$, the matrix coefficients of ${}^t\overline{g}$ are also in the same ring so that in fact all matrix coefficients are in $\mathbb{Z}[\lambda][1/2]$. The other inclusion is clear.

We can therefore identify $\Gamma_1(2) = (\Gamma_1)_{(O_2, O_3, \dots, O_l)}$ where the action of Γ_1 on the right hand side is via the projection pr . The action $(\Gamma_1)_{(O_2, O_3, \dots, O_l)} \rightsquigarrow \Delta_1 \times \{(O_2, O_3, \dots, O_l)\}$ is isomorphic to the action $\Gamma_1(2) \rightsquigarrow \Delta_1$. Furthermore reduction mod 7 is compatible with the embedding of $\Gamma_1(2) \hookrightarrow \Gamma_1$: restricting η to $(\Gamma_1)_{(O_2, O_3, \dots, O_l)}$ gives the reduction map mod 7 on $\Gamma_1(2)$. From here on we identify $\Gamma_1(2)$ with a subgroup of Γ_1 and $\Gamma(2)$ with a subgroup of Γ .

- (4) $\eta(\Gamma_1) \subset \text{SL}_2(\mathbb{F}_7)$. We will in fact prove $\Gamma_1 = \langle \Gamma_1(2), \rho_{p_1}, \rho_{p_2}, \dots, \rho_{p_l} \rangle$ where ρ_{p_i} are the group elements given in equation 3.1. This will complete the proof because it is already proved by Mumford that $\eta(\Gamma_1(2)) \subset \text{SL}_2(\mathbb{F}_7)$, and for the elements ρ_p it can easily be verified¹ that in the basis $1, \zeta - 1, (\zeta - 1)^2$:

$$(3.2) \quad \eta(\rho_p) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{5}{\overline{P}} \frac{P - \overline{P}}{\sqrt{-7}} & 1 \end{pmatrix}.$$

Set $\Gamma'_1 = \langle \Gamma_1(2), \rho_{p_1}, \rho_{p_2}, \dots, \rho_{p_l} \rangle$, both Γ_1 and Γ'_1 have the same vertex stabilizers acting on the product $\Delta_2 \times \Delta_3 \times \dots \times \Delta_l$; namely, $(\Gamma'_1)_{(O_2, O_3, \dots, O_l)} = (\Gamma_1)_{(O_2, O_3, \dots, O_l)} = \Gamma_1(2)$. In order to show that they are isomorphic we only have to show that Γ'_1 acts transitively on the vertices of $\Delta_2 \times \dots \times \Delta_l$. The vertex $(O_2, O_3, \dots, O_l) \in \Delta_2 \times \dots \times \Delta_l$ has 2^{l-1} different types of vertices among its nearest neighbors, and we can reach at least one representative of each of these types using the elements $\rho_2, \rho_3, \dots, \rho_l$ and their adjoints. By weak approximation theorem $\Gamma_1(2)$ acts transitively on the nearest neighbors of the same type, so we can reach all the nearest neighbors of (O_2, O_3, \dots, O_l) . This suffices to prove transitivity.

- (5) Γ_1 **acts transitively on the vertices of Y** . Let $V = (V_1, V_2, \dots, V_l)$ be a vertex of Y . By (2) there is an element $\gamma \in \Gamma_1$ such that $\gamma V_i = O_i \forall 2 \leq i \leq l$. By Mumfords' theorem (proposition 3.5 above) and by the identification $\Gamma_1(2) = (\Gamma_1)_{(O_2, \dots, O_l)}$ we can find an element $\delta \in (\Gamma_1)_{(O_2, \dots, O_l)}$ such that $\delta \gamma V = O$.

¹In Mumfords notation [Mum79] the matrix is transposed to this one

- (6) **Γ acts freely.** To prove that Γ acts freely on the geometric cubes it is enough to show that Γ acts freely on the vertices, because even the ambient group $\mathrm{PGL}_3(\mathbb{Q}_p)$ does not invert edges in $\Delta_p^{(1)}$ (note that this is no longer true for 2-simplexes in Δ_p).

By Mumfords' theorem (proposition 3.5) the Γ stabilizer of O is trivial:

$$\begin{aligned}\Gamma_O &= (\Gamma_{(O_2, O_3, \dots, O_l)})_{O_1} \\ &= (\Gamma(2))_{O_1} \\ &= \langle e \rangle\end{aligned}$$

Since Mumfords' proposition is true for any choice of a 2-Sylow subgroup we know even more: $\eta((\Gamma_1)_O)$ intersects trivially any 2-Sylow subgroup of \hat{G} . This together with the transitivity of the action of Γ_1 (proposition 5) proves that every vertex stabilizer is trivial. In fact, given another vertex V , there is a $\gamma \in \Gamma_1$ such that $\gamma O = V$, and by our observation above $\eta((\Gamma_1)_V) = \eta(g)\eta((\Gamma_1)_O)\eta(g^{-1})$ intersects trivially with \hat{H} .

- (7) **Γ acts transitively.** The number of orbits for the action of Γ on the vertices of Y is exactly $\frac{[\Gamma_1 : \Gamma]}{|(\Gamma_1)_O|}$. Similarly, by Mumfords' proposition, we have $\frac{[\Gamma_1(2) : \Gamma(2)]}{|(\Gamma_1(2))_{O_1}|} = 1$. Dividing and using the equality of the denominators we obtain:

$$\begin{aligned}\# \text{ of orbits} &= \frac{[\Gamma_1 : \Gamma]}{[\Gamma_1(2) : \Gamma(2)]} \\ &= \frac{[\eta(\Gamma_1) : \eta(\Gamma)]}{[\eta(\Gamma_1(2)) : \eta(\Gamma(2))]} \\ &= \frac{[\mathrm{SL}(\mathbb{F}_7) : H]}{[\mathrm{SL}_2(\mathbb{F}_7) : H]} = 1\end{aligned}$$

Where in the third equality we have used the fact (see [Mum79]) that $\eta(\Gamma_1(2)) = \mathrm{SL}_2(\mathbb{F}_7)$ and consequently $\eta(\Gamma(2)) = H$, 2-Sylow subgroup of $\mathrm{SL}_2(\mathbb{F}_7)$.

- (8) **The general case.** If $2 \notin \{p_1, \dots, p_l\}$ then we can add it artificially by setting $p_0 = 2$. Now by identifying the two actions: $\Gamma(p_1, p_2, \dots, p_l) \rightsquigarrow \Delta_1 \times \Delta_2 \times \dots \times \Delta_l$ and $(\Gamma(p_0, p_1, \dots, p_l))_{O_0} \rightsquigarrow \{O_0\} \times \Delta_1 \times \Delta_2 \times \dots \times \Delta_l$ we deduce that the latter action satisfies the desired properties. This completes the proof of proposition 3.6

□

3.3.2. *A Kazhdan automaton.* The example below is a particular example of a lattice acting on a product of trees, coming from an arithmetic

lattice in higher rank p -adic groups. These lattices play an important role in the classification of lattices in automorphism groups of products of trees that admit infinite, finite-dimensional representations in characteristic 0 [BMZ]. A similar construction appears also in [BM00].

Proposition 3.7. *Let $\Gamma < \mathrm{PGL}_3(\mathbb{Q}_p) \times \mathrm{PGL}_3(\mathbb{Q}_q)$ be an irreducible lattice that acts vertex transitively and freely on the (geometric cells of the) product of the one skeletons of the buildings $Y = \Delta_p^{(1)} \times \Delta_q^{(1)}$. Then $\Sigma = \Gamma \backslash Y$ is a VH-T-square complex with one vertex and $G(A^T(\Sigma)) = \Gamma_{O_p}$ is a lattice in $\mathrm{PGL}_3(\mathbb{Q}_q)$. In particular it has property T.*

Note: The situation is symmetric, so by changing p and q we can show that $G(A^T(\Sigma)^*) = \Gamma_{O_q}$ is a lattice in $\mathrm{PGL}_3(\mathbb{Q}_p)$ and has property T as well.

Proof. It is clear that Σ is a VH-T-complex and therefore its universal cover is a product of trees $\tilde{\Sigma} = \tilde{Y} = T_h \times T_v$, where T_h is a $2(p^2 + p + 1)$ -regular tree and T_v is a $2(q^2 + q + 1)$ -regular tree. In fact we obtain the following tower of regular coverings:

$$(T_h \times T_v, \overline{O}) \rightarrow (Y, \tilde{O}) \rightarrow (\Sigma, O)$$

Here $\overline{O} = (\overline{O}_1, \overline{O}_2)$ and $\tilde{O} = (\tilde{O}_1, \tilde{O}_2)$ are base vertices. The covering groups fit into a short exact sequence:

$$1 \rightarrow \pi_1(Y, \tilde{O}) \rightarrow \Theta \stackrel{\text{def}}{=} \pi_1(\Sigma, O) \xrightarrow{p} \Gamma \rightarrow 1,$$

where we have defined $\Theta \stackrel{\text{def}}{=} \pi_1(\Sigma, O)$.

Let us denote by $\mathrm{pr}_1 : \mathrm{Aut}(T_h) \times \mathrm{Aut}(T_v) \rightarrow \mathrm{Aut}(T_h)$ the projection map that forgets the action on the T_v coordinate. By abuse of notation we will denote by the same notation also the projection $\mathrm{pr}_1 : \mathrm{Aut}(\Delta_p^{(1)}) \times \mathrm{Aut}(\Delta_q^{(1)}) \rightarrow \mathrm{Aut}(\Delta_p^{(1)})$. By proposition (2.13) we are interested in the group $(\mathrm{pr}_1(\Theta))_{\overline{O}_1}$. First, let us note the following

Lemma 3.8.

$$\ker(\mathrm{pr}_1) \cap \Theta = \pi_1(\Delta_1^{(1)}, \tilde{O}_2)$$

Proof. Indeed assume that $\theta \in \ker(\mathrm{pr}_1) \cap \Theta$, then $p(\theta) = (e, \rho) \in \ker(\mathrm{pr}_1) \cap \Gamma$. By weak approximation one can find elements $\gamma = (\gamma_1, \gamma_2) \in \Gamma$ with γ_2 arbitrarily close, but not equal to ρ and therefore:

$$[\gamma, p(\theta)] = [(1, [\gamma_2, \rho])] \xrightarrow{\gamma_2 \rightarrow \rho} e$$

Since Γ is discrete this implies that $p(\theta) = e$, and $\rho \in \pi_1(\Delta_1^{(1)}, \tilde{O}_2)$. Since the opposite inclusion is obvious this concludes the proof of the lemma. \square

Since $\pi_1(\Delta_p^{(1)})$ acts freely on T_h we have:

$$\begin{aligned} (\text{pr}_1(\Theta))_{\overline{O_1}} &= (p(\text{pr}_1(\Theta)))_{\tilde{O}_1} \\ &= p\left(\frac{\Theta}{\pi_1(\Delta_p^{(1)}, \tilde{O}_1)}\right)_{\tilde{O}_1} \\ &= \left(\frac{\Theta}{\pi_1(\Delta_p^{(1)}, \tilde{O}_1)\pi_1(\Delta_q^{(1)}, \tilde{O}_2)}\right)_{\tilde{O}_1} \\ &= \Gamma_{\tilde{O}_1} \end{aligned}$$

completing the proof of proposition 3.7. \square

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