A note on doubles of groups

Nadia Benakli [*] Oliver T. Dasbach [†‡] Yair Glasner [§] Brian Mangum [¶]

Abstract

Recently, Rips produced an example of a double of two free groups which has unsolvable generalized word problem. In this paper, we show that Rips's example fits into a large class of doubles of groups, each member of which contains $F_2 \times F_2$ and therefore has unsolvable generalized word problem and is incoherent.

By a result of Mihailova [Mih66] every group which contains the direct product of two (non-abelian) free groups does not admit a solution to the generalized word problem. Recently Rips showed [Rip96] - without using Mihailova's result - that the double of two free groups of rank 2 over a certain normal subgroup of index 3 does not have a generalized word problem solution. It was, however, already shown by Gersten [Ger81] that such a group must contain $F_2 \times F_2$, and thus Rips's result follows from Gersten's and Mihailova's.

The purpose of this paper is to clarify the situation. We will show that Rips's example, and in fact any double of a virtually free finitely generated group over a finite index subgroup, is virtually the direct product of two free groups. Thus Rips's proof yields a new proof for Mihailova's result. Furthermore we will show that a more general construction of a double of a group must always contain $F_2 \times F_2$.

One might naively think that every finitely presented group which admits a solution to the word problem but which is neither coherent nor does it admit a solution to the generalized word problem must contain $F_2 \times F_2$.

A counterexample follows from a construction of Rips [Rip82]. He constructed for every finitely presented group G a short exact sequence

$$1 \longrightarrow K \longrightarrow H \longrightarrow G \longrightarrow 1$$

so that H is a finitely presented word hyperbolic group, and K is - as a group - finitely generated. (In fact, Rips's construction yields for every λ a finitely presented group H satisfying the small cancellation condition $C'(\lambda)$. In particular, λ can be chosen so that H is word hyperbolic.)

It follows that if either G is not coherent or G does not admit a solution to the generalized word problem, then the same is true for H.

Since H is word hyperbolic, it has a solution to the word problem but cannot contain $F_2 \times F_2$ or even $\mathbb{Z} \times \mathbb{Z}$.

[*] e-mail: *benakli@math.columbia.edu*
[†] The work of the second author was supported by the Deutsche Forschungsgemeinschaft (DFG)
[‡] e-mail: *kasten@math.columbia.edu*, *http://www.math.uni-duesseldorf.de/home/kasten*
[§] e-mail: *yairgl@math.huji.ac.il*
[¶] e-mail: *mangum@math.columbia.edu*

Rips's construction starts with a group G that is already known to be incoherent and which does not admit a solution to the generalized word problem. Very recently, however, Wise [Wis98] gave a construction of an incoherent word hyperbolic group without referring to a group which contains $F_2 \times F_2$.

Definition 1.1. *For a group G one defines the following properties:*

1. *The group G is coherent if every finitely generated subgroup is finitely presented.*

2. *It is LERF (locally extended residually finite) or subgroup separable if for every finitely generated subgroup H of G and every $g \in G$, $g \notin H$ there is a subgroup of finite index in G that contains H but does not contain g. In particular, a LERF group is residually finite.*

3. *The group G admits a solution to the generalized word problem or occurrence problem if there is an algorithm that decides for every element $g \in G$ and every finitely generated subgroup $H \subset G$ whether g is in H or not.*

As for a finitely presented group G residual finiteness implies that the word problem is solvable, it is easy to see that if G is LERF then the generalized word problem for G is solvable.

Moreover, it follows more or less directly from the definition that if a group G has one of these properties then so does every subgroup and every finite extension of G.

An important obstacle for a group to be coherent, LERF or to admit a solution to the generalized word problem is given by

Proposition 1.2. *The group $F_2 \times F_2$ is neither coherent ([Gru78]) nor does it admit a solution to the generalized word problem (Mihailova, see [Mih66, MI71] and compare with [Gru78]) and is thus not LERF. Therefore every group which contains $F_2 \times F_2$ has also none of these properties.*

Let G be a group and H a subgroup. By the double of G over H we mean the group that we get by the free product of G with itself amalgamated over H. Here we will show that a large family of group doubles contain $F_2 \times F_2$ as a subgroup.

Lemma 1.3. *Let G be a group and $H < G$. Let $L = G *_H G$ be the double of G over H. Let $\phi_1 : L \to G$ be the homomorphism obtained by identifying the two factors of L. Then $K = ker(\phi_1)$ is a free group. If H is of finite index in G then the rank of K is $[G : H] - 1$.*

Proof. The group L is an amalgam of groups and has a natural action on its Bass-Serre tree T (which is locally finite iff $[G : H] < \infty$). By definition K intersects trivially the two copies of G in L, and since K is normal, it intersects trivially all their conjugates, which are all the vertex stabilizers. We conclude that K acts freely and effectively on T. (Note that L does not necessarily act effectively on T.) Therefore $K \cong \pi_1(K\backslash T, \{pt\})$ is a free group (see also [Ser80]). Now ϕ_1 is a surjective homomorphism and therefore the index of K in L is $[L : K] = \#\,|G|$ (note that G might be infinite). The graph $K\backslash T$ is therefore a $\#\,|G|\,-$fold cover of the edge of groups corresponding to the amalgam L (in the sense of Bass [Bas93]). The only possibility of such a covering graph is shown in Figure 1. The rank of K as a free group is now easily verified as K is the fundamental group of the covering graph in Figure 1. $\square$

Figure 1: The covering of graphs corresponding to $K \hookrightarrow L$

The idea for the proof of the following theorem was inspired by the notion of reducible lattices acting on products of trees, defined by Burger, Mozes and Zimmer in ([BMZ, BM97])

Theorem 1.4. *Let G be a group, $H < G$ be a subgroup with $3 \leq [G : H] \leq \infty$.*

1. *Assume that there exists a subgroup $N < H$ with the following properties.*

   (a) N contains a non-abelian, free subgroup.

   (b) N is normal in G.

   In particular, this assumption is fulfilled if G contains a non-abelian free group and H has finite index in G. Then the double $L \stackrel{\text{def}}{=} G *_H G$ contains $F_2 \times F_2$ as a subgroup and is thus neither coherent nor LERF nor does it admit a solution to the generalized word problem.

2. *Assume G is a finitely generated virtually (non-abelian) free group, and H has finite index in G. The double $G *_H G$ is virtually the direct product $F_{r_1} \times F_{r_2}$ of two non-abelian free groups of finite rank. Moreover, $F_2 \times F_2$ and $G *_H G$ are commensurable.*

Proof. The group N is normal in both copies of G in L and is therefore normal in L. We consider the following homomorphism of L into a product:

$$\phi = (\phi_1, \phi_2) : L \to G \times L/N \tag{1.1}$$

Here ϕ_1 is induced by identifying the two factors of L, and ϕ_2 is just the projection to the quotient. First notice that $ker(\phi_2) = N$ and that N, as a subgroup of G, injects into G under ϕ_1. Therefore $\ker(\phi) = \ker(\phi_1) \cap \ker(\phi_2)$ is trivial. So ϕ is injective and it will be enough to prove that $\phi(L)$ contains a copy of $F_2 \times F_2$. Define $K_i \stackrel{\text{def}}{=} \phi_i(\ker(\phi_{3-i})), i = 1, 2$, and look at the embedding $K_1 \times K_2 \hookrightarrow \phi(L)$. We will show that each one of the K_i contains a non-abelian free group and thus conclude the proof of Part 1. The group $K_1 = N$ contains a non-abelian free group by assumption. In order to deal with the other factor, consider the map $\overline{\phi_1} : G/N *_{H/N} G/N \to G/N$ (the homomorphism obtained by identifying the two copies of G/N). The fact that $ker(\phi_1) \cap ker(\phi_2) = \langle e \rangle$ and the commutativity of the following diagram:

$$
\begin{array}{ccc}
L \cong G *_H G & \xrightarrow{\phi_2} & L/N \cong G/N *_{H/N} G/N \\
\phi_1 \Big\downarrow & & \overline{\phi_1} \Big\downarrow \\
G & \longrightarrow & G/N
\end{array}
$$

imply that $K_2 = \ker(\overline{\phi_1})$. Now, by the Lemma 1.3, K_2 is a free group of rank

$$[G/N : H/N] - 1 = [G : H] - 1.$$

The assumption that this number is at least 2 shows that K_2 is non-abelian.

In the case where G is a virtually non-abelian free group, $G *_H G$ contains $K_1 \times K_2$ as a subgroup of finite index where K_1 is virtually free and K_2 is free. Any non-abelian free group of finite rank is isomorphic to a subgroup of finite index in F_2. Therefore, $G *_H G$ and $F_2 \times F_2$ are commensurable. This concludes the theorem. $\square$

Remark: In this theorem, if we take G to be F_2 and take H to be the normal subgroup in Rips's example ([Rip96]), then the double $G *_H G$ and $F_2 \times F_2$ are shown to be commensurable. Rips has shown that $G *_H G$ has no solution to the generalized word problem. Therefore, $F_2 \times F_2$ has the same property. This gives a new proof of Mihailova's result.

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Addresses:

Columbia University, Department of Mathematics, New York, NY 10027

Columbia University, Department of Mathematics, New York, NY 10027 and
Heinrich-Heine-Universität, Mathematisches Institut, Abt. M.+M.D., D-40225 Düsseldorf

Institute of Mathematics, The Hebrew University, Jerusalem 91904, Israel

Barnard College, Columbia University, Department of Mathematics, New York, NY 10027

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