Ramanujan graphs with small girth.

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Abstract: We construct an infinite family of $(q+1)-$regular Ramanujan graphs X_n of girth 1. We also give covering maps $X_{n+1} \to X_n$ such that the minimal common covering of all the X_n's is the universal covering tree.

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1 Introduction

Ramanujan graphs were first introduced by Lubotzky Phillips and Sarnak (LPS) in [LPS88] as graphs satisfying the "asymptotically optimal" bound on the size of the second eigenvalue. A $(q+1)-$regular graph X is called *Ramanujan* if $|\lambda| \leq 2\sqrt{q}$ for every non-trivial (i.e. $\neq \pm(q+1)$) eigenvalue λ of its adjacency matrix. By saying that the bound $2\sqrt{q}$ is *asymptotically optimal* we mean that trying to impose any lower bound gives rise only to finite families of graphs; this is the content of Alon-Boppana theorem [Nil91]. It is an open question weather there exist infinite families of $(q+1)-$regular Ramanujan graphs for a general number q. Explicit constructions in the case where q is prime were given by Lubotzky Phillips and Sarnak (LPS graphs), these were later extended to the case where q is a prime power ([Mor94, JL97]) [1]. It is not difficult to describe these examples, but the proof of the Ramanujan property relies on very deep theorems from number theory.

LPS graphs exhibit many interesting combinatorial properties, some of these are a direct consequence of the Ramanujan property and others are independent of the spectral properties of the graph. Two examples are:

- It is a direct consequence of the Ramanujan property that LPS graphs are good expanders.
- It can be proved in an elementary way, independent of the Ramanujan property, that LPS graphs have very large girth. In fact the bi-partite LPS graphs satisfy $\text{girth}(X) \geq \frac{4}{3}\log(|X|)$.

Lubotzky, in his book [Lub94, Question 10.7.1], poses the question of clarifying the connection between the Ramanujan property and the girth. There are some theorems showing a correlation between the eigenvalue distribution and the existence of small circuits, but they are all rather weak. For example Greenberg (see [Gre95],[Lub94, theorem 4.5.7]) proves that an infinite family of Ramanujan graphs $\ldots X_n \to X_{n-1} \to \ldots \to X_1$ covering each other, with $X_n \to X_1$ a regular covering map, must satisfy $\text{girth}(X_n) \to \infty$ (In other words an example like the one given in theorem (1.2) below is not possible when the covering maps $X_n \to X_1$ are assumed to be regular). In the other direction McKay (see [McK81]) shows that infinite families of $(q+1)-$regular

[1] It should be emphasized that even non constructive methods or methods of probabilistic nature for proving existence of Ramanujan graphs are not known. The best known results so far are due to Joel Friedman ([Fri91]) who shows that the non-trivial eigenvalues of a random $(q+1)-$regular graph with n vertices satisfy $|\lambda| \leq 2\sqrt{q} + \log(q) + C$ almost surely (i.e. with a probability that tends to 1 with n). The bound here is strictly larger than the Ramanujan bound and independent of n.

graphs with asymptotically few circuits have most of their eigenvalues concentrated in the Ramanujan interval $[-2\sqrt{q}, 2\sqrt{q}]$.

In this paper we give the following example, proving that the Ramanujan property does not imply large girth.

Definition 1.1. *An infinite sequence of graphs covering each other* $\ldots \to X_{n+1} \to X_n \to X_{n-1} \to \ldots \to X_1$ *will be called a tower of graphs.*

Theorem 1.2. *There exists a tower of regular Ramanujan graphs* X_n *satisfying the following properties:*

1. *There is a common bound* M *on the girth of all the graphs.*

2. *The minimal common covering of all graphs is the universal covering tree.*

We also give an explicit description of one such family of graphs, similar to the explicit constructions given by Lubotzky Phillips and Sarnak in ([LPS88]). In fact in section (4) we show that our graphs can be realized as families of Schreier graphs of $PSL_2(\mathbb{Z}/q^n\mathbb{Z})$ (or closely related groups) with respect to some Cartan subgroups. The Ramanujan graphs obtained in these examples have girth 1, (i.e. they all contain loops).

There are good reasons to expect that examples such as the one given in theorem 1.2 should exist. Indeed, the property of being a Ramanujan graph is only asymptotically optimal. It is not difficult to find small graphs, satisfying better bounds on the second eigenvalue. Given such a "better than Ramanujan" graph we can make a local change to the graph: creating a small circuit while introducing only minor changes in the eigenvalues and retaining the Ramanujan property. To obtain an infinite family of Ramanujan graphs with small girth, we have to start with an infinite family of better then Ramanujan graphs with some precise estimates on their second eigenvalues, which seems very difficult. We do not take this approach, instead we go back and introduce a minor change in the construction of LPS graphs and this yields the desired family of Ramanujan graphs with small girth.

If theorem 1.2(2) does not hold then 1.2(1) automatically does hold because any closed path in the common covering of all Ramanujan graphs will appear in each one of them. This happens exactly when the fundamental groups of our graphs have a non-trivial intersection. In [LPS88] Ramanujan graphs are constructed as quotients of the Bruhat-Tits tree T of $PGL_2(\mathbb{Q}_q)$ by torsion free congruence subgroups of a $\{q\}$-arithmetic lattice $\Gamma < PGL_2(\mathbb{Q}_q)$. It is customary to use principal congruence subgroups but, as the intersection of an infinite family of such is always trivial, we replace them by an infinite family of non-principal congruence subgroups. In order to achieve a minimal common covering which is a tree (1.2(2)), we change the covering morphisms $X_{n+1} \to X_n$.

Remark: We start from the classical construction of Ramanujan graphs due to Lubotzky Phillips and Sarnak and modify it slightly. This construction involves definite quaternion algebras defined over $\mathbb{Q}$, and is described in [Lub94] and in section (2.4). This approach has the advantage of simplifying the presentation and the disadvantage of yielding only $(q+1)-$regular graphs, where q is prime. The regularity restriction is not really necessary, our method will work also for variants of the LPS construction which yield $(q+1)-$regular Ramanujan graphs for every prime power q. For example Morgenstern's construction in positive characteristic [Mor94] or the construction of Jordan Livné involving totally definite quaternion algebras over number fields [JL97].

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2 Review

2.1 Covering theory for graphs

We define a *graph* X to be a set of *vertices* VX, a set of *edges* EX together with a fixed-point-free involution $^- : EX \to EX$ (associating with every edge e an edge $\bar{e}$ called its inverse) and two maps called *the origin and terminus maps* $o, t : EX \to VX$ satisfying $o\bar{e} = te, \quad t\bar{e} = oe$. We think of our graphs as non-directed but we represent each *geometric edge* $[e]$ by a pair of directed edges $\{e, \bar{e}\}$. All our graphs might contain loops or multiple edges. All the graphs in this paper are assumed connected. Notions like graph *morphisms, automorphisms, etc ...* are all defined in the obvious way. All automorphisms of a given graph X form a group denoted by $\mathrm{Aut}(X)$.

Let G be a group acting (on the left, by graph automorphisms) on a graph X. We say that the action is *without inversion* if no element of G takes an edge to its inverse. Whenever G acts without inversion on a graph X there exists a well defined *quotient graph* denoted $G\backslash X$ and a *quotient morphism* $p : X \to G\backslash X$. To see this one should check that the structure maps $\{^-, o, t\}$ of the graph X induce well defined structure maps on the quotient sets $G\backslash VX$ and $G\backslash EX$. We say that a group G *acts* *freely* on the graph X, if all vertex stabilizers are trivial.

A *link of a vertex* $v \in VX$ is the set $\mathrm{Lk}(v) \overset{\mathrm{def}}{=} \{e \in EX | oe = v\}$. A graph morphism $\phi : X \to Y$ induces a map $\phi_v : \mathrm{Lk}(v) \to \mathrm{Lk}(\phi(v))$ $\forall v \in VX$. A *covering map of graphs* $\phi : X \to Y$ is, by definition, a (surjective) map that induces a bijection on every vertex link. If a group G acts freely and without inversion on a graph X, the quotient map $p : X \to G\backslash X$ is a covering map. Covering maps obtained in this way are called *regular*. Every graph is regularly covered by a tree:

Theorem 2.1. *For every graph Y there exists a unique regular covering map $\pi : \tilde{Y} \to Y$ with $\tilde{Y}$ a tree.*

The tree $\tilde{Y}$ is called *the universal covering tree of* Y. The group that acts freely and without inversion to yield Y as a quotient is denoted by $\pi_1(Y, \cdot)$, it is called *the fundamental group of* Y. The uniqueness statement in the theorem means that the pair $(\pi_1(Y, \cdot), \tilde{Y})$ is uniquely determined by the graph Y (up to a naturally defined notion of isomorphism of group actions on graphs). It turns out that fundamental groups of graphs are always free groups.

If a group G acts freely and without inversion on a graph X with a quotient graph $Y = G\backslash X$, and if $H < G$ is a subgroup there is a natural covering map of graphs $X \to H\backslash X \to G\backslash X = Y$. It turns out that if we take $X = \tilde{Y}$ and $G = \pi_1(Y, \cdot)$ all possible covering graphs of Y are obtained in this way:

Theorem 2.2. *There is a bijective correspondence, called the Galois correspondence, between the subgroups of $\pi_1(Y, \cdot)$ and intermediate covering graphs $\tilde{Y} \to Z \to Y$. The Galois correspondence associates with a subgroup H of $\pi_1(Y, \cdot)$ the intermediate covering:*

$$\tilde{Y} \to H\backslash\tilde{Y} \to \left(Y = \pi_1(Y, \cdot)\backslash\tilde{Y}\right). \tag{2.1}$$

Inclusion of subgroups $H_2 < H_1 < \pi_1(Y, \cdot)$ is transformed to covering of graphs $H_2\backslash\tilde{Y} \to H_1\backslash\tilde{Y}$. Normal subgroups of $\pi_1(Y, \cdot)$ correspond to regular coverings of Y. Explicitly: if $N \lhd \pi_1(Y, \cdot)$ the action of $\pi_1(Y, \cdot)$ on $\tilde{Y}$ induces a free action without inversion of $\pi_1(Y, \cdot)/N$ on $N\backslash\tilde{Y}$; and there is a natural isomorphism $(\pi_1(X, \cdot)/N)\backslash(N\backslash\tilde{X}) \cong Y$.

The $(q + 1)-$regular graphs are exactly the graphs whose universal covering tree is $T = T_{q+1}$, the $(q + 1)-$regular tree. Let $q + 1 = 2n$ and let X be the graph with one vertex and n edges, such a graph is sometimes called a *wedge of n circles*. The Galois correspondence can be described very explicitly for the graph X:

Example 2.3.

- *Let Γ be a free group on n letters, $S = \{\gamma_1, \gamma_2, \ldots, \gamma_n, \gamma_1^{-1}, \gamma_2^{-1}, \ldots, \gamma_n^{-1}\} \subset \Gamma$ a set of free generators and their inverses, and $T = X(\Gamma, S)$ the right Cayley graph of Γ with respect to S. $T = T_{2n}$ is the $2n-$regular tree and Γ acts freely transitively and without inversion on T, thus $X = \Gamma\backslash T$ is a wedge of n circles and the pair (Γ, T) can be identified with $\left(\pi_1(X, \cdot), \tilde{X}\right)$.*

- *If $N \triangleleft \Gamma$, the graph $N\backslash T$ can be identified with the (right) Cayley graph of Γ/N with respect to the symmetric set of generators $\{\overline{\gamma_1}, \overline{\gamma_2}, \ldots, \overline{\gamma_n}^{-1}\}$*

- *If $N < H < \Gamma$ are subgroups with $N \triangleleft \Gamma$, the graph $H\backslash T$ can be identified with the Schreier graph of the group pair $(\Gamma/N, H/N)$ with respect to the same set of generators. Explicitly: we can identify the vertices of $H\backslash T$ with the right cosets of (H/N) in (Γ/N). The directed edges will be of the form $\{(\gamma N, \gamma\gamma_i^{\pm 1} N)\}_{\gamma \in \Gamma, \gamma_i^{\pm 1} \in S}$.*

2.2 Classification of tree automorphisms

Theorem 2.4. *(See [Ser80]) Let T be a regular tree, d the standard metric on VT, $\sigma \in Aut(T)$, $l = l(\sigma) = \min\{d(v, \sigma v) : v \in VT\}$ and $X = X(\sigma) = \{v \in VT | d(v, \sigma v) = l(\sigma)\}$, then exactly one of the following three possibilities hold:*

1. *σ is an inversion: There exists an edge $e \in ET$ with $\sigma e = \overline{e}$. in this case e is uniquely determined, $l = 1$ and X consists of the two points $\{oe, te\}$.*

2. *σ is elliptic: $l = 0$ and X is a convex subset of T consisting of all the points fixed by σ.*

3. *σ is hyperbolic: $l \geq 0$ and X is a bi-infinite line on which σ acts by a translation of length l.*

2.3 Bruhat-Tits theory

A rich source of examples for group actions on trees comes from Bruhat-Tits theory. The basic tool is a natural action of the group $G = PGL_2(\mathbb{Q}_q)$ on the $(q+1)-$regular tree $T = T_{q+1}$ (an excellent exposition of this action can be found in Serre's book [Ser80]). We will use the following properties of this action:

1. The subgroup $K = PGL_2(\mathbb{Z}_q)$ is the stabilizer of a vertex $O \in VT$. G acts transitively on the vertices and on the directed edges of T. In particular all vertex stabilizers are conjugate to K.

2. The stabilizer of the sphere $S_T(O,n)$ is the group $K(q^n) \overset{\text{def}}{=} ker\{PGL_2(\mathbb{Z}_q) \overset{\psi_n}{\to} PGL_2(\mathbb{Z}/q^n\mathbb{Z})\}$. Thus the action of K on $S_T(O,n)$ factors through an action of $PGL_2(\mathbb{Z}/q^n\mathbb{Z})$ on $S_T(O,n)$. It turns out that this action can be identified with the action of $PGL_2(\mathbb{Z}/q^n\mathbb{Z})$ on the projective line $\mathbb{P}^1(\mathbb{Z}/q^n\mathbb{Z})$ by Möbius transformation. Consequently, the action of $PGL_2(\mathbb{Z}/q^n\mathbb{Z})$ on $S_T(O,n)$ is transitive and the point stabilizers are all conjugate to the (Borel) subgroup of upper triangular matrices. Since the action of $PGL_2(\mathbb{Z}/q^n\mathbb{Z})$ on $S_T(O,n)$ comes from an action on the ball $B_T(O,n)$ it is impossible for this action to be $2-$transitive, it is transitive however on the pairs of points $\{(x,y) \in S_T(O,n) \times S_T(O,n) |$ The path [x,y] contains the point $O\}$ and the stabilizers of such pairs are conjugate to the diagonal (Cartan) subgroup [2].

3. The set of infinite rays emerging from O, is called the *boundary of the tree* and the group K acts transitively on it. In fact this action on the boundary can be identified with the action of $PGL_2(\mathbb{Z}_q)$ on the projective line $\mathbb{P}^1(\mathbb{Z}_q) \cong \mathbb{P}^1(\mathbb{Q}_q)$ by Möbius transformations [3].

2.4 The construction of LPS graphs

By theorem (2.2) $(q+1)-$regular graphs are equivalent to groups Γ acting freely and with finitely many orbits on the $(q+1)-$regular tree T_{q+1}. A wider class of groups is the class of *uniform lattices* namely groups acting with a finite number of orbits and with finite vertex stabilizers. One method for constructing uniform lattices inside $PGL_2(\mathbb{Q}_q)$ is the arithmetic construction due to Borel and Harish-Chandra (see [BHC62, BHC61]). Uniform lattices constructed in this way are referred to as $\{q\}-$arithmetic lattices or sometimes just as arithmetic lattices. The extensive knowledge available on arithmetic lattices and their properties has enabled Lubotzky Phillips and Sarnak ([LPS88]) to prove that $\Gamma(n)\backslash T$ are Ramanujan graphs for certain infinite families of arithmetic lattices $\{\Gamma(n)\}_{n\in\mathbb{N}}$. We proceed to describe these lattices.

Let $\mathbb{H} = \mathbb{H}_{u,v}$ be a *quaternion algebra* defined over $\mathbb{Q}$ (i.e. $u, v \in \mathbb{Q}$). $\mathbb{H}$ is a four dimensional algebra over $\mathbb{Q}$ spanned by the four symbols $1, i, j, k$ satisfying the relations $i^2 = -u, j^2 = -v;\ ij = -ji = k$. If $k/\mathbb{Q}$ is a field, $\mathbb{H}(k) \overset{\text{def}}{=} \mathbb{H} \otimes_{\mathbb{Q}} k$ (or in other words $\mathbb{H}(k)$ is defined in the same way only as an algebra over k). Let G be the

[2] In fact the action is also transitive on the set of triplets of points satisfying a similar geometric condition, the stabilizer of such a triplet is trivial

[3] This is proved by noticing that everything that was said about the action on the sphere $S(n)$ is compatible with the natural map $S(n+1) \to S(n)$ and then passing to the inverse limit.

$\mathbb{Q}$-algebraic group $\mathbb{H}^*/Z\mathbb{H}^*$, one can think of G as a way of associating a group $G(k)$ to every field $k/\mathbb{Q}$ by letting $G(k) = \mathbb{H}(k)^*/Z\mathbb{H}(k)^*$ (The group of invertible elements modulo the center). If $R < \mathbb{C}$ is a ring we will think of $G(R)$ as the group of elements in $G(\mathbb{C})$ that have representatives all of whose coefficients are elements of R.

There is a dichotomy saying that $H(k)$ is either isomorphic to $M_2(k)$ or is a division algebra [4]. In the first case we say that $\mathbb{H}$ *ramifies* over k and in the second that $\mathbb{H}$ *splits* over k. If $\mathbb{H}$ splits over k then $G(k) \cong PGL_2(k)$.

We say that $\mathbb{H}$ splits (resp. ramifies) at the prime q if it splits (resp. ramifies) over $\mathbb{Q}_q$, (including the case $q = \infty,\ \mathbb{Q}_q = \mathbb{R}$). Every quaternion algebra $\mathbb{H}$ ramifies over a finite set of primes (of even cardinality) and splits over all other primes. $\mathbb{H}$ is called a *definite* quaternion algebra if it ramifies at ∞.

Assume that a definite quaternion algebra $\mathbb{H}$ splits at q, then $G(\mathbb{Z}[1/q]) < G(\mathbb{Q}_q) \cong PGL_2(\mathbb{Q}_q)$ is an example of a $\{q\}-$ arithmetic lattice in $PGL_2(\mathbb{Q}_q)$. If N is any integer prime to q the homomorphism $\psi_N : \mathbb{Z}[1/q] \to \mathbb{Z}[1/q]/N\mathbb{Z}[1/q] \cong \mathbb{Z}/N\mathbb{Z}$ gives rise to a homomorphism

$$\psi_N : \Gamma = G(\mathbb{Z}[1/q]) \to G(\mathbb{Z}[1/q]/N\mathbb{Z}[1/q]) \cong PGL_2(\mathbb{Z}/N\mathbb{Z}). \tag{2.2}$$

Definition 2.5. *Principal congruence subgroups of Γ are groups of the form $\Gamma(N) = ker(\psi_N)$. A subgroup of Γ is called a congruence subgroup if it contains a principal congruence subgroup.*

If N is big enough then the group $\Gamma(N) < PGL_2(\mathbb{Q}_q)$ acts freely on the Bruhat-Tits tree T_{q+1}.

Theorem 2.6. *(Lubotzky, Phillips, Sarnak). Let $\mathbb{H} = \mathbb{H}_{u,v}$ be a definite quaternion algebra defined over $\mathbb{Q}$, G the $\mathbb{Q}-$algebraic group $\mathbb{H}^*/Z\mathbb{H}^*$ and assume that $\mathbb{H}$ splits over q. If a congruence subgroup $\Gamma' < \Gamma = G(\mathbb{Z}[1/q])$ acts freely on $T = T_{q+1}$. then the graph $\Gamma' \backslash T$ is a $(q+1)-$regular Ramanujan graph.*

Remark: Theorem (2.6) is usually stated for principal congruence subgroups. The general case is easily deduced: by definition every congruence subgroup Γ' contains a principal congruence subgroup $\Gamma(N)$ giving rise to a covering map $\eta : \Gamma(N)\backslash T \to \Gamma'\backslash T$. If f is an eigenfunction for the adjacency operator on $\Gamma'\backslash T$ then $f \circ \eta$ will be an eigenfunction on $\Gamma(N)\backslash T$ with the same eigenvalue. The graph $\Gamma(N)\backslash T$ will, therefore, inherit all the eigenvalues of $\Gamma'\backslash T$ proving that the later must be Ramanujan if the former is.

[4]It is easy to see that $\mathbb{H}$ is a central simple algebra and from the structure theory of such algebras $\mathbb{H}(k) \cong M_n(D)$ for some division ring D/k, the dichotomy now follows because $dim_k\mathbb{H}(k) = 4$.

3 Ramanujan Graphs of Small Girth

Let $\mathbb{H} = \mathbb{H}_{u,v}$ be a definite quaternion algebra defined over $\mathbb{Q}$ ($0 < u, v \in \mathbb{Q}$, $i^2 = -u, j^2 = -v, ij = -ji = k$, $\mathbb{H}$ ramifies at ∞). Let G be the $\mathbb{Q}$ algebraic group $\mathbb{H}^*/Z\mathbb{H}^*$, q_1 a prime such that $\mathbb{H}$ splits at q_1 and Γ the $\{q_1\}-$arithmetic lattice $G(\mathbb{Z}[1/q_1]) < PGL_2(\mathbb{Q}_{q_1})$.

We give a geometric description of some congruence subgroups of Γ which is based on ideas coming from the theory of lattices acting on products of trees ([BM00, JL00]). Let $q_2 \neq q_1$ be a second prime such that $\mathbb{H}$ splits over q_2 and $T_2 = T_{q_2+1}$ the Bruhat-Tits tree corresponding to $PGL_2(\mathbb{Q}_{q_2})$. The group Γ acts on T_2 through its embedding in $G(\mathbb{Q}_{q_2}) \cong PGL_2(\mathbb{Q}_{q_2})$, in fact here Γ is a subgroup of $PGL_2(\mathbb{Z}_{q_2})$ so it fixes a vertex $O_2 \in VT_2$. Furthermore by 2.3(2) we can identify $\Gamma(q_2^n) = \Gamma_{B_{T_2}(O_2,n)}$.

Given any finite set of vertices $O_2 \in C \subset VT_2$ there exist an $n \in \mathbb{N}$ such that $\Gamma_C \supset \Gamma_{B_{T_2}(O_2,n)}$. By 2.3(2) $\Gamma_{B_{T_2}(O_2,n)} = \Gamma(q_2^n)$ is a principal congruence subgroup, so the pointwise stabilizer $\Gamma_C < \Gamma$ is a congruence subgroup of Γ, and the corresponding graph $X_C \overset{\text{def}}{=} \Gamma_C \backslash T_1$ is Ramanujan by theorem (2.6). If we have two such sets $C_1 \subset C_2$ then $\Gamma_{C_2} < \Gamma_{C_1}$ and by the Galois correspondence 2.2 there is a covering map $X_{C_2} \to X_{C_1}$.

An ascending sequence of finite subsets $O_2 \in C_1 \subset \ldots C_{n-1} \subset C_n \subset C_{n+1} \ldots$ gives rise a tower of Ramanujan graphs $\{X_n \overset{\text{def}}{=} X_{C_n}\}_{n \in \mathbb{N}}$. If $C = \bigcup_n C_n \subset VT_2$, then $\bigcap_n \Gamma_{C_n} = \Gamma_C$. The minimal graph that covers all the X_n's is $X_C \overset{\text{def}}{=} \Gamma_C \backslash T_1$. it will be a tree and theorem 1.2(2) will be satisfied iff $\Gamma_C = \langle e \rangle$.

We first construct a tower of Ramanujan graphs whose minimal common covering is not a tree and consequently the girth of all graphs is bounded. Let Σ be the $\{q_1, q_2\}-$arithmetic lattice $\Sigma \overset{\text{def}}{=} G(\mathbb{Z}[1/q_1, 1/q_2])$ and identify Γ with the subgroup of Σ consisting of these elements whose coefficients contain only denominators which are powers of q_1, which is exactly the subgroup of Σ fixing the vertex $O_2 \in VT_2$.

$$\Gamma = \Sigma \cap PGL_2(\mathbb{Z}_{q_2}) = \Sigma_{O_2} \tag{3.1}$$

Without loss of generality, we may assume that Γ acts freely on T_1 by replacing both Σ and Γ by $\Sigma(N)$ and $\Gamma(N)$ - the principal congruence subgroups mod N, where N is a large enough number such that $(N, q_1q_2) = 1$.

Definition 3.1. *We say that the action of the group Σ on $T_1 \times T_2$ contains a torus if there are two infinite geodesics (=infinite paths without backtracking) $l_i \subset T_i$ and a subgroup $\mathbb{Z}^2 \cong \langle \gamma, \delta \rangle < \Sigma$ fixing setwise the tessellated plain $A = l_1 \times l_2 \subset T_1 \times T_2$, and acting on it freely and co-compactly.*

Figure 1: A torus

The action of the group Σ on $T_1 \times T_2$ does contain a torus. This is a theorem due to Prasad (see [Pra79]), another, more geometrical, proof for the existence of a torus is given by Mozes in ([Moz95, see the discussion below prposition 2.11]). Both proofs are given in a much greater generality. Shahar Mozes indicated to me that his proof, when it is adopted to our particular case, becomes very simple. I sketch his argument here for the convenience of the readers. In the specific example, described in section (4), it is easy to find a torus explicitly.

Proposition 3.2. *(Prasad) The action of the group Σ on $T_1 \times T_2$ contains a torus.*

Proof. The reader should refer to figure (1) to understand this proof. Consider the square complex $X \stackrel{\text{def}}{=} \Sigma \backslash (T_1 \times T_2)$, let $p : T_1 \times T_2 \to X$ be the covering morphism and $O = (O_1, O_2) \in T_1 \times T_2$ a base vertex. Choose two closed paths α_1 and α_2 in the horizontal and vertical $1-$skeletons of the complex X respectively. Both α_i should start at the base vertex $p(O) \in X$. Now lift the bi-infinite path $\alpha \stackrel{\text{def}}{=} \ldots \alpha_1 \cdot \alpha_2 \cdot \alpha_1 \cdot \alpha_2 \ldots$ to a path $\tilde{\alpha}$ in $T_1 \times T_2$ passing through O. The convex hull of the path $\tilde{\alpha}$ in $T_1 \times T_2$ is a tessellated plane which we denote by A, it is obviously invariant under the deck transformation $\gamma \stackrel{\text{def}}{=} \alpha_1 \cdot \alpha_2$, viewed as an element of $\pi_1(X, p(O))$ acting on $T_1 \times T_2$. Now consider all the zig-zag lines parallel to $\tilde{\alpha}$. By the invariance under γ, the restriction of p to such a line is determined by a finite segment of a fixed length. So it is possible to find two such lines $\tilde{\beta}_1, \tilde{\beta}_2$ such that $p|_{\tilde{\beta}_1} = p|_{\tilde{\beta}_2}$. Let $\delta : A \to A$ be an affine transformation taking $\tilde{\beta}_1$ to $\tilde{\beta}_2$. Since A is the convex hull of each of the $\tilde{\beta}_i$, the mapping $p|_A$ is determined by its restriction to β_i. This implies that δ is also a deck transformation $p \circ \delta = p$. The desired $\mathbb{Z}^2$ is now generated by $\mathbb{Z}^2 = \langle \gamma, \delta \rangle$ and we are done. $\square$

We may assume without loss of generality that the action $\mathbb{Z}^2 \cong \langle \gamma, \delta \rangle \rightsquigarrow A$ is the standard action: γ acts by translation of length $M_1 = l^{T_1}(\gamma)$ on the first coordinate and fixes the second coordinate and the other way around for δ. γ will thus be elliptic on T_2 and hyperbolic on T_1 and the opposite for δ. We can also assume that $\gamma \in \Gamma$ i.e. that O_2, the vertex fixed by $PGL_2(\mathbb{Z}_{q_2})$, is contained in l_2.

The element γ, acting on T_2, fixes pointwise the bi-infinite line $l_2 \in T_2$, in particular it fixes pointwise any finite subset of l_2. Taking $\{O_2 \in C_n \subset l_2\}_{n \in \mathbb{N}}$ to be symmetric line segments of length $2n$ around O_2 we obtain a tower of Ramanujan graphs $X_n \stackrel{\text{def}}{=} X_{C_n} = \Gamma_{C_n} \backslash T_1$ with bounded girth. In fact we can explicitly describe the circle common to all these graphs, the infinite line $l_1 \subset T_1$ is mapped to a circle of length $\leq M_1$ in the graph $X_C = \Gamma_C \backslash T_1$ which in turn covers all the finite graphs X_n.

We now modify our example so that 1.2(2) is satisfied, i.e so that the minimal common covering of the tower of Ramanujan graphs is the tree T_1. Let $l \subset T_2$ be an infinite geodesic with the following properties:

- $O_2 \in l \bigcap l_2$.
- $\Gamma_l \stackrel{\text{def}}{=} \text{Stab}_\Gamma(l) = \langle e \rangle$

The existence of such a geodesic is clear by counting considerations: Each non identity element $g \in PGL_2(\mathbb{Q}_{q_2})$ fixes at most three points on the boundary of T_2 ([Ser80],2.3(3)), but Γ is countable and the boundary is not. By 2.3(3) there exists an element $g \in PSL_2(\mathbb{Z}_{q_2})$ such that $gl_2 = l$. The weak approximation theorem [Lub94], says that $\Sigma \cap PSL_2(\mathbb{Q}_{q_2})$ is dense[5] in $PSL_2(\mathbb{Q}_{q_2})$, and since $PSL_2(\mathbb{Z}_{q_2})$ is open $\Gamma \cap PSL_2(\mathbb{Z}_{q_2}) < PSL_2(\mathbb{Z}_{q_2})$ is dense. For a given n we can find an element $\gamma_n \in \Gamma$ which is close enough to g that $\gamma_n C_n = gC_n \subset l$. It follows that $\Gamma_{gC_n} = \gamma_n \Gamma_{C_n} \gamma_n^{-1}$ and that the graphs $X_n \cong X_{gC_n}$ are isomorphic. In particular, the graphs X_{gC_n} are also Ramanujan graphs with bounded girth. The intersection

$$\bigcap_n \Gamma_{gC_n} = \Gamma_{gC} = \langle e \rangle \tag{3.2}$$

[5]The topology here is the $q_2 - adic$ topology on $PGL_2(\mathbb{Q}_{q_2})$. This is the same as the compact open topology coming from the action on the tree T_2: two elements are close if their restrictions to some large finite subset of VT_2 coincide.

is now trivial and therefore the tower of graphs $\{X_{gC_n}\}_{n\in\mathbb{N}}$ satisfies all the properties stated in theorem (1.2). Note that we have not changed the isomorphism type of the graphs but merely the covering maps between them, in order to satisfy equation (3.2). □

4 Explicit description

Here we specialize to a very concrete example, which enables us to give an explicit description of a tower of Ramanujan graphs in terms of Schreier graphs.

Theorem 4.1. *(Compare [Lub94, theorem 7.4.3]). Let q_1, q_2 be two primes both congruent to* 1 *mod* 4*, n any integer,*

$$\begin{array}{rcl} L(n) & = & \begin{cases} PSL_2(\mathbb{Z}/q_2^n\mathbb{Z}) & \textit{if } q_1 \textit{ is a quadratic residue mod } q_2 \\ PGL_2(\mathbb{Z}/q_2^n\mathbb{Z}) & \textit{otherwise} \end{cases} \\ A(n) & < & L(n) \qquad \textit{the diagonal group.} \\ S(n) & = & \left\{ \begin{pmatrix} x_0 + x_1\sqrt{-1} & x_2 + x_3\sqrt{-1} \\ -x_2 + x_3\sqrt{-1} & x_1 - x_2\sqrt{-1} \end{pmatrix} \in L(n) \;\middle|\; \begin{array}{l} \sum_{i=1}^4 x_i^2 = q_1 \\ 0 < x_0 = 1 (\mathrm{mod}\, 2) \\ x_1 = x_2 = x_3 = 0 (\mathrm{mod}\, 2) \end{array} \right\} \end{array}$$

Then, the set S is symmetric and contains exactly $(q_1 + 1)$ elements. The right Schreier graphs

$$X(L(n), A(n), S(n)) \tag{4.1}$$

form a tower of $(q_1 + 1)-$regular Ramanujan graphs with girth 1 *(i.e. all graphs contain a loop).*

Proof. This is just spelling out theorem 1.2 when $\mathbb{H} = \mathbb{H}_{1,1}$ is the (standard) Hamilton quaternion algebra. $\mathbb{H}$ splits at all primes except for $\{2, \infty\}$, when q is a prime congruent to 1 mod 4 there is a $\sqrt{-1} \in \mathbb{Q}_q$ and the splitting is explicitly described by [6]:

$$\phi_q : a + bi + cj + dk \to \begin{pmatrix} a + b\sqrt{-1} & c + d\sqrt{-1} \\ -c + d\sqrt{-1} & a - b\sqrt{-1} \end{pmatrix} \tag{4.2}$$

Let $G = \mathbb{H}^*/Z\mathbb{H}^*$ be the algebraic group of invertible elements in $\mathbb{H}$ modulo the center. For the lattices [7] Γ (resp. Σ) we take the principal congruence subgroup mod

[6] One can check that this is an isomorphism by solving the linear equations for the matrix entries.

[7] Γ is a uniform lattice in $PGL_2(\mathbb{Q}_{q_1})$, Σ is a uniform lattice under its diagonal embedding in $PGL_2(\mathbb{Q}_{q_1}) \times PGL_2(\mathbb{Q}_{q_2})$. This means that it acts with a finite number of orbits and finite vertex stabilizers on the product $T_1 \times T_2$.

2, of $G(\mathbb{Z}[1/q_1])$ (resp. $G(\mathbb{Z}[1/q_1, 1/q_2])$).

$$\Gamma \stackrel{\text{def}}{=} \left\{ [x_0 + x_1 i + x_2 j + x_3 k] \;\middle|\; \begin{array}{l} x_i \in \mathbb{Z} \\ \sum_{i=0}^{3} x_i^2 = q_1^m \text{ for some } m \in \mathbb{N} \\ x_0 = 1 (\text{mod}\, 2) \\ x_2 = x_3 = x_4 = 0 (\text{mod}\, 2) \end{array} \right\}$$

$$\Sigma \stackrel{\text{def}}{=} \left\{ [x_0 + x_1 i + x_2 j + x_3 k] \;\middle|\; \begin{array}{l} x_i \in \mathbb{Z} \\ \sum_{i=0}^{3} x_i^2 = q_1^m q_2^n \text{ for some } m, n \in \mathbb{N} \\ x_0 = 1 (\text{mod}\, 2) \\ x_2 = x_3 = x_4 = 0 (\text{mod}\, 2) \end{array} \right\} \tag{4.3}$$

Where the square brackets stand for equivalence class modulo the center $Z\mathbb{H}$.

The group Γ acts freely, transitively and without inversion on the tree T_1 (see [Lub94, Lemma 7.4.1]), this makes all the details of example (2.3) applicable, so we identify the graphs X_n with the Schreier graphs

$$X_n = X(\Gamma/\Gamma(q_2^n), \Gamma_{C_n}/\Gamma(q_2^n), S') \tag{4.4}$$

Here S' is the natural set of generators of Γ, which makes T_1 into the Cayley graph of Γ, $\{C_n\}_{n \in \mathbb{N}}$ is an ascending sequence of segments of length $2n$ around O_2 and $\Gamma(q_2^n) = \Gamma_{B_{T_2}(O_2, n)}$ is a natural choice for a normal subgroup contained in Γ_{C_n}.

The group $\Gamma/\Gamma(q_2^n) = \Gamma/\Gamma_{B_{T_2}(O_2,n)}$ is identified in [Lub94, Remark 7.4.4] as the group $L(n)$ (one can see from 2.3(2) that it is a subgroup of $PGL_2(\mathbb{Z}/q_2^n\mathbb{Z})$). We have seen in the previous section, that as long as we are only interested in the isomorphism type of the graphs X_n the precise choice of the segments C_n does not matter. We make the choice that will give, using 2.3(2), $\Gamma_{C_n}/\Gamma_{B_{T_2}(O_2,n)} = A(n)$. To identify S', we choose a base vertex $O_1 \in VT_1$ - the vertex stabilized by $PGL_2(\mathbb{Z}_{q_1})$, and let

$$S' = \left\{ [x_0 + x_1 i + x_2 j + x_3 k] \in \Gamma \;\middle|\; \begin{array}{l} x_i \in \mathbb{Z} \\ \sum_{i=1}^{4} x_i^2 = q_1 \\ 0 < x_0 = 1 (\text{mod}\, 2) \\ x_1 = x_2 = x_3 = 0 (\text{mod}\, 2) \end{array} \right\} \tag{4.5}$$

be the symmetric set of generators of Γ taking O_1 to its $(q_1 + 1)$ neighbors. $S(n)$ is exactly the image of S' in $L(n)$ under the map $\psi_{q_1^n} \circ \phi_{q_2}$.

After making all these identifications equation 4.4 gives the desired equation 4.1. From the previous section we know that $\{X_n\}_{n \in \mathbb{N}}$, thus defined is a tower of Ramanujan graphs of bounded girth. In order to show that the girth is actually 1 and in order to find the minimal common covering of all graphs we must explicitly identify the torus $\langle \gamma, \delta \rangle$.

Each q_i can be represented as a sum of two squares $q_i = a_i^2 + b_i^2$. Without loss of generality we may assume that a_i is positive and odd and that b_i is even. As our pair of commuting elements we can take $\gamma = a_1 + b_1 i, \delta = a_2 + b_2 i$. We let $l_i \subset T_i$ be the bi-infinite geodesic which is stabilized (setwise) by the diagonal (Cartan) subgroup $A \subset PGL_2(\mathbb{Q}_{q_i})$ then γ fixes l_2 pointwise and acts on l_1 by translation of length 1. Since by our choice the C_n's are all subsets of l_2, the line l_1 will be mapped into a loop (=circle of length 1) in each of the graphs X_n.

The minimal common covering of all the X_n's (with the natural covering morphisms) is not T_1. In order to obtain a tower whose minimal common covering is T_1 one has to replace $A(n)$ by $g(n)A(n)g(n)^{-1}$ where $g(n) = \psi(q_2^n(g))$ is the reduction mod q_2^n of any element $g \in PGL_2(\mathbb{Q}_{q_2})$ such that $\Gamma_{gl_2} = \langle e \rangle$. For example any element with entries that are not algebraic with respect to each other will do, but one can also find concrete algebraic examples. □

Remark: Another variant would be to replace C_n by paths of length n starting at the vertex O_2. The graphs obtained in this way would be $X(S(n), B(n), S(n))$ where $B(n) < L(n)$ is the upper triangular (Borel) subgroup. These graphs can be identified with the graphs coming from the action of $L(n)$ on the projective line $\mathbb{P}^1(\mathbb{Z}/q_2^n\mathbb{Z})$.

5 Remarks and open questions

- A similar construction can be carried out for surfaces. If we replace $\mathbb{H}$ with a non-definite quaternion algebra which ramifies over $\mathbb{Q}$, and Σ by a mixed irreducible lattice in a product of a $p-$adic and a real Lie group then we can obtain a tower of compact surfaces S_n. By the Jacket-Langlands correspondence all S_n satisfy Selberg's $\lambda_1 \geq 3/16$ (and conjecturally $\lambda_1 \geq 1/4$) theorem, and in addition they will all share the same closed geodesic.

  For example pick the quaternion algebra $\mathbb{H} = \mathbb{H}_{-2,-3} \stackrel{\text{def}}{=} \langle 1, i, j, k | ij = -ji = k, i^2 = 2, j^2 = 3 \rangle$. $\mathbb{H}$ splits at 7 and at ∞ via the splitting

$$x_0 + x_1 i + x2_j + x_3 k \to \begin{pmatrix} x_0 + x_1\sqrt{2} & x_2 + x_3\sqrt{2} \\ 3(x_2 - x_3\sqrt{2}) & x_0 - x_1\sqrt{2} \end{pmatrix} \tag{5.1}$$

  Let G be the algebraic group $\mathbb{H}^*/Z\mathbb{H}^*$ defined over $\mathbb{Q}$ and consider the lattice $\Lambda \stackrel{\text{def}}{=} G(\mathbb{Z}[1/7])(2) \in PGL_2(\mathbb{R}) \times PGL_2(\mathbb{Q}_7)$ (i.e. the principal congruence subgroup mod 2 of $G(\mathbb{Z}[1/7])$). Lambda is torsion free. The element $\gamma \in \Gamma$

given by

$$\gamma \stackrel{\text{def}}{=} 3 + 2i \sim \begin{pmatrix} 3+2\sqrt{2} & 0 \\ 0 & 3-2\sqrt{2} \end{pmatrix} \tag{5.2}$$

fixes pointwise an infinite geodesic l_2 on the tree associated to $PGL_2(\mathbb{Q}_7)$; it is also hyperbolic, acting as a translation of length $\log(17+12\sqrt{2})$ on the axis $l_1 = i\mathbb{R}$, as an element of $PGL_2(\mathbb{R})$ acting on the hyperbolic plane $\mathfrak{h}$. We obtain a tower of Riemann surfaces $\{\Lambda_{C_n}\backslash\mathfrak{h}\}_n$ where $C_n \subset l_2$ are finite segments with length going to infinity. All these surfaces will contain a closed geodesic of length $\log(17+12\sqrt{2})$. As in the combinatorial case the minimal common covering of all these surfaces will be $\langle\gamma\rangle\backslash\mathfrak{h}$, and by twisting the covering morphisms we can arrange for the minimal common covering to be $\mathfrak{h}$.

- In retrospect the fact that there exist congruence subgroups with non-trivial intersection seems obvious. It is interesting to note however that this intersection can not be too large. This is the content of the following lemma, the proof of which was indicated to me by Alex Lubotzky:

Lemma 5.1. *Let $\Gamma < PGL_2(\mathbb{Q}_q)$ be an arithmetic lattice, $\Delta < \Gamma$ an infinite index torsion free subgroup which is an intersection of congruence subgroups. Then Δ is either trivial or $\mathbb{Z}$.*

Proof. If Δ is torsion free then it is free (because it acts freely on the tree). If it is not Abelian it must be Zariski dense because PGL_2 does not have any proper, non solvable algebraic subgroups. But Δ is closed in the congruence topology on Γ so by the strong approximation theorem ([Wei84, Nor87]) Δ is open in the congruence topology on Γ and therefore of finite index. □

Geometrically this means that for any infinite tower of LPS graphs the minimal common covering graph contains at most one circle. In fact, this statement will hold for all known constructions of Ramanujan graphs of constant degree. As far as I know all these constructions come from congruence subgroups in arithmetic lattices. One is lead to ask the following question:

Question 5.2. *Can the minimal common covering of a tower of Ramanujan graphs have more then one circle (i.e. have a non Abelian fundamental group)?*

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