CONTINUITY OF THE SIGNATURE

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Here is my solution of the riddle from 12/3/2014 (Theorem 1 below), with a lot of explanations, to be understandable by undergraduate math students. The proof relies on Theorem 3, where I show that the Gram-Schmidt process can be done on an open neighborhood of each point in the space of symmetric invertible matrices.

Comments are welcome. There are much quicker solutions to the riddle, and maybe they will be communicated by other people.

Let me start with the story. Recently, when reading some old paper, I saw a claim that the signature of a nondegenerate symmetric bilinear form on a real vector space is continuous. This was something I had never considered before. At first I thought that continuity refers to the Zariski topology. However this was quickly seen to be false, already in dimension 1 (see Example 2 below). What is true is that the signature is continuous in the classical topology.

Now for some notation. Take $n \in \mathbb{N}$, and consider the affine space $\mathbb{R}^n$. This is a topological space with the classical topology, in which the open sets are unions of open balls in the Euclidean metric.

Let $\mathrm{Mat}_n(\mathbb{R})$ be the set of $n \times n$ real matrices. This set is isomorphic to $\mathbb{R}^{n^2}$, and hence it gets a topology (the classical topology). Note that there are several obvious bijections $\mathrm{Mat}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$, but as long as we choose an $\mathbb{R}$-linear bijection we get the same induced topology. Any subset of $\mathrm{Mat}_n(\mathbb{R})$ acquires the subspace topology. We are interested in the set $\mathrm{GL}_n(\mathbb{R})$ of invertible matrices, and the set $\mathrm{S}_n(\mathbb{R})$ of symmetric invertible matrices.

Take any $\boldsymbol{b} \in \mathrm{S}_n(\mathbb{R})$. The Gram-Schmidt process (tweaked a bit to handle negative numbers) says that we can find a matrix $\boldsymbol{g} \in \mathrm{GL}_n(\mathbb{R})$ such that

$$\boldsymbol{g}^{\mathrm{t}} \cdot \boldsymbol{b} \cdot \boldsymbol{g} = \mathrm{diag}(d_1, \ldots, d_n),$$

a diagonal matrix with entries $d_i \in \{\pm 1\}$. Let us call a matrix such as $\mathrm{diag}(d_1, \ldots, d_n)$ a *signature matrix*. The Sylvester Inertia Theorem says that the integer

$$\mathrm{sig}(\boldsymbol{b}) := \sum_i d_i$$

is independent of the matrix $\boldsymbol{g}$. The integer $\mathrm{sig}(\boldsymbol{b})$ is called the *signature* of $\boldsymbol{b}$.

Let X be a topological space and Z a set. Recall that a function $f : X \to Z$ is called *locally constant* if there is an open covering $X = \bigcup_i U_i$ such that $f|_{U_i}$ is constant. This is equivalent to saying that f is continuous when the set Z is given the discrete topology.

The riddle was to prove this:

Theorem 1. *The function*

$$\mathrm{sig} : \mathrm{S}_n(\mathbb{R}) \to \mathbb{Z}$$

is locally constant.

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The next example proves the case $n = 1$. The general case is done later; it will follow from a stronger result: Theorem 3.

Example 2. The set $\mathrm{S}_1(\mathbb{R})$ is just the set of nonzero real numbers. For $\boldsymbol{b} = [b] \in \mathrm{S}_1(\mathbb{R})$ we have $\mathrm{sig}(b) = b/|b|$, so that $\mathrm{sig}(b) = 1$ iff $b > 0$, and $\mathrm{sig}(b) = -1$ iff $b < 0$. Since the intervals $U_+ := (0, \infty)$ and $U_- := (-\infty, 0)$ are open in the classical topology, and $\mathrm{S}_1(\mathbb{R}) = U_+ \cup U_-$, we see that sig is indeed locally constant.

For those who know the Zariski topology, consider this topology on $\operatorname{Spec} \mathbb{R}[t]$, and the induced subspace topology on

$$\mathrm{S}_1(\mathbb{R}) \subset \mathbf{A}^1(\mathbb{R}) \subset \operatorname{Spec} \mathbb{R}[t].$$

Both sets U_+ and U_- are dense in $\mathrm{S}_1(\mathbb{R})$. Therefore the function sig is not locally constant for the Zariski topology.

Now for the main result of this note. (Presumably it is known to experts, although I do not know a reference.)

Theorem 3. *Let X be some topological space, and let $\boldsymbol{b} : X \to \mathrm{S}_n(\mathbb{R})$ be a continuous function. Take any point $x \in X$. Then there is an open neighborhood U of x, a continuous function $\boldsymbol{g} : U \to \mathrm{GL}_n(\mathbb{R})$, and numbers $d_i \in \{\pm 1\}$, such that*

$$\boldsymbol{g}(y)^{\mathrm{t}} \cdot \boldsymbol{b}(y) \cdot \boldsymbol{g}(y) = \operatorname{diag}(d_1, d_2, \ldots, d_n)$$

for every point $y \in U$.

The proofs of the theorems are at the end of the note, after some preparation.

In what follows X is some topological space. Let us denote by $\mathrm{C}(X)$ the ring of continuous functions $f : X \to \mathbb{R}$. It is a commutative ring, and there is a ring homomorphism $\mathbb{R} \to \mathrm{C}(X)$, sending a number a to the constant function $a(x) = a$. If $g : Y \to X$ is a continuous map of topological spaces, then there is an induced $\mathbb{R}$-ring homomorphism $g^* : \mathrm{C}(X) \to \mathrm{C}(Y)$, namely $g^*(f) := f \circ g$. In case Y is a subset of X (with the subspace topology), and $g : Y \to X$ is the inclusion, then $g^*(f) = f|_Y$, the restriction. When $Y = \{x\}$ is a single point, then $\mathrm{C}(Y) = \mathbb{R}$ and $f|_Y = f(x) \in \mathbb{R}$.

The commutative ring $\mathrm{C}(X)$ is viewed as an algebraic object (i.e. it has no topology). There is the noncommutative ring $\mathrm{Mat}_n(\mathrm{C}(X))$ of $n \times n$ matrices, and inside it we have the group $\mathrm{GL}_n(\mathrm{C}(X))$ of invertible matrices, and the set $\mathrm{S}_n(\mathrm{C}(X))$ of symmetric invertible matrices. Note that $\mathrm{GL}_n(\mathrm{C}(X))$ is precisely the set of invertible elements of the ring $\mathrm{Mat}_n(\mathrm{C}(X))$. A continuous map of topological spaces $g : Y \to X$ induces an $\mathbb{R}$-ring homomorphism $g^* : \mathrm{Mat}_n(\mathrm{C}(X)) \to \mathrm{Mat}_n(\mathrm{C}(Y))$, a group homomorphism $g^* : \mathrm{GL}_n(\mathrm{C}(X)) \to \mathrm{GL}_n(\mathrm{C}(Y))$, and a function of sets $g^* : \mathrm{S}_n(\mathrm{C}(X)) \to \mathrm{S}_n(\mathrm{C}(Y))$.

There is an obvious bijection

$$\mathrm{Mat}_n(\mathrm{C}(X)) \cong \{\text{continuous functions } X \to \mathrm{Mat}_n(\mathbb{R})\}; \tag{4}$$

namely if $[a_{i,j}] \in \mathrm{Mat}_n(\mathrm{C}(X))$, then the corresponding continuous function $\boldsymbol{a} : X \to \mathrm{Mat}_n(\mathbb{R})$ is $\boldsymbol{a}(x) := [a_{i,j}(x)]$.

Here is another way to interpret Theorem 3. It will be used in the proof. The bijection (4) allows us to view the function $\boldsymbol{b}$ as an element of the set $\mathrm{S}_n(\mathrm{C}(X))$. The theorem says that on a sufficiently small open neighborhood U of x, there is a matrix $\boldsymbol{g} \in \mathrm{GL}_n(\mathrm{C}(U))$, such that the matrix $\boldsymbol{g}^{\mathrm{t}} \cdot \boldsymbol{b}|_U \cdot \boldsymbol{g} \in \mathrm{Mat}_n(\mathrm{C}(U))$ is a signature matrix.

Here are two easy lemmas that we will need.

Lemma 5. *Let $\boldsymbol{g} \in \mathrm{Mat}_n(\mathrm{C}(X))$. The following conditions are equivalent*:

(i) *The matrix $\boldsymbol{g}$ is invertible in the ring $\mathrm{Mat}_n(\mathrm{C}(X))$.*

(ii) *The element* $\det(\boldsymbol{g})$ *invertible in the ring* $\mathrm{C}(X)$.
(iii) *For every point* $x \in X$ *the number* $\det(\boldsymbol{g})(x) \in \mathbb{R}$ *is nonzero.*
(iv) *For every point* $x \in X$ *the matrix* $\boldsymbol{g}(x)$ *is invertible in the ring* $\mathrm{Mat}_n(\mathbb{R})$.

Proof. (i) $\Leftrightarrow$ (ii): This is true for any commutative ring C, including $C := \mathrm{C}(X)$. Indeed, given $\boldsymbol{g} \in \mathrm{Mat}_n(C)$, let $\boldsymbol{h} \in \mathrm{Mat}_n(C)$ be the classical adjoint matrix of $\boldsymbol{g}$, namely the matrix whose (i, j)-th entry is $(-1)^{ij}$ times the (j, i)-th minor of $\boldsymbol{g}$. Then $\boldsymbol{g} \cdot \boldsymbol{h} = \boldsymbol{h} \cdot \boldsymbol{g} = \det(\boldsymbol{g}) \cdot \mathbf{1}$, which is invertible in the ring $\mathrm{Mat}_n(C)$ iff $\det(\boldsymbol{g})$ is invertible in the ring C.

(ii) $\Leftrightarrow$ (iii): This is clearly true for any $g \in \mathrm{C}(X)$, including $g := \det(\boldsymbol{g})$.

(iii) $\Leftrightarrow$ (iv): For every x this is a special (and trivial) case of "(i) $\Leftrightarrow$ (ii)". $\square$

Lemma 6. *Let* $\boldsymbol{g} \in \mathrm{Mat}_n(\mathrm{C}(X))$*, and let* $x \in X$ *be a point. The following conditions are equivalent*:

(i) *The matrix* $\boldsymbol{g}(x) \in \mathrm{Mat}_n(\mathbb{R})$ *is invertible.*
(ii) *There is an open neighborhood* U *of* X *such that the matrix* $\boldsymbol{g}|_U \in \mathrm{Mat}_n(\mathrm{C}(U))$ *is invertible.*

Proof. By Lemma 5 and continuity of the determinant. $\square$

We now talk about bilinear forms. A *bilinear form* on the $\mathrm{C}(X)$-module $\mathrm{C}(X)^n$ is a $\mathrm{C}(X)$-bilinear function

$$\beta : \mathrm{C}(X)^n \times \mathrm{C}(X)^n \to \mathrm{C}(X).$$

The form β is *symmetric* if $\beta(w, v) = \beta(v, w)$ for all pairs of vectors $v, w \in \mathrm{C}(X)^n$.

There is a canonical bijection between bilinear forms β on $\mathrm{C}(X)^n$ and matrices $\boldsymbol{b} \in \mathrm{Mat}_n(\mathrm{C}(X))$, sending a matrix $\boldsymbol{b}$ to the form

$$\beta(v, w) := v^{\mathrm{t}} \cdot \boldsymbol{b} \cdot w. \tag{7}$$

(We consider vectors in $\mathrm{C}(X)^n$ as columns.) The form β is symmetric iff the corresponding matrix $\boldsymbol{b}$ is symmetric.

Exercise 8. Give good definitions of these notions:

(1) A nondegenerate bilinear form on $\mathrm{C}(X)^n$.
(2) A basis of $\mathrm{C}(X)^n$.
(3) A "signed" orthonormal basis of $\mathrm{C}(X)^n$ with respect to a bilinear form β.

Let β be a bilinear form on $\mathrm{C}(X)^n$. For any subset $Y \subset X$ we get an induced bilinear form $\beta|_Y$ on $\mathrm{C}(Y)^n$, as follows. Let $\boldsymbol{b} \in \mathrm{Mat}_n(\mathrm{C}(X))$ be the matrix corresponding to $\boldsymbol{b}$, as in (7). By restriction we get a matrix $\boldsymbol{b}|_Y \in \mathrm{Mat}_n(\mathrm{C}(Y))$, and the bilinear form $\beta|_Y$ is defined to be the one that corresponds to the matrix $\boldsymbol{b}|_Y$. When $Y = \{x\}$, a point, we write $\beta(x) := \beta|_Y$. Since $\mathrm{C}(\{x\}) = \mathbb{R}$, we see that $\beta(x)$ is a bilinear form on $\mathbb{R}^n$.

Proof of Theorem 3. The proof is by induction on $n \geq 1$. We will devise an enhanced version of the Gram-Schmidt process.

Let $n \geq 1$, and assume that either $n = 1$, or $n \geq 2$ and the theorem is true for $n - 1$. Consider the bilinear form β on $\mathrm{C}(X)^n$ corresponding to $\boldsymbol{b}$. The form $\beta(x)$ on $\mathbb{R}^n$ corresponds to the matrix $\boldsymbol{b}(x) \in \mathrm{S}_n(\mathbb{R})$, so it is nondegenerate. As in the usual proof of Gram-Schmidt (tweaked for negative numbers), there is a vector $v_1 \in \mathbb{R}^n$ such that $\beta(x)(v_1, v_1) \neq 0$.

Using the embedding $\mathbb{R}^n \subset \mathrm{C}(X)^n$ we can view v_1 as a constant vector in $\mathrm{C}(X)^n$. Let $b_1 := \beta(v_1, v_1) \in \mathrm{C}(X)$, and let $d_1 \in \{\pm 1\}$ be the sign of the nonzero real number $b_1(x)$. The function $d_1 b_1 \in \mathrm{C}(X)$ satisfies $d_1 b_1(x) > 0$. Therefore there is an open neighborhood

U' of x, and a multiplicatively invertible function $a_1 \in \mathrm{C}(U')$, such that $a_1^2 = d_1 b_1|_{U'}$. Define the vector $v_1' := a_1^{-1} v_1 \in \mathrm{C}(U')^n$. Then $\beta|_{U'}(v_1', v_1') = d_1$, as functions on U'.

If $n = 1$ then the open set $U := U'$ and the matrix $\boldsymbol{g} := [a_1^{-1}] \in \mathrm{GL}_1(\mathrm{C}(U))$ satisfy $\boldsymbol{g}^{\mathrm{t}} \cdot \boldsymbol{b}|_U \cdot \boldsymbol{g} = [d_1]$. So the proof is done.

If $n \geq 2$ we proceed like this. Let $(v_2, \ldots, v_n)$ be a basis of the orthogonal complement of v_1 in the $\mathbb{R}$-module $\mathbb{R}^n$, with respect to the form $\beta(x)$. Using the embedding $\mathbb{R}^n \subset \mathrm{C}(X)^n$ we get a sequence $(v_1, \ldots, v_n)$ of vectors in $\mathrm{C}(X)^n$.

For $i \geq 2$ we define

$$v_i' := v_i - d_1 \cdot \beta|_{U'}(v_i, v_1') \cdot v_1' \in \mathrm{C}(U')^n.$$

Observe that $\beta|_{U'}(v_1', v_i') = 0$ for $i \geq 2$, and that the sequence $(v_1'(x), \ldots, v_n'(x))$ is a basis of $\mathbb{R}^n$.

Define the symmetric matrix

$$\boldsymbol{b}' := [\beta|_{U'}(v_{i-1}', v_{j-1}')]_{2 \leq i,j \leq n} \in \mathrm{Mat}_{n-1}(\mathrm{C}(U')).$$

And let $\boldsymbol{h}' \in \mathrm{Mat}_n(\mathrm{C}(U'))$ be the matrix whose columns are $v_1', \ldots, v_n'$. We get

$$\boldsymbol{h}'^{\mathrm{t}} \cdot \boldsymbol{b}|_{U'} \cdot \boldsymbol{h}' = \begin{bmatrix} d_1 & 0 \\ 0 & \boldsymbol{b}' \end{bmatrix} \tag{9}$$

in $\mathrm{Mat}_n(\mathrm{C}(U'))$. Since $\boldsymbol{h}'(x)$ and $\boldsymbol{b}(x)$ are invertible, by Lemma 6 there is an open neighborhood U'' of x in U', such that $\boldsymbol{h}'|_{U''}$ and $\boldsymbol{b}|_{U''}$ are invertible. This implies $\boldsymbol{b}'|_{U''} \in \mathrm{S}_{n-1}(\mathrm{C}(U''))$.

By the induction hypothesis, applied to $x \in U''$ and $\boldsymbol{b}'|_{U''} \in \mathrm{S}_{n-1}(\mathrm{C}(U''))$, there is an open neighborhood U of x in U'', and a matrix $\boldsymbol{g}' \in \mathrm{GL}_{n-1}(\mathrm{C}(U))$, such that

$$\boldsymbol{g}'^{\mathrm{t}} \cdot \boldsymbol{b}'|_U \cdot \boldsymbol{g}' = \mathrm{diag}(d_2, \ldots, d_n)$$

for some $d_2, \ldots, d_n \in \{\pm 1\}$. Let

$$\boldsymbol{g} := \boldsymbol{h}'|_U \cdot \begin{bmatrix} 1 & 0 \\ 0 & \boldsymbol{g}' \end{bmatrix} \in \mathrm{GL}_n(\mathrm{C}(U)).$$

Then

$$\boldsymbol{g}^{\mathrm{t}} \cdot \boldsymbol{b}|_U \cdot \boldsymbol{g} = \mathrm{diag}(d_1, \ldots, d_n).$$

□

Corollary 10. *Let X be some topological space, and let $\boldsymbol{b} : X \to \mathrm{S}_n(\mathbb{R})$ be a continuous function. Then the function*

$$\mathrm{sig} \circ \boldsymbol{b} : X \to \mathbb{Z}$$

is locally constant.

Proof. On any open set U as in Theorem 3 the function $\mathrm{sig} \circ \boldsymbol{b}$ is constant. □

Proof of Theorem 1. Take the topological space $X := \mathrm{S}_n(\mathbb{R})$ and the identity map $\boldsymbol{b}$, and apply Corollary 10. □

Remark 11. For those who know sheaf theory, let $\mathcal{O}_X$ be the sheaf of continuous real valued functions on X. The stalk $\mathcal{O}_{X,x}$ at x is the ring of germs of continuous functions. What I really prove in Theorem 3 is that the Gram-Schmidt process can be done inside $\mathrm{Mat}_n(\mathcal{O}_{X,x})$. Namely given a matrix $\boldsymbol{b} \in \mathrm{S}_n(\mathcal{O}_{X,x})$, there exists $\boldsymbol{g} \in \mathrm{GL}_n(\mathcal{O}_{X,x})$ such that $\boldsymbol{g}^{\mathrm{t}} \cdot \boldsymbol{b} \cdot \boldsymbol{g}$ is a signature matrix.

Exercise 12. For any $0 \leq r \leq n$ we can consider the space $\mathrm{S}_{n,r}(\mathbb{R})$ of symmetric $n \times n$ matrices of rank r. So $\mathrm{S}_{n,0}(\mathbb{R}) = \{0\}$, and $\mathrm{S}_{n,n}(\mathbb{R}) = \mathrm{S}_n(\mathbb{R})$. Try to generalize Theorem 3 to $\mathrm{S}_{n,r}(\mathbb{R})$; of course here $d_i \in \{0, \pm 1\}$. Use it to deduce Theorem 1 for $\mathrm{S}_{n,r}(\mathbb{R})$.

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