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# Canonical Deformations of de Rham Complexes

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## 1. INTRODUCTION AND STATEMENT OF RESULTS

### 1.1. *Background*

Let $k$ be a field, and let $A = k[x_1, ..., x_n]$ be the commutative polynomial ring. Suppose $q = [q_{ij}]$ is an $n \times n$ matrix with entries in $k$, satisfying $q_{ij} q_{ji} = q_{i,i} = 1$. Let $k\langle x_1, ..., x_n\rangle$ be the free associative algebra.

DEFINITION 1.1.1. The *multi-paramater quantum deformation* of $A$ is the $k$-algebra

$$A_q = \mathcal{O}(\mathbf{A}^n_q) := k\langle x_1, ..., x_n\rangle/(x_j x_i - q_{ji} x_i x_j \mid i, j = 1, ..., n).$$

This type of deformation was investigated in [AST; Su1; Re]. It was shown that there is a corresponding flat deformation $\mathcal{O}(M_{n,q,\lambda})$ of the ring $\mathcal{O}(M_n)$ of functions on $n \times n$ matrices. $\mathcal{O}(M_{n,q,\lambda})$ is a bialgebra coacting on $A_q$, and $\lambda \in k^\times$ is an extra deformation parameter.

The bialgebra $\mathcal{O}(M_{n,q,\lambda})$ actually coacts on the *deformed de Rham complex* $\Omega^*_{q,\lambda} = \Omega^*(\mathbf{A}^n_{q,\lambda}) = \bigoplus_{i=0}^n \Omega^i_{q,\lambda}$. This is the differential graded algebra (DGA) given by

$$\Omega^*_{q,\lambda} := k\langle x_1, ..., x_n, \xi_1, ..., \xi_n\rangle/(\text{relations}) \tag{1.1.1}$$

$$\text{relations} = \begin{cases} x_j x_i - q_{ji} x_i x_j, & \text{if } i<j \\ \xi_j \xi_i + \lambda^{-1} q_{ji} \xi_i \xi_j, & \text{if } i<j \\ \xi_i^2 & \\ \xi_i x_i - \lambda x_i \xi_i & \\ x_j \xi_i - \lambda^{-1} q_{ji} \xi_i x_j, & \text{if } i<j \\ \xi_j x_i - q_{ji} x_i \xi_j - (1-\lambda^{-1}) q_{ji} \xi_i x_j, & \text{if } i<j \end{cases} \tag{1.1.2}$$

and $\mathrm{d}(x_i) = \xi_i$. This is a deformation of the DGA $\Omega^* := \Omega^*_{A/k}$, the usual de Rham complex of $A$ relative to $k$, and $\Omega^0_{q,\lambda} = A_q$. Such a deformation was first considered by Woronowicz [Wo] and Wess and Zumino [WZ].



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The deformation $\Omega^*_{q,\lambda}$ is also called a *quantum differential calculus*, since it defines the *quantum vector fields* $\partial_i$, $i = 1, ..., n$, which satisfy

$$\mathrm{d}(a) = \sum_{i=1}^{n} \xi_i \partial_i(a), \qquad a \in A_q \tag{1.1.3}$$

(cf. [Ma2, Section 2.1]). As pointed out by Manin in [Ma1], the "missing relations" of the bialgebra coacting on $A_q$ (i.e., those relations needed to give the correct Hilbert function) are recovered by insisting on a simultaneous coaction on some other algebra; in this case it is the "algebra of differentials with constant coefficients"

$$B_{q,\lambda} := k[\xi_1, ..., \xi_n] \subset \Omega^*_{q,\lambda}. \tag{1.1.4}$$

Maltsiniotis [M1] went on to show that there is a DGA bialgebra $\Omega^*(M_{n,q,\lambda})$ which is a flat deformation of $\Omega^*(M_n)$, and which coacts, as a DGA, on $\Omega^*_{q,\lambda} = \Omega^*(\mathbf{A}^n_{q,\lambda})$. This result was later generalized by Sudbery [Su2].

We are also interested in another type of deformation of $A$. Assume $n \geq 2$, and consider the projective space $\mathbf{P} = \mathbf{P}^{n-1}_k$, which we identify with Proj $A$. We also identify $\mathrm{PGL}_n(k)$ with $\mathrm{Aut}_k(\mathbf{P}^{n-1}_k)$.

Definition 1.1.2. Given $\sigma \in \mathrm{PGL}_n(k)$, let $A_\sigma$ be the *skew homogeneous coordinate ring* associated to the triple $(\mathbf{P}, \sigma, \mathcal{O}_{\mathbf{P}}(1))$ (see [AV; Ye1, Section 6]).

For $\sigma = 1$ we have $A_\sigma = A$, and as $\sigma$ varies this is a flat deformation. The algebra $A_\sigma$ is called a *projective linear deformation* of $A$. In degree 1 there is equality: $(A_\sigma)_1 = \Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(1)) = A_1$, so the elements $x_1, ..., x_n$ belong to $A_\sigma$. In fact the $x_i$ generate $A_\sigma$; see Section 3.4.

## 1.2. *First Results*

Denote the parameter space of the quantum deformations $\mathrm{Spec}\, k[q_{11}, ..., q_{nn}]/(q_{ij}q_{ji} - 1, q_{ii} - 1)$ by $Q_n$. Let us write $A_q$ for a deformation of either type: $q \in Q_n(k)$ or $q \in \mathrm{PGL}_n(k)$. As mentioned earlier, the work of previous authors provided a differential calculus over the $k$-algebra $A_q$. We shall introduce *integration* into the picture. It is well known that algebraically, the only integration is contour integration of meromorphic forms, i.e., *residues*. In algebraic geometry the residue map plays a role in Serre duality (and more generally, Grothendieck duality). In [Ye1] the formalism of duality was extended to noncommutative algebras, and we will relate this duality of the integration over $A_q$ in Section 1.3. In this section we start by describing the integral.

Here are some conventions on gradings to be used throughout the paper. Let $M = \bigoplus_{(i,j) \in \mathbb{Z}^2} M^{(i,j)}$ be a $\mathbb{Z}^2$-graded abelian group. Define a $\mathbb{Z}$-grading on $M$, called total degree, by setting $M_l := \bigoplus_{i+j=l} M^{(i,j)}$, $l \in \mathbb{Z}$. Define another $\mathbb{Z}$-grading, called dimension, by setting $M^i := \bigoplus_j M^{(i,j)}$, $i \in \mathbb{Z}$. Finally define a $\mathbb{Z}/(2)$-grading on $M$, called parity, by setting $M^{[l]} := \bigoplus_{i \in [l]} M^i$, $[l] = l + 2\mathbb{Z} \in \mathbb{Z}/(2)$. The various gradings are denoted by $\deg_{\mathbb{Z}^2}$, deg, dim, and par, respectively. The base ring $k$ is always assumed to be pure of $\deg_{\mathbb{Z}^2} = 0$. Unless otherwise sdpecified, if $M$ is a $\mathbb{Z}$-graded group, then this grading will be the total degree, and $M$ will be pure of $\dim = 0$ (hence, even).

The de Rham complex $\Omega^* = \Omega^*_{A/k}$ is a $\mathbb{Z}^2$-graded $k$-algebra, with $\deg_{\mathbb{Z}^2} x_i = (0, 1)$. The derivative d has degree $(1, -1)$, so $\deg_{\mathbb{Z}^2}(\mathrm{d}x_i) = (1, 0)$. Write $\Omega^{(i,j)}$ for the $(i, j)$th-homogeneous component of $\Omega^*$; so according to the conventions $\Omega^i = \bigoplus_j \Omega^{(i,j)}$, and $\Omega^* = \bigoplus_i \Omega^i$ is the decomposition w.r.t. dimension. In terms of superalgebras, $\Omega^*$ is supercommutative, and d is an odd derivarion.

The Euler derivation on $A$ is multiplication by the total degree: $a \mapsto ia$ for $a \in A_i$. It corresponds to the $A$-linear map $\eta : \Omega^* \to \Omega^{*-1}$, $\eta(\mathrm{d}x_i) = x_i$, which is in fact an odd derivation. Moreover, $\eta^2 = 0$, and $(\Omega^*, \eta)$ is a Koszul complex for $k = A/\mathfrak{m}$, where $\mathfrak{m} = (x_1, \ldots, x_n)$ is the augmentation ideal. Using $(\Omega^*, \eta)$ to compute $\mathrm{Ext}^{\cdot}_A(k, \Omega^n)$, the identity map $\Omega^n \to \Omega^n$ represents a nonzero element of $\mathrm{Ext}^n_A(k, \Omega^n)$, and hence a canonical isomorphism $\int : \mathrm{Ext}^n_A(k, \Omega^n) \xrightarrow{\simeq} k$. The theorem below says that all this structure deforms.

THEOREM 1.2.1. *Consider a deformation $A_q$ of $A$, with either $q \in Q_n(k)$ or $q \in \mathrm{PGL}_n(k)$. Then there is a $\mathbb{Z}^2$-graded k-algebra $\Omega^*_q = \bigoplus_{(i,j) \in \mathbb{Z}^2} \Omega^{(i,j)}_q$, and k-linear endomorphisms* d, $\eta$ *of $\Omega^*_q$, having the following properties*:

(1) *$\Omega^0_q = A_q$ as $\mathbb{Z}$-graded k-algebras, with respect to total degree.*

(2) *The endomorphisms* d, $\eta$ *are homogeneous of degrees $(1, -1)$, $(-1, 1)$, respectively. Moreover,* d *and* $\eta$ *are odd derivations,* $\mathrm{dd} = \eta\eta = 0$, *and* $[\mathrm{d}, \eta]_{\text{super}} = \mathrm{d}\eta + \eta\mathrm{d}$ *is multiplication by the total degree.*

(3) *Let $B_q = \bigoplus_{i=0}^n B^i_q$ be the k-subalgebra of $\Omega^*_q$ generated by* $\mathrm{d}x_1, \ldots, \mathrm{d}x_n$. *Then multiplication induces isomorphisms*

$$\Omega^*_q \cong A_q \otimes_k B_q \cong B_q \otimes_k A_q.$$

*The structures* $(\Omega^*_q, \mathrm{d}, \eta)$ *form a $\mathbb{Z}^2$-graded flat family over the parameter space, specializing to* $(\Omega^*, \mathrm{d}, \eta)$.

The proof of the theorem is in Section 2.2 for $q \in Q_n(k)$, and in Section 3.4 for $q \in \mathrm{PGL}_n(k)$.

The DGA $\Omega^*_q$ is called the *canonical deformation* of $\Omega^*$. For $q \in Q_n(k)$ it is no other than the deformation $\lambda = 1$ of (1.1.1). Property (3) is what Manin calls a skew product. It guarantees that the $\Omega^i_q$ are free left and right

$A_q$-modules. By (2) we have $\eta(A_q)=0$, so $\eta$ is a map of $A_q$-bimodules. It follows that $(\Omega_q^*, \eta)$ is a Koszul complex for $k = A_q/\mathfrak{m}_q$, and we get the following.

COROLLARY 1.2.2. *The derivation $\eta$ induces a canonical isomorphism*

$$\int : \mathrm{Ext}^n_{A_q}(k, \Omega^n_q) \xrightarrow{\simeq} k. \tag{1.2.1}$$

We are using the notational conventions of [Ye1]; accordingly $\mathrm{Hom}_{A_q}(-,-)$ means Hom as $\mathbb{Z}$-graded left $A_q$-modules, and similarly for Ext. The module $\Omega^n_q$ is $\mathbb{Z}$-graded by total degree, so its generator $\mathrm{d}x_1 \cdots \mathrm{d}x_n$ has $\deg = n$. There is a left–right symmetry for the integral, which shall be discussed in Section 4.1.

A few words about *flatness*. By a "flat deformation" of a graded $k$-algebra we simply mean that the Hilbert function remains unchanged. In the examples considered in this paper there is actually a flat family of algebras over the parameter space, in the sence of algebraic geometry. Thus if $Q$ is the parameter space, there is a flat graded $\mathcal{O}_Q$-algebra $\mathbf{A}$, such that the fibre $\mathbf{A} \otimes k(q) = A_q$ for every $q \in Q(k)$. The canonical deformation of the de Rham complex is also at flat family $(\mathbf{\Omega}^*, \mathrm{d}, \eta)$. One can show that the integral is defined globally: there is an isomorphism

$$\int : \mathbf{Ext}^n_{\mathbf{A}}(\mathcal{O}_Q, \mathbf{\Omega}^n) \xrightarrow{\simeq} \mathcal{O}_Q \tag{1.2.2}$$

of sheaves over $Q$, specializing to (1.2.1) for every $q \in Q(k)$. For a discussion of this, turn to Subsections 2.2, 3.2, and 4.1.

1.3. *Balanced Dualizing Complexes*

Let $C = \bigoplus_{i \geq 0} C_i$ be a noetherian (noncommutative) $\mathbb{Z}$-graded $k$-algebra, with $C_0 = k$. In [Ye1] the notion of a dualizing complex over $C$ was introduced. Denote by $C^{\circ}$ (resp. $C^{\mathrm{e}}$) the opposite (resp. enveloping) algebra of $C$; thus $C^{\mathrm{e}} = C \otimes_k C^{\circ}$. Right $C$-modules (resp. $C$-bimodules) are identified with left $C^{\circ}$ (resp. $C^{\mathrm{e}}$)-modules. Let $\mathbf{GrMod}(C)$ be the category of graded left $C$-modules and degree 0 homomorphisms, and let $\mathbf{D}(C)$ be its derived category. In $\mathbf{D}(C)$ there are the standard full subcategories, such as $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(C)$: bounded complexes with coherent (=finitely generated) cohomology modules. Let $\mathrm{Hom}_C(M, N)_i$ denote the group of degree $i$ homomorphisms between the modules $M$ and $N$, and let $\mathrm{Hom}_C(-,-) := \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_C(-,-)_i$. So $\mathrm{Hom}_C(-,-)_0$ is the categorical Hom in $\mathbf{GrMod}(C)$. Define $\mathrm{RHom}_C^{\cdot}$ to be the derived functor of $\mathrm{Hom}_C$, in its various possible domains and ranges. One possibility is:

$$\mathrm{RHom}^{\cdot}_C : \mathbf{D}(C)^{\circ} \times \mathbf{D}^+(C^{\mathrm{e}}) \to \mathbf{D}(C^{\circ}).$$

See [Ye1, Section 2] for details.

DEFINITION 1.3.1 ([Ye1, Definition 3.3]. Given a complex $R^{\cdot} \in \mathbf{D}^{+}(C^{e})$, define the associated duality functors

$$D := \mathrm{RHom}^{\cdot}_{C}(-, R^{\cdot}), \qquad D^{\circ} := \mathrm{RHom}^{\cdot}_{C^{\circ}}(-, R^{\cdot}).$$

The complex $R^{\cdot}$ is called dualizing if

(1) The functors $D$ and $D^{\circ}$ restrict to functors $D : \mathbf{D}^{b}_{c}(C)^{\circ} \to \mathbf{D}^{b}_{c}(C^{\circ})$ and $D^{\circ} : \mathbf{D}^{b}_{c}(C^{\circ})^{\circ} \to \mathbf{D}^{b}_{c}(C)$.

(2) The morphisms of adjunction $1 \to D^{\circ}D$ and $1 \to DD^{\circ}$ are isomorphisms on $\mathbf{D}^{b}_{c}(C)$ and $\mathbf{D}^{b}_{c}(C^{0})$, respectively.

It is known that a dualizing complex is unique, up to a shift in dimensions and tensoring with an invertible bimodule. To get rid of these indeterminacies one considers torsion. For a $C$-module $M$ let $\Gamma_{\mathfrak{m}}(M)$ be the submodule of left $\mathfrak{m}$-torsion elements, where $\mathfrak{m} \bigoplus_{i>0} C_i$ is the augmentation ideal. Let $\mathrm{R}\Gamma_{\mathfrak{m}}$ be its derived functor. Similarly we define $\mathrm{R}\Gamma_{\mathfrak{m}^{\circ}}$ as the derived functor of right $\mathfrak{m}$-torsion. Let $C' := \mathrm{Hom}_k(C, k)$, which is a $C$-bimodule.

DEFINITION 1.3.2 [Ye1, Definition 4.1]. A dualizing complex $R^{\cdot} \in \mathbf{D}^{+}(C)$ is called balanced if there exist isomorphisms

$$\mathrm{R}\Gamma_{\mathfrak{m}}(R^{\cdot}) \cong \mathrm{R}\Gamma_{\mathfrak{m}^{\circ}}(R^{\cdot}) \cong C'$$

in $\mathbf{D}(C^{e})$.

A balanced dualizing complex $R^{\cdot}$ is unique up to isomorphism [Ye1, Section 4]. Since $\mathrm{End}_{\mathbf{D}(C^{e})}(R^{\cdot}) = k$, an isomorphism $\tau : \mathrm{Ext}^{0}_{C}(k, R^{\cdot}) = \mathrm{H}^{0}\, Dk \xrightarrow{\cong} k$ completely rigidifies the complex. This rigidification is symmetric, since we obtain from it $\tau^{\circ} := D^{\circ}(\tau) : D^{\circ}k \xrightarrow{\cong} D^{\circ}\, Dk \cong k$.

Now let us consider the deformed algebra $A_q$, with either $q \in Q_n(k)$ or $q \in \mathrm{PGL}_n(k)$. Since $A_q$ is an Artin–Schelter regular algebra, it has a balanced dualizing complex (cf. [Ye1, Section 4]). In fact this complex can be chosen to be a single invertible bimodule. Our main result is the following.

THEOREM 1.3.3. *Let $\Omega^{*}_{q}$ be the canonical deformation of the Rham complex of Theorem* 1.2.1. *Then $\Omega^{n}_{q}[n]$ is a balanced dualizing complex over $A_q$.*

The proof of this therem is also in Section 2.2 for $q \in Q_n(k)$, and in Section 3.4 for $q \in \mathrm{PGL}_n(k)$.

The significance of the theorem is that the abstract invertible bimodule, whose existence is guaranteed by duality theory, is realized as:

$$\Omega^{n}_{q} = A_q \cdot \mathrm{d}x_1 \cdots \mathrm{d}x_n = \mathrm{d}x_1 \cdots \mathrm{d}x_n \cdot A_q \subset \Omega^{*}_{q}.$$

Observe that $\Omega_q^n[n]$ is rigidified canonically by the integral $\int$ of (1.2.1), which may be viewed as isomorphism $\int : \mathrm{RHom}^{\cdot}_{A_q}(k, \Omega_q^n[n]) \xrightarrow{\simeq} k$ in $\mathbf{D}(A_q^e)$.

The paper is organized as follows: Section 2 treats the case of the multiparameter quantum deformation. Section 3 treats the case of the projective linear deformation. Section 4 contains additional remarks and some questions.

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## 2. The Multiparameter Quantum Case

### 2.1. *Twisting by a 2-Cocycle*

Let $k$ be a field, let $G$ be an abelian group, and let $c : G \times G \to k^{\times}$ be a 2-cocycle; i.e.,

$$c(g, h)\, c(f, g+h) = c(f+g, h)\, c(f, g) \tag{2.1.1}$$

for all $f, g, h \in G$. Assume $c(0, 0) = 1$. Let $A = \bigoplus_{g \in G} A_g$ be a $G$-graded $k$-algebra. For $a \in A_g$, $a \neq 0$, write $\deg_G a = g$. Following [AST] we define a new $k$-algebra $A^{\sim c}$, the twist of $A$ by $c$, as follows. The underlying $k$-module of $A^{\sim c}$ is that of $A$. Multiplication is given by

$$a \odot b := c(f, g)\, a \cdot b, \qquad a \in A_f,\ b \in A_g. \tag{2.1.2}$$

The cocycle condition ensures the associativity of the product $\odot$. Note that twisting by a cocycle is a reversible operation: $A \cong (A^{\sim c})^{\sim c^{-1}}$.

We shall need to generalize the twisting operation. Say $M_1$, $M_2$, and $N$ are $G$-graded $k$-modules, and say dot: $M_1 \times M_2 \to N$, $(m_1, m_2) \mapsto m_1 \cdot m_2$, is a $G$-homogeneous, $k$-bilinear pairing. Set $M_i^{\sim c} := M_i$, $N^{\sim c} := N$, and define a pairing $\odot : M_1^{\sim c} \times M_2^{\sim c} \to N^{\sim c}$, the twist of dot by $c$, as in (2.1.2). The pairing dot may be viewed as the universal pairing $M_1 \times M_2 \to M_1 \otimes_k M_2$, followed by a degree 0, $k$-linear homomorphism $M_1 \otimes_k M_2 \to N$. Thus the universal twisted pairing is $\odot : M_1^{\sim c} \times M_2^{\sim c} \to (M_1 \otimes^k M_2)^{\sim c}$, $m_1 \odot m_2 = c(f, g)\, m_1 \otimes m_2$. The twisted pairing is associative; let $M_3$ be another $G$-graded $k$-module. Then the cocyle condition guarantees that $m_1 \odot (m_2 \odot m_3) = (m_1 \odot m_2) \odot m_3 \in (M_1 \otimes_k M_2 \otimes_k M_3)^{\sim c}$ for any $m_i \in M_i$.

Denote by $\mathbf{GrMod}(G; A)$ the category of $G$-graded left $A$-modules and degree 0 homomorphisms.

Lemma 2.1.1. *Any $G$-graded left $A$-module $M$ can be twisted into a left $A^{\sim c}$-module $M^{\sim c}$, by modifying the multiplication as in* (2.1.2). *The assignment $M \mapsto M^{\sim c}$ is an equivalence $\mathbf{GrMod}(G; A) \to \mathbf{GrMod}(G; A^{\sim c})$. Similarly for right modules and bimodules.*

Given $g \in G$ and a graded $k$-module $M$, its twist by $g$ is the graded module $M(g)$ such that $M(g)_f = M_{g+f}$. Let $N$ be another $G$-graded left $A$-module. For $g \in G$ let $\mathrm{Hom}_A(M, N)_g$ be the set of degree $g$ $A$-linear maps, and let $\mathrm{Hom}_A(M, N) := \bigoplus_{g \in G} \mathrm{Hom}_A(M, N)_g$, a $G$-graded $k$-module. We shall view $\mathrm{Hom}_A(M, N)$ as acting on the right. In other words, the action is a pairing

$$M \times \mathrm{Hom}_A(M, N) \to N. \tag{2.1.3}$$

This allows $A$ to act on the left, as it should. This pairing is evidently of the kind considered above, and so it may be twisted by $c$. Thus for every $\phi \in \mathrm{Hom}_A(M, N)$ we get a $k$-linear map

$$M^{\sim c} \to N^{\sim c}, \qquad m \mapsto m \odot \phi. \tag{2.1.4}$$

If $M$ (resp. $N$, resp. $M$ and $N$) happens to be a bimodule, then $\mathrm{Hom}_A(M, N)$ is endowed with a left (resp. right, resp. bi-) $A$-module structure.

LEMMA 2.1.2. *Let $M$ and $N$ be $G$-graded left $A$-modules. The action of* (2.1.4) *gives a bijection*

$$H_c : \mathrm{Hom}_A(M, N)^{\sim c} \xrightarrow{\simeq} \mathrm{Hom}_{A^{\sim c}}(M^{\sim c}, N^{\sim c}).$$

*If $M$ (resp. $N$) is a bimodule then this is an isomorphism of left (resp. right) $A^{\sim c}$-modules.*

*Proof.* All assertions boil down to the associativity of $\odot$. Consider $\phi \in \mathrm{Hom}_A(M, N)^{\sim c}$; we must show that the map $M^{\sim c} \to N^{\sim c}$, $m \mapsto m \odot \phi$, is (left) $A^{\sim c}$-linear. But given $a \in A^{\sim c}$ we have $(a \odot m) \odot \phi = a \odot (m \odot \phi)$. We leave the rest to the reader. ▮

Suppose $\phi \in \mathrm{Hom}_A(M, N)_0$, so that $\phi : M \to N$ is a morphism in **GrMod**$(G; A)$. By assumption $c(0, 0) = 1$, and a simple calculation shows that $c(g, 0) = c(0, g) = 1$ for all $g \in G$. Therefore, as maps of $k$-modules, $H_c(\phi) = \phi : M^{\sim c} = M \to N^{\sim c} = N$, in accordance with Lemma 2.1.1.

Let $\mathrm{Ext}^i_A(-, -)$ be the derived functors of $\mathrm{Hom}_A(-, -)$ on the categories of graded modules.

PROPOSITION 2.1.3. *Let $M$ and $N$ be $G$-graded $A$-bimodules. Then for all $i \geq 0$ there is an isomorphism of $G$-graded $A^{\sim c}$-bimodules*

$$H_c : \mathrm{Ext}^i_A(M, N)^{\sim c} \xrightarrow{\simeq} \mathrm{Ext}^i_{A^{\sim c}}(M^{\sim c}, N^{\sim c}).$$

*This isomorphism is natural in $M$ and $N$, and for $i = 0$ it coincides with $H_c$ of Lemma* 2.1.2.

*Proof.* We may calculate $\mathrm{Ext}^i_A(M, N)$ either using a projective resolution $\cdots \to P^{-1} \to P^0 \to M \to 0$, or using an injective resolution $0 \to N \to I^0 \to I^1 \to \cdots$ in $\mathbf{GrMod}(G; A)$. Now $\cdots \to (P^{-1})^{\sim c} \to (P^0)^{\sim c} \to M^{\sim c} \to 0$ is a projective resolution of $M^{\sim c}$ in $\mathbf{GrMod}(G; A^{\sim c})$. By Lemma 2.1.2 we get isomorphisms of right $A^{\sim c}$-modules $H_c: \mathrm{Hom}_A(P^i, N)^{\sim c} \xrightarrow{\simeq} \mathrm{Hom}_{A^{\sim c}}((P^i)^{\sim c}, N^{\sim c})$, which pass on to the $\mathrm{Ext}^i$. The same $H_c$ are obtained using the injective resolution, and now they are isomorphisms of left $A^{\sim c}$-modules. ▮

Given $M \in \mathbf{GrMod}(G; A^e)$ and $V \in \mathbf{GrMod}(G; k)$, the graded $k$-module $\mathrm{Hom}_k(M, V)$ has a structure of an $A$-bimodule. Let us write $\phi \cdot m = m \cdot \phi = \phi(m) \in V$ for $\phi \in \mathrm{Hom}_k(M, V)$ and $m \in M$. In terms of $G$-graded $k$-bilinear pairings, for any $a, b \in A$ there is associativity, and also cyclic commutativity: $a \cdot \phi \cdot b \cdot m = \phi \cdot b \cdot m \cdot a = \cdots$.

PROPOSITION 2.1.4. *Suppose the cocycle $c$ satisfies $c(g, -g) = 1$ for all $g \in G$. Assume that $V$ is a $G$-graded $k$-module, pure of degree 0 (i.e., $V_g = 0$ if $g \neq 0$). Then there is an isomorphism of $A^{\sim c}$-bimodules,*

$$I_c : \mathrm{Hom}_k(M, V)^{\sim c} \xrightarrow{\simeq} \mathrm{Hom}_k(M^{\sim c}, V),$$

*given by $I_c(\phi)(m) = \phi \odot m = m \odot \phi$, for $m \in M$, $\phi \in \mathrm{Hom}_k(M, V)$.*

*Proof.* Say $\phi \in \mathrm{Hom}_k(M, V)_g$ and $m \in M_{-g}$. Then

$$\phi \odot m = c(g, -g)\phi \cdot m = \phi \cdot m = m \cdot \phi = c(-g, g) m \cdot \phi = m \odot \phi \qquad (2.1.5)$$

so $I_c$ is well defined. Next let $\phi \in \mathrm{Hom}_k(M, V)_g$, let $a \in A_f$, and let $m \in M_{-f-g}$. Then

$$\begin{aligned} I_c(a \odot \phi)(m) &= (a \odot \phi) \odot m \\ &= a \odot (\phi \odot m) && \text{(associativity)} \\ &= (\phi \odot m) \odot a && \text{(as in (2.15))} \\ &= \phi \odot (m \odot a) && \text{(associativity)} \\ &= (a \odot I_c(\phi))(m). \end{aligned}$$

So $I_c$ is left $A^{\sim c}$-linear and similarly for right linearity. ▮

2.2. *Proofs of Theorems* 1.2.1 *and* 1.3.3

Let $A = k[x_1, ..., x_n]$ be a commutative polynomial ring over the field $k$. Let $\Omega^* = \Omega^*_{A/k}$ be the de Rham complex of $A$ relative to $k$, with exterior derivative d and Euler derivative $\eta$. Given $q = [q_{ij}] \in Q_n(k)$, define $\Omega^*_q$ to be the algebra $\Omega^*_{q,\lambda}$ of (1.1.1), with $\lambda = 1$. Thus $\Omega^*_q$ has generators $x_1, ..., x_n, \xi_1, ..., \xi_n$ and relations

$$x_j x_i - q_{ji} x_i x_j, \qquad \xi_j \xi_i + q_{ji} \xi_i \xi_j, \qquad \xi_i^2, \qquad x_j \xi_i - q_{ji} \xi_i x_j \tag{2.2.1}$$

for $i, j = 1, ..., n$. In particular, taking $q_{ij} = 1$, the original de Rham complex $\Omega^*$ is recovered, with $\xi_i = \mathrm{d}\, x_i$.

The free algebra $k\langle x_1, ..., x_n, \xi_1, ..., \xi_n \rangle$ is graded by the group $G = \bigoplus_{i=1}^n \mathbb{Z} \cdot g_i \cong \mathbb{Z}^n$, with $\deg_G x_i = \deg_G \xi_i = g_i$. It is also $\mathbb{Z}^2$-graded with $\deg_{\mathbb{Z}^2} x_i = (0, 1)$ and $\deg_{\mathbb{Z}^2} \xi_i = (1, 0)$. Since the relations (2.2.1) are homogeneous with respect to both the $G$-grading and the $\mathbb{Z}^2$-grading, these gradings pass on to the quotient $\Omega^*_q$.

According to [AST, Proposition 1 and Lemma 4], there exists a 2-cocycle $c : G \times G \to k^\times$ such that

$$\frac{c((g_i, g_j)}{c(g_j, g_i)} = q_{ij}, \qquad i, j = 1, ..., n, \tag{2.2.2}$$

$$c(g, -g) = 1, \qquad g \in G. \tag{2.2.3}$$

Fix one such cocycle.

Twisting $\Omega^*$ by $c$ we get a $G$-graded $k$-algebra $(\Omega^*)^{\sim c}$. The $\mathbb{Z}^2$-grading on $(\Omega^*)^{\sim c}$ (and, hence, parity) is retained. We claim that $\mathrm{d}, \eta : (\Omega^*)^{\sim c} \to (\Omega^*)^{\sim c}$ are still odd derivations. Let us verify this for d. It is $k$-linear of $G$-degree 0, so for $\alpha, \beta \in (\Omega^*)^{\sim c}$ with $\deg_G \alpha = f$, $\deg_G \beta = g$, and $\operatorname{par} \alpha = i$ (mod 2), it holds:

$$\begin{aligned} \mathrm{d}(\alpha \odot \beta) &= \mathrm{d}(c(f, g) \alpha \cdot \beta) \\ &= c(f, g)(\mathrm{d}(\alpha) \cdot \beta + (-1)^i \alpha \cdot \mathrm{d}(\beta)) \\ &= \mathrm{d}(\alpha) \odot \beta + (-1)^i \alpha \odot \mathrm{d}(\beta). \end{aligned}$$

Since d and $\eta$ are odd derivations, $[\mathrm{d}, \eta]_{\text{super}} = \mathrm{d}\eta + \eta \mathrm{d}$ is an even derivation. Clearly $\alpha \mapsto (\deg \alpha) \cdot \alpha$ is an even derivation. In order to prove these two derivations are equal it suffices to check on the generators $x_1, ..., \xi_1, ...$. But the generators have total degree 1, and

$$\begin{aligned} (\mathrm{d}\eta + \eta \mathrm{d})(x_i) &= \eta \mathrm{d}(x_i) = x_i, \\ (\mathrm{d}\eta + \eta \mathrm{d})(\xi_i) &= \mathrm{d}\eta(\xi_i) = \xi_i. \end{aligned}$$

From Formula (2.2.2) and the equality of the Hilbert functions we get a $k$-algebra isomorphism $\Psi : \Omega^*_q \xrightarrow{\simeq} (\Omega^*)^{\sim c}$, with $\Psi(x_i) = x_i$, $\Psi(\xi_i) = \xi_i$.

Here is the proof of Theorem 1.2.1; property (1) is verified directly. Define d and $\eta$ on $\Omega^*_q$ using the isomorphism $\Psi$; so property (2) holds.

As for property (3), it holds for $q_{ij}=1$; $A \otimes_k B \cong B \otimes_k A \cong \Omega^*$ via multiplication. Since $A$ and $B$ are $G$-graded, it follows that $A^{\sim c} \otimes_k B^{\sim c} \cong B^{\sim c} \otimes_k A^{\sim c} \cong (\Omega^*)^{\sim c}$ (although the maps on the underlying $k$-modules depend on $c$!).

As for Theorem 1.3.3, it is equivalent to the statement that $(\Omega^n)^{\sim c}[n]$ is a balanced dualizing complex over $A^{\sim c}$. According to [Ye1; Theorem 4.8] it suffices to show that

$$\operatorname{Ext}^n_{A^{\sim c}}(A^{\sim c}/(\mathfrak{m}^{\sim c})^{i+1}, (\Omega^n)^{\sim c}) \cong \operatorname{Hom}_k(A^{\sim c}/(\mathfrak{m}^{\sim c})^{i+1}, k)$$

as $A^{\sim c}$-bimodules, for $i=0, 1$. Here $\mathfrak{m}^{\sim c} \subset A^{\sim c}$ is the augmentation ideal. As $A^{\sim c}$-bimodules, $A^{\sim c}/(\mathfrak{m}^{\sim c})^{i+1} \cong (A/\mathfrak{m}^{i+1})^{\sim c}$, and

$$\begin{aligned}
&\operatorname{Ext}^n_{A^{\sim c}}((A/\mathfrak{m}^{i+1})^{\sim c}, (\Omega^n)^{\sim c}) \\
&\quad \cong \operatorname{Ext}^n_A(A/\mathfrak{m}^{i+1}, \Omega^n)^{\sim c} && \text{(via } H_c\text{)} \\
&\quad \cong \operatorname{Hom}_k(A/\mathfrak{m}^{i+1}, k)^{\sim c} && (\Omega^n \text{ is balanced over } A) \\
&\quad \cong \operatorname{Hom}_k((A/\mathfrak{m}^{i+1})^{\sim c}, k) && \text{(via } I_c\text{)}.
\end{aligned}$$

To conclude this section, let us describe the flat family $\mathbf{\Omega}^*$ on $Q_n$. Consider the free algebra $\mathcal{O}_{Q_n}\langle x_1, ..., \xi_1, ...\rangle$, and in it the ideal sheaf $\mathbf{I}$ generated by the relations (2.2.1). Let $\mathbf{\Omega}^*$ be the quotient $\mathcal{O}_{Q_n}\langle x_1, ..., \xi_1, ...\rangle/\mathbf{I}$. For each closed point $q \in Q_n$ we get $\mathbf{\Omega}^* \otimes k(q) = [\Omega^* \otimes_k k(q)]_q$, and this implies that $\mathbf{\Omega}^*$, being a direct sum of coherent sheaves, is flat over $Q_n$.

## 3. The Projective Linear Case

### 3.1. *Equivariant Sheaves on* $P^n_k$

Let $X$ be a scheme over the field $k$, let $\mathcal{L}$ be an invertible sheaf on $X$ and let $\sigma \in \operatorname{Aut}_k(X)$. For an $\mathcal{O}_X$-module $\mathcal{M}$ write $\mathcal{M}^\sigma := \sigma^*\mathcal{M}$. Given $i \in \mathbb{Z}$, define the $i$th twist of $\mathcal{M}$ (with respect to $\mathcal{L}$ and $\sigma$) to

$$\mathcal{M}(i) = \mathcal{M}(i, \sigma, \mathcal{L}) := \begin{cases} \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \cdots \otimes \mathcal{L}^{\sigma^{i-1}} \otimes \mathcal{M}^{\sigma^i} & \text{if } i>0 \\ \mathcal{M} & \text{if } i=0 \\ \mathcal{L}^{-\sigma^i} \otimes \cdots \otimes \mathcal{L}^{-\sigma^{-2}} \otimes \mathcal{L}^{-\sigma^{-1}} \otimes \mathcal{M}^{\sigma^i} & \text{if } i<0 \end{cases}$$

and set

$$\Gamma^\sigma_* \mathcal{M} := \bigoplus_{i \in \mathbb{Z}} \Gamma(X, \mathcal{M}(i, \sigma, \mathcal{L})).$$

Then $A_\sigma := \Gamma^\sigma_* \mathcal{O}_X$ is the skew homogeneous coordinate ring associated to the triple $(X, \mathcal{L}, \sigma)$ (cf. [AV; Ye1, Section 6]), and the assignment $\mathcal{M} \mapsto \Gamma^\sigma_* \mathcal{M}$ is a functor $\mathbf{Mod}(X) \to \mathbf{GrMod}(A_\sigma)$.

A $\sigma$-equivariant $\mathcal{O}_X$-module is a pair $(\mathcal{M}, e)$, where $\mathcal{M}$ is an $\mathcal{O}_X$-module, and $e : \sigma^* \mathcal{M} \xrightarrow{\simeq} \mathcal{M}$ is an $\mathcal{O}_X$-linear isomorphism. The $\sigma$-equivariant $\mathcal{O}_X$-modules form an abelian category. An equivariance $e : \mathcal{M}^\sigma \xrightarrow{\simeq} \mathcal{M}$ induces, for every integer $i$, an equivariance $e^{\sigma^i} : \mathcal{M}^{\sigma^{i+1}} \xrightarrow{\simeq} \mathcal{M}^{\sigma^i}$; by abuse of notation we shall write $e = e^{\sigma^i}$. Hence, for any $i, j \in \mathbb{Z}$ there is an isomorphism $\sigma^{i-j} : \mathcal{M}^{\sigma^j} \xrightarrow{\simeq} \mathcal{M}^{\sigma^i}$.

An equivariance $e$ on $\mathcal{M}$ induces a $k$-linear isomorphism

$$\gamma := e\sigma^* : \Gamma(X, \mathcal{M}) \to \Gamma(X, \sigma^* \mathcal{M}) \to \Gamma(X, \mathcal{M}).$$

In [GIT] $\gamma$ is called the dual action of $\sigma$.

In [Ye1, Lemma 6.16] it was shown that an equivariance $e$ on an $\mathcal{O}_X$-module $\mathcal{M}$ induces an $A_\sigma$-bimodule structure on $\Gamma^\sigma_* \mathcal{M}$, and that $\Gamma^\sigma_*$ is a functor ($\sigma$-equivariant $\mathcal{O}_X$-modules) $\to \mathbf{GrMod}(A^e_\sigma)$. We shall require an extension of this result to $\sigma$-equivariant $\mathcal{O}_X$-algebras, which are pairs $(\mathcal{C}, e)$, with $\mathcal{C}$ an $\mathcal{O}_X$-algebra, and with $e : \sigma^* \mathcal{C} \xrightarrow{\simeq} \mathcal{C}$ an algebra isomorphism. Similarly, we define equivariant $\mathcal{C}$-modules.

LEMMA 3.1.1. *Let $(\mathcal{C}, e)$ be a $\sigma$-equivariant $\mathcal{O}_X$-algebra. Then $\Gamma^\sigma_*(\mathcal{C}, e)$ is a graded $k$-algebra. Let $(\mathcal{M}, e_{\mathcal{M}})$ be a $\sigma$-equivariant $\mathcal{C}$-module. Then $\Gamma^\sigma_*(\mathcal{M}, e_{\mathcal{M}})$ is a graded $\Gamma^\sigma_*(\mathcal{C}, e)$-bimodule. $\Gamma^\sigma_*$ is functorial in $(\mathcal{C}, e)$ and $(\mathcal{M}, e_{\mathcal{M}})$.*

*Proof.* The product on $\Gamma^\sigma_*(\mathcal{C}, e)$ is

$$\begin{aligned}
\Gamma(X, \mathcal{C}(i)) \times \Gamma(X, \mathcal{C}(j)) &\xrightarrow{1 \times (\sigma^*)^i} \Gamma(X, \mathcal{C}(i)) \times \Gamma(X, \mathcal{C}(j)^{\sigma^i}) \\
&\xrightarrow{\otimes} \Gamma(X, \mathcal{O}_X(i+j) \otimes \mathcal{C}^{\sigma^i} \otimes \mathcal{C}^{\sigma^{i+j}}) \\
&\xrightarrow{1 \otimes e^{-j} \otimes 1} \Gamma(X, \mathcal{O}_X(i+j) \otimes \mathcal{C}^{\sigma^{i+j}} \otimes \mathcal{C}^{\sigma^{i+j}}) \\
&\xrightarrow{1 \otimes \mathrm{mult}} \Gamma(X, \mathcal{O}_X(i+j) \otimes \mathcal{C}^{\sigma^{i+j}}) \\
&= \Gamma(X, \mathcal{C}(i+j)), \qquad (3.1.1)
\end{aligned}$$

where mult is the product in $\mathcal{C}$ (cf. [Ye1, Formula (6.15)]). Associativity is proved as in [Ye1, Lemma 6.16] and similarly for modules. ∎

Now let $X = \mathbf{P} := \mathbf{P}^n_k$, $n \geqslant 1$, and let $\mathcal{L} := \mathcal{O}_{\mathbf{P}}(1)$. Changing the setup slightly from Section 1.1, we set $A := k[x_0, ..., x_n]$, and identify $\mathbf{P} = \operatorname{Proj} A$. Consider $\Omega^* = \Omega^*_{A/k}$ as a $\mathbb{Z}$-graded $A$-algebra, with respect to total degree:

$(\Omega^*)_l = \bigoplus_{i+j=l} \Omega^{(i,j)}$. Let $\mathscr{C}^\cdot = \bigoplus_{i=0}^{n+1} \mathscr{C}^i$ be the sheafification on $\mathbf{P}$ of $\Omega^*$. It is an $\mathscr{O}_{\mathbf{P}}$-algebra. Since the Euler derivation $\eta$ has total degree 0, it passes to an $\mathscr{O}_{\mathbf{P}}$-linear, odd derivation $\tilde{\eta} : \mathscr{C}^\cdot \to \mathscr{C}^{\cdot -1}$. Note that as a complex, $(\mathscr{C}^\cdot, \tilde{\eta})$ is exact.

Choose an equivariance $e : \sigma^*\mathscr{O}_{\mathbf{P}}(1) \xrightarrow{\simeq} \mathscr{O}_{\mathbf{P}}(1)$. Under the identifications $\mathrm{Aut}_k(\mathbf{P}) = \mathrm{PGL}_{n+1}(k)$ and $\mathrm{Aut}_k(A_1) = \mathrm{Aut}_k(\Gamma(X, \mathscr{O}_{\mathbf{P}}(1))) = \mathrm{GL}_{n+1}(k)$, we get an element $\gamma := e\sigma^* \in \mathrm{GL}_{n+1}(k)$, lifting $\sigma$. $\gamma$ extends to an isomorphism $\gamma : \Omega^* \xrightarrow{\simeq} \Omega^*$ which commutes with d. This allows us to define a canonical equivariance $\varepsilon_{\mathscr{C}}$ on $\mathscr{C}^\cdot$. Let $P \in \mathbf{P}$. Any section of $\sigma^*\mathscr{C}^i = \mathscr{O}_{\mathbf{P}} \otimes_{\sigma^{-1}\mathscr{O}_{\mathbf{P}}} \sigma^{-1}\mathscr{C}^i$ near $P$ is a sum of sections of the form $b \otimes a^{-1}\alpha$, with $b$ a section of $\mathscr{O}_{\mathbf{P}}$ near $P$, with $a \in A_l$ not vanishing at $\sigma(P)$ and with $\alpha \in (\Omega^i)_l = \Omega^{i,l-i}$. Define $\varepsilon_{\mathscr{C}}(b \otimes a^{-1}\alpha) := b\gamma(a)^{-1}\gamma(\alpha)$, which is a section of $\mathscr{C}^i$ near $P$. This map is well defined and is an isomorphism of sheaves. Furthermore, since $e$ and $\gamma$ are determined only up to a scalar in $k^\times$, this indeterminacy is cancelled out, so $\varepsilon_{\mathscr{C}}$ is canonical. On $\mathscr{C}^0 = \mathscr{O}_{\mathbf{P}}$ the equivariance $\varepsilon_{\mathscr{C}}$ is the identity, so $\Gamma_*^\sigma(\mathscr{C}^0, \varepsilon_{\mathscr{C}}) = \Gamma_*^\sigma \mathscr{O}_{\mathbf{P}} = A_\sigma$.

DEFINITION 3.1.2. The canonical deformation of $(\Omega^*, \eta)$ at $\sigma \in \mathrm{Aut}_k(\mathbf{P}_k^n)$ is the pair $(\Omega_\sigma^*, \eta)$, where $\Omega_\sigma^*$ is the $\mathbb{Z}^2$-graded $k$-algebra $\Gamma_*^\sigma(\mathscr{C}^\cdot, \varepsilon_{\mathscr{C}})$ and where $\eta$ is the odd derivation $\Gamma_*^\sigma(\tilde{\eta}) : \Omega_\sigma^* \to \Omega_\sigma^{*-1}$.

The Euler sequence on $\mathbf{P}_k^n$ is the exact sequence of sheaves

$$0 \to \Omega_{\mathbf{P}/k}^1 \to \mathscr{C}^1 \xrightarrow{\eta} \mathscr{O}_{\mathbf{P}} \to 0, \tag{3.1.2}$$

where $\Omega_{\mathbf{P}/k}^1 \to \mathscr{C}^1$ comes from the derivation $\mathscr{O}_{\mathbf{P}} \to \mathscr{C}^1$, which is the sheafification of the de Rham derivative $\mathrm{d} : A \to \Omega^1$ (cf. [Li, Section 8]). Denoting by $\varepsilon$ the canonical $\sigma$-equivariance on $\Omega_{\mathbf{P}/k}^*$, we find that the sequence (3.1.2) is $\sigma$-equivariant. Taking top exterior powers there is a canonical isomorphism of $\sigma$-equivariant $\mathscr{O}_{\mathbf{P}}$-modules

$$(\mathscr{C}^{n+1}, \varepsilon_{\mathscr{C}}) \cong (\Omega_{\mathbf{P}/k}^n, \varepsilon), \tag{3.1.3}$$

and an application of the functor $\Gamma_*^\sigma$ gives an isomorphism of $A_\sigma$-bimodules

$$\Omega_\sigma^{n+1} = \Gamma_*^\sigma(\mathscr{C}^{n+1}, \varepsilon_{\mathscr{C}}) \cong \Gamma_*^\sigma(\Omega_{\mathbf{P}/k}^n, \varepsilon). \tag{3.1.4}$$

### 3.2. *The Sheaf* $\mathbf{\Omega}^*$ *on* $\mathrm{PGL}_n$

In this subsection we indicate a global approach to the deformations $A_\sigma$ and $\Omega_\sigma^*$. Let $G := \mathrm{PGL}_{n+1}$, $n \geq 0$, which as a scheme is the affine open subscheme of $\mathrm{Proj}\, k[a_{00}, ..., a_{nn}]$ defined by $\det[a_{ij}] \neq 0$. The action of $G$ on $\mathbf{P} = \mathbf{P}_k^n$ is a $k$-morphism $\mu : G \times \mathbf{P} \to \mathbf{P}$, determined by

$$\begin{aligned} \mu^*\mathcal{O}_{\mathbf{P}}(1) &\xrightarrow{\simeq} p_1^*\mathcal{O}_G(1)\otimes p_2^*\mathcal{O}_{\mathbf{P}}(1) \\ \mu^*(x_i) &\mapsto \sum_{j=0}^{n} a_{ij}\otimes x_j, \end{aligned} \tag{3.2.1}$$

where $p_1$ and $p_2$ are the projections from $G\times\mathbf{P}$ to its factors (see [GIT]).

For $l\in\mathbf{N}$ let $\mu_l : G\times\mathbf{P}\to\mathbf{P}$ correspond to $\sigma^l : \mathbf{P}\to\mathbf{P}$, $\sigma\in G(k)$. Thus $\mu_0=p_2$ and $\mu_1=\mu$. Define an $\mathcal{O}_G$-module

$$\mathbf{A}_l := p_{1*}[\mu_0^*\mathcal{O}_{\mathbf{P}}(1)\otimes\cdots\otimes\mu_{l-1}^*\mathcal{O}_{\mathbf{P}}(1)].$$

There is a product $\mathbf{A}_i\times\mathbf{A}_j\to\mathbf{A}_{i+j}$ given by applying $(p_1\times\mu_i)^*$ to the second factor and then tensoring. According to the cohomology and base change theorem, $\mathbf{A}:=\bigoplus_{l\in\mathbf{N}}\mathbf{A}_l$ is a flat $\mathcal{O}_G$-algebra. For each $\sigma\in G(k)$ one has $\mathbf{A}\otimes k(\sigma)=A_\sigma$.

Next define the $\mathcal{O}_G$-module,

$$\mathbf{\Omega}^{(i,j)} := p_{1*}[\mu_0^*\mathcal{O}_{\mathbf{P}}(1)\otimes\cdots\otimes\mu_{i+j-1}^*\mathcal{O}_{\mathbf{P}}(1)\otimes\mu_{i+j}^*\mathcal{C}^i].$$

Set $\mathbf{\Omega}^i := \bigoplus_{j\geq 0}\mathbf{\Omega}^{(i,j)}$ and $\mathbf{\Omega}^* := \bigoplus_{i=0}^{n}\mathbf{\Omega}^i$. Again $\mathbf{\Omega}^*$ is a flat $\mathcal{O}_G$-module. To define the product on $\mathbf{\Omega}^*$ we need to specify an equivariance (a *linearization* in the language of [GIT]):

$$\varepsilon_{\mathcal{C}} : (p_1\times\mu)^*\, p_2^*\mathcal{C}^{\cdot} = \mu^*\mathcal{C}^{\cdot} \xrightarrow{\simeq} p_2^*\mathcal{C}^{\cdot}.$$

We shall do this for $\mathcal{C}^1$, which suffices, since $\mathcal{C}^i = \Lambda^i\mathcal{C}^1$.

Let $V := \bigoplus_{i=0}^{n} k\cdot v_i$ be pure of total degree 0, so $\Omega^1\cong A(-1)\otimes_k V$ as graded $A$-modules, and $\mathcal{C}^1\cong\mathcal{O}_{\mathbf{P}}(-1)\otimes_k V$. Define

$$\phi : \begin{cases} \mu^*[\mathcal{O}_{\mathbf{P}}\otimes_k V] = \mathcal{O}_{G\times\mathbf{P}}\otimes_k V \to p_1^*\mathcal{O}_G(1)\otimes_k V \\ 1\otimes v_i \mapsto \sum_{j=0}^{n} a_{ij}\otimes v_j \end{cases}$$

and let

$$\psi : \mu^*\mathcal{O}_{\mathbf{P}}(-1) \xrightarrow{\simeq} p_1^*\mathcal{O}_G(-1)\otimes p_2^*\mathcal{O}_{\mathbf{P}}(-1)$$

be the inverse of the dual of (3.2.1). Then

$$\begin{aligned} \varepsilon_{\mathcal{C}} := \psi\otimes\phi : \mu^*\mathcal{C}^1 &\cong \mu^*\mathcal{O}_{\mathbf{P}}(-1)\otimes\mu^*[\mathcal{O}_{\mathbf{P}}\otimes_k V] \\ &\xrightarrow{\simeq} p_2^*\mathcal{O}_{\mathbf{P}}(-1)\otimes_k V \cong p_2^*\mathcal{C}^1 \end{aligned} \tag{3.2.2}$$

is the desired equivariance. For each $\sigma\in G(k)$, the $\sigma$-equivariance $(\mathcal{C}^{\cdot}, \varepsilon_{\mathcal{C}})$ of Section 3.1 is recovered in the fibre.

### 3.3. *Twisting by a Linear Action*

Let $\langle\gamma\rangle$ be an infinite cyclic group, and let $k\langle\gamma\rangle$ be the group ring. Consider $k\langle\gamma\rangle$ as a $\mathbb{Z}$-graded ring, pure of degree 0. Let $M$, $M'$, and $N$

be $\mathbb{Z}$-graded $k\langle\gamma\rangle$-modules, and let dot: $M \times M' \to$ , $(m, m') \mapsto m \cdot m'$, be a $k$-bilinear, $\gamma$-invariant, degree 0 pairing. Thus for $m \in M_i$ and $m' \in M'_j$, one has $m \cdot m' \in N_{i+j}$ and $\gamma m \cdot \gamma m' = \gamma(m \cdot m')$. Define a new pairing

$$\odot : \begin{cases} M \times M' \to N \\ (m, m' \mapsto m \cdot \gamma^i m') \qquad \text{for} \quad m \in M_i. \end{cases} \tag{3.3.1}$$

The pairing dot: $M \times M' \to N$ corresponds to a $k\langle\gamma\rangle$-linear homomorphism $M \otimes_k M' \to N$, where as usual $\gamma(m \otimes m') = \gamma m \otimes \gamma m'$. Thus taking $N := M \otimes_k M'$ and $m \cdot m' := m \otimes m'$, we get the universal $\gamma$-twisted pairing $\odot : M \times M' \to M \otimes_k M'$, $(m, m') \mapsto m \odot m' = m \otimes \gamma^i m'$ for $m \in M_i$. The twisted pairing is associative: given graded $k\langle\gamma\rangle$-modules $M$, $M'$, and $M''$, and given $(m, m', m'') \in M_i \times M'_{i'} \times M''_{i''}$, we have

$$\begin{aligned} (m \odot m') \odot m'' &= (m \otimes \gamma^i m') \otimes \gamma^{i+i'} m'' \\ &= m \otimes \gamma^i (m' \otimes \gamma^{i'} m'') \\ &= m \odot (m' \odot m'') \in (M \otimes_k M' \otimes_k M'')_{i+i'+i''}. \end{aligned}$$

Let $A$ be a $\mathbb{Z}$-graded $k$-algebra, with multiplication dot: $A \times A \to A$, and let $\gamma : A \to A$ be an automorphism of graded $k$-algebras. We may view $\gamma : A \to A$ as an action by $k\langle\gamma\rangle$, and view dot: $A \times A \to A$ as an invariant pairing. Define the twist of $A$ by $\gamma$ to be the $k$-algebra $A^{\sim\gamma}$, whose underlying $k$-module is $A$, and whose multiplication is $\odot : A \times A \to A$, the twisted pairing of (3.3.1) (cf. [ATV2, Section 8]).

Suppose $M$ is a $\mathbb{Z}$-graded left $A$-module, which is also a $k\langle\gamma\rangle$-module, and such that the multiplication pairing dot: $A \times M \to M$ is $\gamma$-invariant. Denote by $\mathbf{GrMod}^{\langle\gamma\rangle}(A)$ the category consisting of such modules, where a morphism $M \to N$ is a degree 0, $A$-linear, $\gamma$-invariant homomorphism.

LEMMA 3.3.1. *Let $M$ be an object of $\mathbf{GrMod}^{\langle\gamma\rangle}(A)$. Then $M$ can be twisted into an $A^{\sim\gamma}$-module $M^{\sim\gamma}$ by modifying the multiplication as in* (3.3.1). *The assignment $M \mapsto M^{\sim\gamma}$ is an equivalence of categories $\mathbf{GrMod}^{\langle\gamma\rangle}(A) \to \mathbf{GrMod}^{\langle\gamma\rangle}(A^{\sim\gamma})$ and similarly for right $A$-modules and for $A$-bimodules.*

Observe that as graded $k\langle\gamma\rangle$-modules, $M = M^{\sim\gamma}$.

Our application is as follows. Let $(X, \sigma, \mathcal{L})$ be a triple as in Section 3.1. Fix some equivariance $e : \sigma^*\mathcal{L} \xrightarrow{\simeq} \mathcal{L}$ (if one exists). Then for any $\sigma$-equivariant $\mathcal{O}_X$-module $(\mathcal{M}, e_{\mathcal{M}})$ the tensor product $(\mathcal{L}^{\otimes i} \otimes \mathcal{M}, e^{\otimes i} \otimes e_{\mathcal{M}})$ is equivariant. This gives a $\gamma$-action on $\Gamma(X, \mathcal{L}^{\otimes i} \otimes \mathcal{M})$, namely $\gamma = (e^{\otimes i} \otimes e_{\mathcal{M}})\sigma^*$. Setting $A = \Gamma_* \mathcal{O}_X := \bigoplus_i \Gamma(X, \mathcal{L}^{\otimes i})$ and $M = \Gamma_* \mathcal{M} := \bigoplus_i \Gamma(X, \mathcal{L}^{\otimes i} \otimes \mathcal{M})$ we find that $M \in \mathbf{GrMod}^{\langle\gamma\rangle}(A^e)$. In [Ye1, Section 6] an untwisting map is defined for all $\mathcal{O}_X$-modules. We shall define another untwisting map, somewhat better behaved, but which works only for equivariant sheaves (cf. [ATV1, Proof of Proposition 7.4]). Given an equivariant $\mathcal{O}_X$-module $(\mathcal{M}, e_{\mathcal{M}})$ and an integer $i$, set

$$U : (\Gamma^{\sigma}_{*}\mathcal{M})_i = \Gamma(X, \mathcal{L} \otimes \mathcal{L}^{\sigma} \otimes \cdots \otimes \mathcal{L}^{\sigma^{i-1}} \otimes \mathcal{M}^{\sigma^{i}})$$

$$\xrightarrow{1 \otimes e \otimes \cdots \otimes e^{i-1} \otimes e^{i}_{\mathcal{M}}} \Gamma(X, \mathcal{L} \otimes \mathcal{L} \otimes \cdots \otimes \mathcal{L} \otimes \mathcal{M}) = (\Gamma_{*}\mathcal{M})_i. \tag{3.3.2}$$

The map $U : A_{\sigma} = \Gamma^{\sigma}_{*}\mathcal{O}_X \to A = \Gamma_{*}\mathcal{O}_X$ is an isomorphism of graded $k$-modules. As for multiplication, see the following.

PROPOSITION 3.3.2. *The map $U : A_{\sigma} \to A^{\sim\gamma}$ is an isomorphism of graded $k$-algebras. Given a $\sigma$-equivariant $\mathcal{O}_X$-module (resp. algebra) $(\mathcal{M}, e_{\mathcal{M}})$, the map $U$: $\Gamma^{\sigma}_{*}\mathcal{M} \to (\Gamma_{*}\mathcal{M})^{\sim\gamma}$ is an isomorphism of graded $A_{\sigma}$-bimodules (resp. graded $k$-algebras). $U$ is functorial in $(\mathcal{M}, e_{\mathcal{M}})$.*

*Proof.* It suffices to check for a bimodule. Say $a \in (A_{\sigma})_l$, $b \in (\Gamma^{\sigma}_{*}\mathcal{M})_m$, and $c \in (A_{\sigma})_n$. Then one has

$$a \cdot b \cdot c = a \otimes (1 \otimes e^{-n})(\sigma^{*})^{l}(b) \otimes (\sigma^{*})^{l+m}(c);$$

$$\begin{aligned} U(a \cdot b \cdot c) = {} & (1 \otimes e \otimes \cdots \otimes e^{l-1})(a) \\ & \otimes (e^{l} \otimes \cdots \otimes e^{l+m-1} \otimes e^{l+m+n}_{\mathcal{M}})(1^{\otimes m} \otimes e_{\mathcal{M}}^{-n})(\sigma^{*})^{l}(b) \\ & \otimes (e^{l+m} \otimes \cdots \otimes e^{l+m+n-1})(\sigma^{*})^{l+m}(c); \end{aligned}$$

$$U(a) = (1 \otimes e \otimes \cdots \otimes e^{l-1})(a);$$

$$\gamma^{l} U(b) = \gamma^{l}((1 \otimes e \otimes \cdots \otimes e^{m-1} \otimes e^{m}_{\mathcal{M}})(b));$$

$$\gamma^{l+m} U(c) = \gamma^{l+m}((1 \otimes e \otimes \cdots \otimes e^{n-1})(c));$$

so that

$$U(a \cdot b \cdot c) = U(a) \odot U(b) \odot U(c). \quad \blacksquare$$

3.4. *Proofs of Theorems* 1.2.1 *and* 1.3.3

Let $X = \mathbf{P} := \mathbf{P}^{n}_{k}$, let $\mathcal{L} := \mathcal{O}_{\mathbf{P}}(1)$, and let $\sigma$ be some $k$-automorphism of $\mathbf{P}$. First let us prove Theorem 1.3.3, for $A = k[x_0, ..., x_n]$. The isomorphism of equivariant sheaves (3.1.3) induces an isomorphism of $A_{\sigma}$-bimodules $\Omega^{n+1}_{\sigma} = \Gamma^{\sigma}_{*}(\mathcal{K}^{n+1}, \varepsilon_{\mathcal{K}}) \cong \Gamma^{\sigma}_{*}(\Omega^{n}_{\mathbf{P}/k}, \varepsilon)$. According to [Ye1, Example 7.14], $\Gamma^{\sigma}_{*}(\Omega^{n}_{\mathbf{P}/k}, \varepsilon)[n+1]$ is a balanced dualizing complex over $A_{\sigma}$

Now let us prove Theorem 1.2.1. Fix some equivariance $e : \sigma^{*}\mathcal{O}_{\mathbf{P}}(1) \xrightarrow{\simeq} \mathcal{O}_{\mathbf{P}}(1)$; this determines an automorphism $\gamma = e\sigma^{*} : A \to A$. According to Proposition 3.3.2 we may work with $A^{\sim\gamma}$ and $(\Omega^{*})^{\sim\gamma}$, instead of with $A_{\sigma}$ and $\Omega^{*}_{\sigma}$. The odd derivation $\eta$ is already defined. Since $\mathrm{d} : \Omega^{*} \to \Omega^{*+1}$ has total degree 0, it is also an odd derivation of $(\Omega^*)^{\sim\gamma}$ (cf. Section 2.2). Property (3) is also easily verified.

Observe that $U: (A_\sigma)_1 = \Gamma(\mathbf{P}, \mathcal{O}_\mathbf{P}(1)) \to A_1 = \Gamma(\mathbf{P}, \mathcal{O}_\mathbf{P}(1))$ is the identity. Therefore $x_0, ..., x_n$ are elements of $A_\sigma$. Set $\xi_i := dx_i \in \Omega_\sigma^{(1,0)}$. Let $[a_{ij}]$ be the matrix of $\gamma: \gamma x_i = \sum_{j=0}^n a_{ij} x_j$. Since d and $\gamma$ commute, $[a_{ij}]$ is also the matrix of $\gamma$ acting on the $\xi_i$. From Proposition 3.3.2 it follows that there is an isomorphism of $\mathbb{Z}^2$-graded $k$-algebras,

$$\Omega_\sigma^* \cong k\langle x_0, ..., x_n, \xi_0, ..., \xi_n\rangle/(\text{relations}) \tag{3.4.1}$$

with

$$\text{relations} = \begin{cases} (\sum_l a_{jl} x_l) x_i - (\sum_l a_{il} x_l) x_j \\ (\sum_l a_{jl} \xi_l) \xi_i + (\sum_l a_{il} \xi_l) \xi_j \\ (\sum_l a_{il} \xi_1) \xi_i \\ (\sum_l a_{jl} x_l) \xi_i - (\sum_l a_{il} \xi_l) x_j \end{cases} \tag{3.4.2}$$

for all $i, j = 0, ..., n$. The ideal $I = (\text{relations}) \subset k[a_{00}, ...]\langle x_0, ..., \xi_0, ...\rangle$ is independent of the equivariance $e$ and is $\mathbb{Z}$-homogeneous in the $a_{ij}$. This means that $I$ descends to a sheaf $\mathbf{I}$ on $G = \mathrm{PGL}_{n+1}$. In fact $\mathbf{I}$ is the kernel of the surjection of sheaves of $\mathcal{O}_G$-algebras $\mathcal{O}_G\langle x_0, ..., \xi_0, ...\rangle \to \mathbf{\Omega}^*$, which sends $x_i \mapsto p_2^*(x_i) \in \Gamma(G, \mathbf{A}_1)$ and $\xi_i \mapsto (1 \otimes \varepsilon_\mathcal{O}^{-1})\, p_2^*(\xi_i) \in \Gamma(G, \mathbf{\Omega}^{1,0})$.

Finally, if $[a_{ij}]$ is a triangular matrix, then conditions (3.4.2) conform to the hypothesis of Theorem B of [Su2]: there is an ordering algorithm for $\Omega_\sigma^*$. This implies the existence of a DGA bialgebra $\Omega^*(M_{n,\sigma})$, which coacts on $\Omega_\sigma^* = \Omega^*(\mathbf{A}_\sigma^n)$ and is a flat deformation of $\Omega^*(M_n)$.

## 4. Some Remarks and Questions

### 4.1. *More on the Integral*

The objects of derived categories are cochain complexes. Accordingly, we shall consider $(\Omega_q^*, \eta)$ as a cochain complex, with $\Omega_q^i$ in dimension $-i$. The augmentation map $A_q \to k = A_q/\mathfrak{m}_q$ induces an isomorphism $(\Omega_q^*, \eta) \xrightarrow{\simeq} k$ in $\mathbf{D}(A_q^e)$. This isomorphism defines not only the "left" integral $\int$ of (1.2.2), but also a "right" integral $\int^\circ : \mathrm{RHom}_{A^\circ}(k, \Omega_q^n[n]) \xrightarrow{\simeq} k$. Denote by $D$ and $D^\circ$ the dualizing functors associated to $\Omega_q^n[n]$. Then $\int : Dk \to k$ is an isomorphism in $\mathbf{D}(A_q^e)$, and it is natural to try to compare $D^\circ(\int): D^\circ k \to D^\circ Dk = k$ with $\int^\circ : D^\circ k \to k$. It turns out that $D^\circ(\int) = \int^\circ$, and we leave this to the reader as an exercise. (*Hint.* Use the DGA structure of $(\Omega_q^*, \eta)$.)

In the projective linear case there is another integral, this one coming from Serre duality on projective space. Assume for simplicity that the field

$k$ is algebraically closed. As usual we set $A := k[x_0, \dots, x_n]$, $n \geq 1$, and $\mathbf{P} := \mathbf{P}^n_k = \operatorname{Proj} A$. Let $\sigma$ be some $k$-automorphism of $\mathbf{P}$. Let $\mathscr{K}^\cdot_{\mathbf{P}}$ be the *Grotendieck residue complex* on $\mathbf{P}$. For all $i$, the sheaves $\mathscr{K}^i_{\mathbf{P}}$ are injective $\mathcal{O}_{\mathbf{P}}$-modules, and they are 0 unless $-n \leq i \leq 0$. At the extreme ends one has $\mathscr{K}_{\mathbf{P}}{}^{-n} = \Omega^n_{\mathbf{P}/k} \otimes k(\mathbf{P})$ and $\mathscr{K}^0_{\mathbf{P}} = \bigoplus_P \operatorname{Hom}^{\mathrm{cont}}_k(\hat{\mathcal{O}}_{\mathbf{P},P}, k)$, the sum running over all closed points $P$ of $\mathbf{P}$. The inclusion $\Omega^n_{\mathbf{P}/k} \subsetneq \mathscr{K}_{\mathbf{P}}{}^{-n}$ induces a quasi-isomorphism $\Omega^n_{\mathbf{P}/k}[n] \to \mathscr{K}^\cdot_{\mathbf{P}}$. At the other end, $\operatorname{Tr} : \Gamma(\mathbf{P}, \mathscr{K}^0_{\mathbf{P}}) \to k$, $\operatorname{Tr}(\sum_P \phi_P) = \sum_P \phi_P(1)$, induces an isomorphism $\operatorname{Tr} : \mathrm{H}^0(\mathbf{P}, \mathscr{K}^\cdot_{\mathbf{P}}) \xrightarrow{\simeq} k$. Together we get Serre's isomorphism $\mathrm{H}^n(\mathbf{P}, \Omega^n_{\mathbf{P}/k}) \xrightarrow{\simeq} k$. See [Ye2] for details.

The residue complex $\mathscr{K}^\cdot_{\mathbf{P}}$ has a canonical $\sigma$-equivariance $\varepsilon$, compatible with the equivariance $\varepsilon$ on $\Omega^n_{\mathbf{P}/k}$. Consider the complex of graded $A_\sigma$-bimodules $\Gamma^\sigma_*(\mathscr{K}^\cdot_{\mathbf{P}}, \varepsilon)$. There is a canonical isomorphism of bimodules $\mathrm{H}^0\Gamma^\sigma_*(\mathscr{K}^\cdot_{\mathbf{P}}, \varepsilon) \xrightarrow{\simeq} A'_\sigma$, where $A'_\sigma = \operatorname{Hom}_k(A_\sigma, k)$. This isomorphism is induced by Tr and is a manifestation of Serre duality (see [Ye1, Lemma 7.9]). Let $\mathscr{K}^\cdot_{A_\sigma}$ be the mapping cone over the homomorphism of complexes $\Gamma^\sigma_*(\mathscr{K}^\cdot_{\mathbf{P}}, \varepsilon) \to A'_\sigma$, so $\mathscr{K}^0_{A_\sigma} = A'_\sigma$, and

$$\Gamma^\sigma_*(\mathscr{K}^\cdot_{\mathbf{P}}, \varepsilon) \to A'_\sigma \to \mathscr{K}^\cdot_{A_\sigma} \to \Gamma^\sigma_*(\mathscr{K}^\cdot_{\mathbf{P}}, \varepsilon)[1]$$

is a triangle in $\mathbf{D}(A^e_\sigma)$. In [Ye1, Section 7] it is proved that all the $\mathscr{K}^i_{A_\sigma}$ are graded-injective over $A_\sigma$, and for $i < 0$ they have no $\mathfrak{m}_\sigma$-torsion. Since the cohomology of an invertible sheaf on $\mathbf{P}$ vanishes in intermediate dimensions, and using (3.1.4), we see that $\Omega^{n+1}_\sigma[n+1] \to \mathscr{K}^\cdot_{A_\sigma}$ is an isomorphism in $\mathbf{D}(A^e_\sigma)$. Define

$$\begin{aligned}\int_{\mathbf{P}} : \operatorname{Ext}^{n+1}_{A_\sigma}(k, \Omega^{n+1}_\sigma) &= \operatorname{Ext}^0_{A_\sigma}(k, \Omega^{n+1}_\sigma[n+1]) \\ &\cong \mathrm{H}^0 \operatorname{Hom}_{A_\sigma}(k, \mathscr{K}^\cdot_{A_\sigma}) \\ &\cong (\mathscr{K}^0_{A_\sigma})_0 \xrightarrow{\mathrm{Tr}} k. \end{aligned} \tag{4.1.1}$$

One can show that

$$\int_{\mathbf{P}} = (-1)^{\lfloor (i+1)/2 \rfloor} \int, \tag{4.1.2}$$

where $\lfloor r \rfloor$ is the integer satisfying $r - 1 < \lfloor r \rfloor \leq r$. Let us sketch a proof of this claim. First reduce the problem to the commutative case, i.e., $\sigma = 1$. This is done using untwisting. Next observe that both integrals correspond to 0-cocycles in the complex $\operatorname{Hom}^\cdot_A(\Omega^*, \mathscr{K}^\cdot_A)$. It remains to write down a coboundary which equals their difference; this is done using the immersions $\mathbf{P}^0 \subset \mathbf{P}^1 \subset \cdots \subset \mathbf{P}$ as the hyperplanes at infinity and induction. Note that this is similar to the calculations in [Li, Section 8], and to Exercise 7.4 in [Ha, Chap. III.7].

Going back to $A = k[x_1, ..., x_n]$, let $Q$ be either of the parameter spaces, $Q_n$ or $\mathrm{PGL}_n$. From the global description of $(\mathbf{\Omega}^*, \eta)$ on $Q$ (at the end of Section 2.2 for $Q_n$; in Section 3.2 for $\mathrm{PGL}_n$) we see that each $\mathbf{\Omega}^i$ is a free $\mathbb{Z}$-graded, left $\mathbf{A}$-module, of rank $\binom{n}{i}$. For $\mathrm{PGL}_n$ the Künneth formula is used. Since $Q$ is an affine scheme, the category of $\mathbb{Z}$-graded, $\mathcal{O}_Q$-quasi-coherent $\mathbf{A}$-modules is equivalent to the category of $\mathbb{Z}$-graded $\Gamma(Q, \mathbf{A})$-modules. Hence $(\mathbf{\Omega}^*, \eta)$ may be used to calculate the sheaves $\mathbf{Ext}^{\cdot}_{\mathbf{A}}(\mathcal{O}_Q, \mathbf{\Omega}^n)$, giving rise to the global integral (1.2.2).

### 4.2. *Uniqueness of the Deformations*

Is it possible to characterize the deformation $\Omega_q^*$ in some meaningful way? For instance, is it universal for some property? Another problem is to characterize the algebra $B_q := k[dx_1, ..., dx_n] \subset \Omega_q^*$. In the multi-parameter quantum case, this algebra is isomorphic to $((A_q)^!)^o$, the opposite of the quadratic dual of $A_q$ (see [Ma1; AST, Section 1]). But this isomorphism is not canonical and seems to be a "coincidence." There is no such isomorphism for $A_\sigma$ if $\sigma \in \mathrm{PGL}_n$ is not diagonalizable.

The knowledge of $B_q$ does determine $\Omega_q^*$, given the data of Theorem 1.2.1 and assuming that char $k \neq 2$. Writing $\xi_i := dx_i$, one has

$$\begin{aligned} x_j \xi_i - q_{ji} \xi_i x_j &= \tfrac{1}{2}[\mathrm{d}, \eta]_{\mathrm{super}}(x_j \xi_i - q_{ji} \xi_i x_j) \\ &= \tfrac{1}{2}\mathrm{d}(x_j x_i - q_{ji} x_i x_j) + \tfrac{1}{2}\eta(\xi_j \xi_i + q_{ji} \xi_i \xi_j) = 0, \end{aligned}$$

so the fourth type of relation in (2.2.1) is a consequence of the first three and similarly for the relations (3.4.2). But the first three types of relations are the relations of $A_q$ and $B_q$.

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