CANONICAL DEFORMATIONS OF DE RHAM COMPLEXES

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1. Introduction and Statement of Results

1.1. Background. Let k be a field, and let $A = k[x_1, \ldots, x_n]$ be the commutative polynomial ring. Suppose $q = [q_{ij}]$ is an $n \times n$ matrix with entries in k, satisfying $q_{ij} q_{ji} = q_{i,i} = 1$. Let $k\langle x_1, \ldots, x_n\rangle$ be the free associative algebra.

Definition 1.1.1. The *multi-parameter quantum deformation* of A is the k-algebra

$$A_q = \mathcal{O}(\mathbf{A}^n_q) := k\langle x_1, \ldots, x_n\rangle / (x_j x_i - q_{ji} x_i x_j \mid i, j = 1, \ldots, n)$$

This type of deformation was investigated in [AST], [Su1] and [Re]. It was shown that there is a corresponding flat deformation $\mathcal{O}(\mathrm{M}_{n,q,\lambda})$ of the ring $\mathcal{O}(\mathrm{M}_n)$ of functions on $n \times n$ matrices. $\mathcal{O}(\mathrm{M}_{n,q,\lambda})$ is a bialgebra coacting on A_q, and $\lambda \in k^\times$ is an extra deformation parameter.

The bialgebra $\mathcal{O}(\mathrm{M}_{n,q,\lambda})$ actually coacts on the *deformed de Rham complex* $\Omega^*_{q,\lambda} = \Omega^*(\mathbf{A}^n_{q,\lambda}) = \bigoplus_{i=0}^n \Omega^i_{q,\lambda}$. This is the differential graded algebra (DGA) given by

$$\Omega^*_{q,\lambda} := k\langle x_1, \ldots, x_n, \xi_1, \ldots, \xi_n\rangle / (\text{relations}) \tag{1.1.1}$$

$$\text{relations} = \begin{cases} x_j x_i - q_{ji} x_i x_j & \text{if } i < j \\ \xi_j \xi_i + \lambda^{-1} q_{ji} \xi_i \xi_j & \text{if } i < j \\ \xi_i^2 & \\ \xi_i x_i - \lambda x_i \xi_i & \\ x_j \xi_i - \lambda^{-1} q_{ji} \xi_i x_j & \text{if } i < j \\ \xi_j x_i - q_{ji} x_i \xi_j - (1 - \lambda^{-1}) q_{ji} \xi_i x_j & \text{if } i < j \end{cases} \tag{1.1.2}$$

and $\mathrm{d}(x_i) = \xi_i$. This is a deformation of the DGA $\Omega^* := \Omega^*_{A/k}$, the usual de Rham complex of A relative to k, and $\Omega^0_{q,\lambda} = A_q$. Such a deformation was first considered by Woronowicz [Wo] and Wess-Zumino [WZ]. The deformation $\Omega^*_{q,\lambda}$ is also called a *quantum differential calculus*, since it defines the *quantum vector fields* $\partial_i, i = 1, \ldots, n$, which satisfy

$$\mathrm{d}(a) = \sum_{i=1}^{n} \xi_i \partial_i(a), \ a \in A_q \tag{1.1.3}$$

(cf. [Ma2] §2.1). As pointed out by Manin in [Ma1], the "missing relations" of the bialgebra coacting on A_q (i.e. those relations needed to give the correct Hilbert function) are recovered by insisting on a simultaneous coaction on some other algebra; in this case it is the "algebra of differentials with constant coefficients"

$$B_{q,\lambda} := k[\xi_1, \ldots, \xi_n] \subset \Omega^*_{q,\lambda} \ . \tag{1.1.4}$$

Maltsiniotis [Ml] went on to show that there is a DGA bialgebra $\Omega^*(\mathrm{M}_{n,q,\lambda})$ which is a flat deformation of $\Omega^*(\mathrm{M}_n)$, and which coacts, as a DGA, on $\Omega^*_{q,\lambda} = \Omega^*(\mathbf{A}^n_{q,\lambda})$. This result was later generalized by Sudbery [Su2].

We are also interested in another type of deformation of A. Assume $n \geq 2$, and consider the projective space $\mathbf{P} = \mathbf{P}^{n-1}_k$, which we identify with $\operatorname{Proj} A$. We also identify $\mathrm{PGL}_{n-1}(k)$ with $\mathrm{Aut}_k(\mathbf{P}^{n-1}_k)$.

Definition 1.1.2. Given $\sigma \in \mathrm{PGL}_{n-1}(k)$, Let A_σ be the the *skew homogeneous coordinate ring* associated to the triple $(\mathbf{P}, \sigma, \mathcal{O}_{\mathbf{P}}(1))$ (see [AV] and [Ye1] §6).

For $\sigma = 1$ we have $A_\sigma = A$, and as σ varies this is a flat deformation. The algebra A_σ is called a *projective linear deformation* of A. In degree 1 there is equality: $(A_\sigma)_1 = \Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(1)) = A_1$, so the elements $x_1, \ldots, x_n$ belong to A_σ. In fact the x_i generate A_σ; see §3.4.

1.2. First Results. Denote the parameter space of the quantum deformations $\operatorname{Spec} k[q_{11}, \ldots q_{nn}]/(q_{ij}q_{ji} - 1, q_{ii} - 1)$ by Q_n. Let us write A_q for a deformation of either type: $q \in Q_n(k)$ or $q \in \mathrm{PGL}_{n-1}(k)$. As mentioned earlier, the work of previous authors provided a differential calculus over the k-algebra A_q. We shall introduce *integration* into the picture. It is well known that algebraically, the only integration is contour integration of meromorphic forms, i.e. *residues*. In algebraic geometry the residue map plays a role in Serre Duality (and more generally, Grothendieck Duality). In [Ye1] the formalism of duality was extended to noncommutative algebras, and we will relate this duality to the integration over A_q in §1.3. In this section we start by describing the integral.

Here are some conventions on gradings to be used throughout the paper. Let $M = \bigoplus_{(i,j) \in \mathbb{Z}^2} M^{(i,j)}$ be a $\mathbb{Z}^2$-graded abelian group. Define a $\mathbb{Z}$-grading on M, called total degree, by setting $M_l := \bigoplus_{i+j=l} M^{(i,j)}$, $l \in \mathbb{Z}$. Define another $\mathbb{Z}$-grading, called dimension, by setting $M^i := \bigoplus_j M^{(i,j)}$, $i \in \mathbb{Z}$. Finally define a $\mathbb{Z}/(2)$-grading on M, called parity, by setting $M^{[l]} := \bigoplus_{i \in [l]} M^i$, $[l] = l + 2\mathbb{Z} \in \mathbb{Z}/(2)$. The various gradings are denoted by $\deg_{\mathbb{Z}^2}$, deg, dim and par, respectively. The base ring k is always assumed to be pure of $\deg_{\mathbb{Z}^2} = 0$. Unless otherwise specified, if M is a $\mathbb{Z}$-graded group, then this grading will be the total degree, and M will be pure of $\dim = 0$ (hence even).

The de Rham complex $\Omega^* = \Omega^*_{A/k}$ is a $\mathbb{Z}^2$-graded k-algebra, with $\deg_{\mathbb{Z}^2} x_i = (0,1)$. The derivative d has degree $(1,-1)$, so $\deg_{\mathbb{Z}^2}(\mathrm{d}\, x_i) = (1,0)$. Write $\Omega^{(i,j)}$ for the (i,j)-th homogeneous component of Ω^*; so according to the conventions $\Omega^i = \bigoplus_j \Omega^{(i,j)}$, and $\Omega^* = \bigoplus_i \Omega^i$ is the decomposition w.r.t. dimension. In terms of superalgebras, Ω^* is supercommutative, and d is an odd derivation.

The Euler derivation on A is multiplication by the total degree: $a \mapsto ia$ for $a \in A_i$. It corresponds to the A-linear map $\eta : \Omega^* \to \Omega^{*-1}$, $\eta(\mathrm{d}\, x_i) = x_i$, which is in fact an odd derivation. Moreover, $\eta^2 = 0$, and (Ω^*, η) is a Koszul complex for $k = A/\mathfrak{m}$, where $\mathfrak{m} = (x_1, \ldots, x_n)$ is the augmentation ideal. Using (Ω^*, η) to compute $\mathrm{Ext}^{\cdot}_A(k, \Omega^n)$, the identity map $\Omega^n \to \Omega^n$ represents a nonzero element of $\mathrm{Ext}^n_A(k, \Omega^n)$, and hence a canonical isomorphism $\int : \mathrm{Ext}^n_A(k, \Omega^n) \xrightarrow{\simeq} k$. The theorem below says that all this structure deforms.

Theorem 1.2.1. *Consider a deformation A_q of A, with either $q \in Q_n(k)$ or $q \in \mathrm{PGL}_{n-1}(k)$. Then there is a $\mathbb{Z}^2$-graded k-algebra $\Omega^*_q = \bigoplus_{(i,j)\in\mathbb{Z}^2} \Omega^{(i,j)}_q$, and k-linear endomorphisms d, η of Ω^*_q, having the following properties:*

(1) *$\Omega^0_q = A_q$ as $\mathbb{Z}$-graded k-algebras, with respect to total degree.*
(2) *The endomorphisms d, η are homogeneous of degrees $(1,-1), (-1,1)$ respectively. Moreover, d and η are odd derivations, $\mathrm{d}\,\mathrm{d} = \eta\eta = 0$, and $[\mathrm{d}, \eta]_{\mathrm{super}} = \mathrm{d}\,\eta + \eta\,\mathrm{d}$ is multiplication by the total degree.*
(3) *Let $B_q = \bigoplus_{i=0}^n B^i_q$ be the k-subalgebra of Ω^*_q generated by $\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n$. Then multiplication induces isomorphisms*

$$\Omega^*_q \cong A_q \otimes_k B_q \cong B_q \otimes_k A_q \ .$$

The structures $(\Omega^*_q, \mathrm{d}, \eta)$ form a $\mathbb{Z}^2$-graded flat family over the parameter space, specializing to $(\Omega^*, \mathrm{d}, \eta)$.

The proof of the theorem is in §2.2 for $q \in Q_n(k)$, and in 3.4 for $q \in \mathrm{PGL}_{n-1}(k)$.

The DGA Ω^*_q is called the *canonical deformation* of Ω^*. For $q \in Q_n(k)$ it is no other then the deformation $\lambda = 1$ of (1.1.1). Property (3) is what Manin calls a skew product. It guarantees that the Ω^i_q are free left and right A_q-modules. By (2) we have $\eta(A_q) = 0$, so η is a map of A_q-bimodules. It follows that (Ω^*_q, η) is a Koszul complex for $k = A_q/\mathfrak{m}_q$, and we get

Corollary 1.2.2. *The derivation η induces a canonical isomorphism*

$$\int : \mathrm{Ext}^n_{A_q}(k, \Omega^n_q) \xrightarrow{\simeq} k \ . \tag{1.2.1}$$

We are using the notational conventions of [Ye1]; accordingly $\mathrm{Hom}_{A_q}(-,-)$ means Hom as $\mathbb{Z}$-graded left A_q-modules, and similarly for Ext. The module Ω^n_q is $\mathbb{Z}$-graded by total degree, so its generator $\mathrm{d}\, x_1 \cdots \mathrm{d}\, x_n$ has $\deg = n$. There is a left-right symmetry for the integral, which shall be discussed in §4.1.

A few words about *flatness*. By a 'flat deformation' of a graded k-algebra we simply mean that the Hilbert function remains unchanged. In the examples considered in this paper there is actually a flat family of algebras over the parameter space, in the sense of algebraic geometry. Thus if Q is the parameter space, there is a flat graded $\mathcal{O}_Q$-algebra $\underline{A}$, such that the fibre $\underline{A} \otimes k(q) = A_q$ for every $q \in Q(k)$. The canonical deformation of the de Rham complex is also a flat family $(\underline{\Omega}^*, \mathrm{d}, \eta)$. One can show that the integral is defined globally: there is an isomorphism

$$\int : \underline{\mathrm{Ext}}^n_{\underline{A}}(\mathcal{O}_Q, \underline{\Omega}^n) \xrightarrow{\simeq} \mathcal{O}_Q \tag{1.2.2}$$

of sheaves over Q, specializing to (1.2.1) for every $q \in Q(k)$. For a discussion of this, turn to Subsections 2.2, 3.2 and 4.1.

1.3. Balanced Dualizing Complexes. Let $C = \bigoplus_{i \geq 0} C_i$ be a noetherian (noncommutative) $\mathbb{Z}$-graded k-algebra, with $C_0 = k$. In [Ye1] the notion of a dualizing complex over C was introduced. Denote by C° (resp. C^{e}) the opposite (resp. enveloping) algebra of C; thus $C^{\mathrm{e}} = C \otimes_k C^{\circ}$. Right C-modules (resp. C-bimodules) are identified with left C°- (resp. C^{e}-) modules. Let $\mathsf{GrMod}(C)$ be the category of graded left C-modules and degree 0 homomorphisms, and let $\mathsf{D}(C)$ be its derived category. In $\mathsf{D}(C)$ there are the standard full subcategories, such as $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(C)$: bounded complexes with coherent (= finitely generated) cohomology modules. Let $\mathrm{Hom}_C(M, N)_i$ denote the group of degree i homomorphisms between the modules M and N, and let $\mathrm{Hom}_C(-,-) := \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_C(-,-)_i$. So $\mathrm{Hom}_C(-,-)_0$ is the categrical Hom in $\mathsf{GrMod}(C)$. Define $\mathrm{RHom}^{\cdot}_C$ to be the derived functor of Hom_C, in its various possible domains and ranges. One possibility is:

$$\mathrm{RHom}^{\cdot}_C : \mathsf{D}(C)^{\circ} \times \mathsf{D}^{+}(C^{\mathrm{e}}) \to \mathsf{D}(C^{\circ}) \ .$$

See [Ye1] §2 for details.

Definition 1.3.1. ([Ye1] Def. 3.3) Given a complex $R^{\cdot} \in \mathsf{D}^{+}(C^{\mathrm{e}})$, define the associated duality functors

$$D := \mathrm{RHom}^{\cdot}_C(-, R^{\cdot}) \ , \ D^{\circ} := \mathrm{RHom}^{\cdot}_{C^{\circ}}(-, R^{\cdot}) \ .$$

The complex $R^{\cdot}$ is called dualizing if

(1) The functors D and D° restrict to functors $D : \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(C)^{\circ} \to \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(C^{\circ})$ and $D^{\circ} : \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(C^{\circ})^{\circ} \to \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(C)$.
(2) The morphisms of adjunction $1 \to D^{\circ} D$ and $1 \to D D^{\circ}$ are isomorphisms on $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(C)$ and $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(C^{\circ})$, respectively.

It is known that a dualizing complex is unique, up to a shift in dimensions and tensoring with an invertible bimodule. To get rid of these indeterminacies one considers torsion. For a C-module M let $\Gamma_{\mathfrak{m}}(M)$ be the submodule of left $\mathfrak{m}$-torsion elements, where $\mathfrak{m} = \bigoplus_{i > 0} C_i$ is the augmentation ideal. Let $\mathrm{R}\,\Gamma_{\mathfrak{m}}$ be its derived functor. Similarly we define $\mathrm{R}\,\Gamma_{\mathfrak{m}^{\circ}}$ as the derived functor of right $\mathfrak{m}$-torsion. Let $C' := \mathrm{Hom}_k(C, k)$, which is a C-bimodule.

Definition 1.3.2. ([Ye1] Def. 4.1) A dualizing complex $R^{\cdot} \in \mathsf{D}^{+}(C)$ is called balanced if there exists isomorphisms

$$\mathrm{R}\,\Gamma_{\mathfrak{m}}(R^{\cdot}) \cong \mathrm{R}\,\Gamma_{\mathfrak{m}^{\circ}}(R^{\cdot}) \cong C'$$

in $\mathsf{D}(C^{\mathrm{e}})$.

A balanced dualizing complex $R^{\cdot}$ is unique up to isomorphism ([Ye1] §4). Since $\mathrm{End}_{\mathsf{D}(C^{\mathrm{e}})}(R^{\cdot}) = k$, an isomorphism $\tau : \mathrm{Ext}^0_C(k, R^{\cdot}) = \mathrm{H}^0\, Dk \xrightarrow{\simeq} k$ completely rigidifies the complex. This rigidification is symmetric, since we obtain from it $\tau^{\circ} := D^{\circ}(\tau) : D^{\circ}k \xrightarrow{\simeq} D^{\circ}Dk \cong k$.

Now let us consider the deformed algebra A_q, with either $q \in Q_n(k)$ or $q \in \mathrm{PGL}_{n-1}(k)$. Since A_q is an Artin-Schelter regular algebra, it has a balanced dualizing complex (cf. [Ye1] §4). In fact this complex can be chosen to be a single invertible bimodule. Our main result is:

Theorem 1.3.3. *Let Ω^*_q be the canonical deformation of the de Rham complex of Thm.* 1.2.1. *Then $\Omega^n_q[n]$ is a balanced dualizing complex over A_q.*

The proof of this theorem is also in §2.2 for $q \in Q_n(k)$, and in 3.4 for $q \in \mathrm{PGL}_{n-1}(k)$.

The significance of the theorem is that the abstract invertible bimodule, whose existence is guaranteed by duality theory, is realized as:

$$\Omega^n_q = A_q \cdot \mathrm{d}\, x_1 \cdots \mathrm{d}\, x_n = \mathrm{d}\, x_1 \cdots \mathrm{d}\, x_n \cdot A_q \subset \Omega^*_q \ .$$

Observe that $\Omega^n_q[n]$ is rigidified canonically by the integral $\int$ of (1.2.1), which may be viewed as isomorphism $\int : \mathrm{RHom}^{\cdot}_{A_q}(k, \Omega^n_q[n]) \xrightarrow{\simeq} k$ in $\mathsf{D}(A^{\mathrm{e}}_q)$.

The paper is organized as follows: Section 2 treats the case of the multiparameter quantum deformation. Section 3 treats the case of the projective linear deformation. Section 4 contains additional remarks and some questions.

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2. The Multiparameter Quantum Case

2.1. Twisting by a 2-Cocycle. Let k be a field, let G be an abelian group, and let $c : G \times G \to k^{\times}$ be a 2-cocycle; i.e.

$$c(g,h)c(f,g+h) = c(f+g,h)c(f,g) \tag{2.1.1}$$

for all $f, g, h \in G$. Assume $c(0,0) = 1$. Let $A = \bigoplus_{g \in G} A_g$ be a G-graded k-algebra. For $a \in A_g$, $a \neq 0$, write $\deg_G a = g$. Following [AST] we define a new k-algebra $A^{\sim c}$, the twist of A by c, as follows. The underlying k-module of $A^{\sim c}$ is that of A. Multiplication is given by

$$a \odot b := c(f,g)a \cdot b \ ; \ a \in A_f \ , \ b \in A_g \ . \tag{2.1.2}$$

The cocycle condition insures the associativity of the product $\odot$. Note that twisting by a cocycle is a reversible operation: $A \cong (A^{\sim c})^{\sim c^{-1}}$.

We shall need to generalize the twisting operation. Say M_1, M_2 and N are G-graded k-modules, and say dot : $M_1 \times M_2 \to N$, $(m_1, m_2) \mapsto m_1 \cdot m_2$, is a G-homogeneous, k-bilinear pairing. Set $M_i^{\sim c} := M_i$, $N^{\sim c} := N$, and define a pairing $\odot : M_1^{\sim c} \times M_2^{\sim c} \to N^{\sim c}$, the twist of dot by c, as in (2.1.2). The pairing dot may be viewed as the universal pairing $M_1 \times M_2 \to M_1 \otimes_k M_2$, followed by a degree

0, k-linear homomorphism $M_1 \otimes_k M_2 \to N$. Thus the universal twisted pairing is $\odot : M_1^{\sim c} \times M_2^{\sim c} \to (M_1 \otimes_k M_2)^{\sim c}$, $m_1 \odot m_2 = c(f,g) m_1 \otimes m_2$. The twisted pairing is associative: let M_3 be another G-graded k-module. Then the cocycle condition guarantees that $m_1 \odot (m_2 \odot m_3) = (m_1 \odot m_2) \odot m_3 \in (M_1 \otimes_k M_2 \otimes_k M_3)^{\sim c}$ for any $m_i \in M_i$.

Denote by $\mathsf{GrMod}(G; A)$ the category of G-graded left A-modules and degree 0 homomorphisms.

Lemma 2.1.1. *Any G-graded left A-module M can be twisted into a left $A^{\sim c}$-module $M^{\sim c}$, by modifying the multiplication as in* (2.1.2)*. The assignment $M \mapsto M^{\sim c}$ is an equivalence $\mathsf{GrMod}(G; A) \to \mathsf{GrMod}(G; A^{\sim c})$. Similarly for right modules and bimodules.*

Given $g \in G$ and a graded k-module M, its twist by g is the graded module $M(g)$ such that $M(g)_f = M_{g+f}$. Let N be another G-graded left A-module. For $g \in G$ let $\mathrm{Hom}_A(M,N)_g$ be the set of degree g A-linear maps, and let $\mathrm{Hom}_A(M,N) := \bigoplus_{g \in G} \mathrm{Hom}_A(M,N)_g$, a G-graded k-module. We shall view $\mathrm{Hom}_A(M,N)$ as acting on the right. In other words the action is a pairing

$$M \times \mathrm{Hom}_A(M,N) \to N \ . \tag{2.1.3}$$

This allows A to act on the left, as it should. This pairing is evidently of the kind considered above, and so may be twisted by c. Thus for every $\phi \in \mathrm{Hom}_A(M,N)$ we get a k-linear map

$$M^{\sim c} \to N^{\sim c} \ , \ m \mapsto m \odot \phi \ . \tag{2.1.4}$$

If M (resp. N, resp. M and N) happens to be a bimodule, then $\mathrm{Hom}_A(M,N)$ is endowed with a left (resp. right, resp. bi-)A-module structure.

Lemma 2.1.2. *Let M and N be G-graded left A-modules. The action of* (2.1.4) *gives a bijection*

$$H_c : \mathrm{Hom}_A(M,N)^{\sim c} \xrightarrow{\simeq} \mathrm{Hom}_{A^{\sim c}}(M^{\sim c}, N^{\sim c}) \ .$$

If M (resp. N) is a bimodule then this is an isomorphism of left (resp. right) $A^{\sim c}$-modules.

Proof. All assertions boil down to the associativity of $\odot$. Consider $\phi \in \mathrm{Hom}_A(M,N)^{\sim c}$; we must show that the map $M^{\sim c} \to N^{\sim c}$, $m \mapsto m \odot \phi$, is (left) $A^{\sim c}$-linear. But given $a \in A^{\sim c}$ we have $(a \odot m) \odot \phi = a \odot (m \odot \phi)$. We leave the rest to the reader. $\square$

Suppose $\phi \in \mathrm{Hom}_A(M,N)_0$, so that $\phi : M \to N$ is a morphism in $\mathsf{GrMod}(G; A)$. By assumption $c(0,0) = 1$, and a simple calculation shows that $c(g,0) = c(0,g) = 1$ for all $g \in G$. Therefore, as maps of k-modules, $H_c(\phi) = \phi : M^{\sim c} = M \to N^{\sim c} = N$, in accordance with lemma 2.1.1.

Let $\mathrm{Ext}^{\cdot}_A(-,-)$ be the derived functors of $\mathrm{Hom}_A(-,-)$ on the categories of graded modules.

Proposition 2.1.3. *Let M and N be G-graded A-bimodules. Then for all $i \geq 0$ there is an isomorphism of G-graded $A^{\sim c}$-bimodules*

$$H_c : \mathrm{Ext}^i_A(M,N)^{\sim c} \xrightarrow{\simeq} \mathrm{Ext}^i_{A^{\sim c}}(M^{\sim c}, N^{\sim c}) \ .$$

This isomorphism is natural in M and N, and for $i = 0$ it coincides with H_c of lemma 2.1.2.

Proof. We may calculate $\mathrm{Ext}^i_A(M,N)$ either using a projective resolution $\cdots \to P^{-1} \to P^0 \to M \to 0$, or using an injective resolution $0 \to M \to I^0 \to I^1 \to \cdots$ in $\mathsf{GrMod}(G;A)$. Now $\cdots \to (P^{-1})^{\sim c} \to (P^0)^{\sim c} \to M^{\sim c} \to 0$ is a projective resolution of $M^{\sim c}$ in $\mathsf{GrMod}(G;A^{\sim c})$. By lemma 2.1.2 we get isomorphisms of right $A^{\sim c}$-modules $H_c : \mathrm{Hom}_A(P^i,N)^{\sim c} \xrightarrow{\simeq} \mathrm{Hom}_{A^{\sim c}}((P^i)^{\sim c}, N^{\sim c})$, which pass on to the Ext^i. The same H_c are obtained using the injective resolution, and now they are isomorphisms of left $A^{\sim c}$-modules. $\square$

Given $M \in \mathsf{GrMod}(G;A)$ and $V \in \mathsf{GrMod}(G;k)$, the graded k-module $\mathrm{Hom}_k(M,V)$ has a structure of an A-bimodule. Let us write $\phi \cdot m = m \cdot \phi = \phi(m) \in V$ for $\phi \in \mathrm{Hom}_k(M,V)$ and $m \in M$. In terms of G-graded k-bilinear pairings, for any $a,b \in A$ there is associativity, and also cyclic commutativity: $a \cdot \phi \cdot b \cdot m = \phi \cdot b \cdot m \cdot a = \ldots$.

Proposition 2.1.4. *Suppose the cocycle c satisfies $c(g,-g) = 1$ for all $g \in G$. Assume that V is a G-graded k-module, pure of degree 0 (i.e. $V_g = 0$ if $g \neq 0$). Then there is an isomorphism of $A^{\sim c}$-bimodules*

$$I_c : \mathrm{Hom}_k(M,V)^{\sim c} \xrightarrow{\simeq} \mathrm{Hom}_k(M^{\sim c}, V)$$

given by $I_c(\phi)(m) = \phi \odot m = m \odot \phi$, for $m \in M$, $\phi \in \mathrm{Hom}_k(M,V)$.

Proof. Say $\phi \in \mathrm{Hom}_k(M,V)_g$ and $m \in M_{-g}$. Then

$$\phi \odot m = c(g,-g)\phi \cdot m = \phi \cdot m = m \cdot \phi = c(-g,g) m \cdot \phi = m \odot \phi \tag{2.1.5}$$

so I_c is well-defined. Next let $\phi \in \mathrm{Hom}_k(M,V)_g$, let $a \in A_f$ and let $m \in M_{-f-g}$. Then

$$\begin{aligned} I_c(a \odot \phi)(m) &= (a \odot \phi) \odot m \\ &= a \odot (\phi \odot m) && \text{[associativity]} \\ &= (\phi \odot m) \odot a && \text{[as in (2.1.5)]} \\ &= \phi \odot (m \odot a) && \text{[associativity]} \\ &= (a \odot I_c(\phi))(m) \ . \end{aligned}$$

So I_c is left $A^{\sim c}$-linear; similarly for right linearity. $\square$

2.2. Proofs of Theorems 1.2.1 and 1.3.3. Let $A = k[x_1, \ldots, x_n]$ be a commutative polynomial ring over the field k. Let $\Omega^* = \Omega^*_{A/k}$ be the de Rham complex of A relative to k, with exterior derivative d and Euler derivative η. Given $q = [q_{ij}] \in Q_n(k)$, define Ω^*_q to be the algebra $\Omega^*_{q,\lambda}$ of (1.1.1), with $\lambda = 1$. Thus Ω^*_q has generators $x_1, \ldots, x_n, \xi_1, \ldots, \xi_n$ and relations

$$x_j x_i - q_{ji} x_i x_j \ , \ \xi_j \xi_i + q_{ji} \xi_i \xi_j \ , \ \xi_i^2 \ , \ x_j \xi_i - q_{ji} \xi_i x_j \tag{2.2.1}$$

for $i, j = 1, \ldots, n$. In particular taking $q_{ij} = 1$, the original de Rham complex Ω^* is recovered, with $\xi_i = \mathrm{d}\, x_i$.

The free algebra $k\langle x_1, \ldots, x_n, \xi_1, \ldots, \xi_n \rangle$ is graded by the group $G = \bigoplus_{i=1}^n \mathbb{Z} \cdot g_i \cong \mathbb{Z}^n$, with $\deg_G x_i = \deg_G \xi_i = g_i$. It is also $\mathbb{Z}^2$-graded with $\deg_{\mathbb{Z}^2} x_i = (0,1)$ and $\deg_{\mathbb{Z}^2} \xi_i = (1,0)$. Since the relations (2.2.1) are homogeneous with respect to both the G-grading and the $\mathbb{Z}^2$-grading, these gradings pass on to the quotient Ω^*_q.

According to [AST] Prop. 1 and Lemma 4, there exists a 2-cocycle $c : G \times G \to k^\times$ such that

$$\frac{c(g_i, g_j)}{c(g_j, g_i)} = q_{ij} \ ; \ i, j = 1, \ldots, n \tag{2.2.2}$$

$$c(g, -g) = 1 \ ; \ g \in G \ . \tag{2.2.3}$$

Fix one such cocycle.

Twisting Ω^* by c we get a G-graded k-algebra $(\Omega^*)^{\sim c}$. The $\mathbb{Z}^2$-grading on $(\Omega^*)^{\sim c}$ (and hence parity) is retained. We claim that $\mathrm{d}, \eta : (\Omega^*)^{\sim c} \to (\Omega^*)^{\sim c}$ are still odd derivations. Let us verify this for d. It is k-linear of G-degree 0, so for $\alpha, \beta \in (\Omega^*)^{\sim c}$ with $\deg_G \alpha = f$, $\deg_G \beta = g$ and $\operatorname{par} \alpha = i \pmod 2$, it holds:

$$\begin{aligned} \mathrm{d}(\alpha \odot \beta) &= \mathrm{d}(c(f,g)\alpha \cdot \beta) \\ &= c(f,g)(\mathrm{d}(\alpha) \cdot \beta + (-1)^i \alpha \cdot \mathrm{d}(\beta)) \\ &= \mathrm{d}(\alpha) \odot \beta + (-1)^i \alpha \odot \mathrm{d}(\beta) \ . \end{aligned}$$

Since d and η are odd derivations, $[\mathrm{d}, \eta]_{\mathrm{super}} = \mathrm{d}\,\eta + \eta\,\mathrm{d}$ is an even derivation. Clearly $\alpha \mapsto (\deg \alpha) \cdot \alpha$ is an even derivation. In order to prove these two derivations are equal it suffices to check on the generators $x_1, \ldots, \xi_1, \ldots$. But the generators have total degree 1, and

$$(\mathrm{d}\,\eta + \eta\,\mathrm{d})(x_i) = \eta\,\mathrm{d}(x_i) = x_i \ ,$$

$$(\mathrm{d}\,\eta + \eta\,\mathrm{d})(\xi_i) = \mathrm{d}\,\eta(\xi_i) = \xi_i \ .$$

From Formula (2.2.2) and the equality of the Hilbert functions it follows that k-algebra isomorphism $\Psi : \Omega^*_q \xrightarrow{\simeq} (\Omega^*)^{\sim c}$, with $\Psi(x_i) = x_i$, $\Psi(\xi_i) = \xi_i$.

Here is the proof of Theorem 1.2.1: property (1) is verified directly. Define d and η on Ω^*_q using the isomorphism Ψ; so property (2) holds. As for property (3), it holds for $q_{ij} = 1$: $A \otimes_k B \cong B \otimes_k A \cong \Omega^*$ via multiplication. Since A and B are G-graded, it follows that $A^{\sim c} \otimes_k B^{\sim c} \cong B^{\sim c} \otimes_k A^{\sim c} \cong (\Omega^*)^{\sim c}$ (although the maps on the underlying k-modules depend on c !).

As for Thm. 1.3.3, it is equivalent to the statement that $(\Omega^n)^{\sim c}[n]$ is a balanced dualizing complex over $A^{\sim c}$. According to [Ye1] Thm. 4.8 it suffices to show that

$$\mathrm{Ext}^n_{A^{\sim c}}(A^{\sim c}/(\mathfrak{m}^{\sim c})^{i+1}, (\Omega^n)^{\sim c}) \cong \mathrm{Hom}_k(A^{\sim c}/(\mathfrak{m}^{\sim c})^{i+1}, k)$$

as $A^{\sim c}$-bimodules, for $i = 0, 1$. Here $\mathfrak{m}^{\sim c} \subset A^{\sim c}$ is the augmentation ideal. As $A^{\sim c}$-bimodules, $A^{\sim c}/(\mathfrak{m}^{\sim c})^{i+1} \cong (A/\mathfrak{m}^{i+1})^{\sim c}$, and

$$\begin{aligned} \mathrm{Ext}^n_{A^{\sim c}}((A/\mathfrak{m}^{i+1})^{\sim c}, (\Omega^n)^{\sim c}) &\cong \mathrm{Ext}^n_A(A/\mathfrak{m}^{i+1}, \Omega^n)^{\sim c} && [\text{via } H_c] \\ &\cong \mathrm{Hom}_k(A/\mathfrak{m}^{i+1}, k)^{\sim c} && [\Omega^n \text{ is balanced over } A] \\ &\cong \mathrm{Hom}_k((A/\mathfrak{m}^{i+1})^{\sim c}, k) && [\text{via } I_c]\ . \end{aligned}$$

To conclude this section, let us describe the flat family $\underline{\Omega}^*$ on Q_n. Consider the free algebra $\mathcal{O}_{Q_n}\langle x_1, \ldots, \xi_1, \ldots \rangle$, and in it the ideal sheaf $\underline{I}$ generated by the relations (2.2.1). Let $\underline{\Omega}^*$ be the quotient $\mathcal{O}_{Q_n}\langle x_1, \ldots, \xi_1, \ldots \rangle / \underline{I}$. For each closed point $q \in Q_n$ we get $\underline{\Omega}^* \otimes k(q) = [\Omega^* \otimes_k k(q)]_q$, and this implies that $\underline{\Omega}^*$, being a direct sum of coherent sheaves, is flat over Q_n.

3. The Projective Linear Case

3.1. Equivariant Sheaves on $\mathbf{P}^n_k$. Let X be a scheme over the field k, let $\mathcal{L}$ be an invertible sheaf on X and let $\sigma \in \mathrm{Aut}_k(X)$. For an $\mathcal{O}_X$-module $\mathcal{M}$ write $\mathcal{M}^\sigma := \sigma^*\mathcal{M}$. Given $i \in \mathbb{Z}$, define the i-th twist of $\mathcal{M}$ (with respect to $\mathcal{L}$ and σ) to be

$$\mathcal{M}(i) = \mathcal{M}(i, \sigma, \mathcal{L}) := \begin{cases} \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \cdots \otimes \mathcal{L}^{\sigma^{i-1}} \otimes \mathcal{M}^{\sigma^i} & \text{if } i > 0 \\ \mathcal{M} & \text{if } i = 0 \\ \mathcal{L}^{-\sigma^i} \otimes \cdots \otimes \mathcal{L}^{-\sigma^{-2}} \otimes \mathcal{L}^{-\sigma^{-1}} \otimes \mathcal{M}^{\sigma^i} & \text{if } i < 0 \end{cases}$$

and set

$$\Gamma^\sigma_* \mathcal{M} := \bigoplus_{i \in \mathbb{Z}} \Gamma(X, \mathcal{M}(i, \sigma, \mathcal{L}))\ .$$

Then $A_\sigma := \Gamma^\sigma_* \mathcal{O}_X$ is the skew homogeneous coordinate ring associated to the triple $(X, \mathcal{L}, \sigma)$ (cf. [AV] and [Ye1] §6), and the assignment $\mathcal{M} \mapsto \Gamma^\sigma_* \mathcal{M}$ is a functor $\mathsf{Mod}(X) \to \mathsf{GrMod}(A_\sigma)$.

A σ-equivariant $\mathcal{O}_X$-module is a pair $(\mathcal{M}, e)$, where $\mathcal{M}$ is an $\mathcal{O}_X$-module, and $e : \sigma^*\mathcal{M} \xrightarrow{\simeq} \mathcal{M}$ is an $\mathcal{O}_X$-linear isomorphism. The σ-equivariant $\mathcal{O}_X$-modules form an abelian category. An equivariance $e : \mathcal{M}^\sigma \xrightarrow{\simeq} \mathcal{M}$ induces, for every integer i, an equivariance $e^{\sigma^i} : \mathcal{M}^{\sigma^{i+1}} \xrightarrow{\simeq} \mathcal{M}^{\sigma^i}$; by abuse of notation we shall write $e = e^{\sigma^i}$. Hence for any $i, j \in \mathbb{Z}$ there is an isomorphism $\sigma^{i-j} : \mathcal{M}^{\sigma^i} \xrightarrow{\simeq} \mathcal{M}^{\sigma^j}$.

An equivariance e on $\mathcal{M}$ induces a k-linear isomorphism

$$\gamma := e\sigma^* : \Gamma(X, \mathcal{M}) \to \Gamma(X, \sigma^*\mathcal{M}) \to \Gamma(X, \mathcal{M})\ .$$

In [GIT] §1.3, γ is called the dual action of σ.

In [Ye1] Lemma 6.16 it was shown that an equivariance e on an $\mathcal{O}_X$-module $\mathcal{M}$ induces an A_σ-bimodule structure on $\Gamma^\sigma_* \mathcal{M}$, and that Γ^σ_* is a functor (σ-equivariant

$\mathcal{O}_X$-modules) $\to \mathsf{GrMod}(A^{\mathrm{e}}_\sigma)$. We shall require an extension of this result to σ-equivariant $\mathcal{O}_X$-algebras, which are pairs $(\mathcal{C}, e)$, with $\mathcal{C}$ an $\mathcal{O}_X$-algebra, and with $e : \sigma^* \mathcal{C} \xrightarrow{\simeq} \mathcal{C}$ an algebra isomorphism. Similarly we define equivariant $\mathcal{C}$-modules.

Lemma 3.1.1. *Let $(\mathcal{C}, e)$ be a σ-equivariant $\mathcal{O}_X$-algebra. Then $\Gamma^\sigma_*(\mathcal{C}, e)$ is a graded k-algebra. Let $(\mathcal{M}, e_{\mathcal{M}})$ be a σ-equivariant $\mathcal{C}$-module. Then $\Gamma^\sigma_*(\mathcal{M}, e_{\mathcal{M}})$ is a graded $\Gamma^\sigma_*(\mathcal{C}, e)$-bimodule. Γ^σ_* is functorial in $(\mathcal{C}, e)$ and $(\mathcal{M}, e_{\mathcal{M}})$.*

Proof. The product on $\Gamma^\sigma_*(\mathcal{C}, e)$ is:

$$\begin{aligned} \Gamma(X, \mathcal{C}(i)) \times \Gamma(X, \mathcal{C}(j)) &\xrightarrow{1 \times (\sigma^*)^i} \Gamma(X, \mathcal{C}(i)) \times \Gamma(X, \mathcal{C}(j)^{\sigma^i}) \\ &\xrightarrow{\otimes} \Gamma(X, \mathcal{O}_X(i+j) \otimes \mathcal{C}^{\sigma^i} \otimes \mathcal{C}^{\sigma^{i+j}}) \\ &\xrightarrow{1 \otimes e^{-j} \otimes 1} \Gamma(X, \mathcal{O}_X(i+j) \otimes \mathcal{C}^{\sigma^{i+j}} \otimes \mathcal{C}^{\sigma^{i+j}}) \\ &\xrightarrow{1 \otimes \mathrm{mult}} \Gamma(X, \mathcal{O}_X(i+j) \otimes \mathcal{C}^{\sigma^{i+j}}) \\ &= \Gamma(X, \mathcal{C}(i+j)) \end{aligned} \tag{3.1.1}$$

where mult is the product in $\mathcal{C}$ (cf. [Ye1] Formula (6.15)). Associativity is proved as in [Ye1] Lemma 6.16. Similarly for modules. $\square$

Now let $X = \mathbf{P} := \mathbf{P}^n_k$, $n \geq 1$, and let $\mathcal{L} := \mathcal{O}_{\mathbf{P}}(1)$. Changing the setup slightly from §1.1, we set $A := k[x_0, \ldots, x_n]$, and identify $\mathbf{P} = \operatorname{Proj} A$. Consider $\Omega^* = \Omega^*_{A/k}$ as a $\mathbb{Z}$-graded A-algebra, with respect to total degree: $(\Omega^*)_l = \bigoplus_{i+j=l} \Omega^{(i,j)}$. Let $\mathcal{C}^{\cdot} = \bigoplus_{i=0}^{n+1} \mathcal{C}^i$ be the sheafification on $\mathbf{P}$ of Ω^*. It is an $\mathcal{O}_{\mathbf{P}}$-algebra. Since the Euler derivation η has total degree 0, it passes to an $\mathcal{O}_{\mathbf{P}}$-linear, odd derivation $\tilde{\eta} : \mathcal{C}^{\cdot} \to \mathcal{C}^{\cdot - 1}$. Note that as a complex, $(\mathcal{C}^{\cdot}, \tilde{\eta})$ is exact.

Choose an equivariance $e : \sigma^* \mathcal{O}_{\mathbf{P}}(1) \xrightarrow{\simeq} \mathcal{O}_{\mathbf{P}}(1)$. Under the identifications $\operatorname{Aut}_k(\mathbf{P}) = \mathrm{PGL}_n(k)$ and $\operatorname{Aut}_k(A_1) = \operatorname{Aut}_k(\Gamma(X, \mathcal{O}_{\mathbf{P}}(1))) = \mathrm{GL}_{n+1}(k)$, we get an element $\gamma := e\sigma^* \in \mathrm{GL}_{n+1}(k)$, lifting σ. γ extends to an isomorphism $\gamma : \Omega^* \xrightarrow{\simeq} \Omega^*$ which commutes with d. This allows us to define a canonical equivariance $\epsilon_{\mathcal{C}}$ on $\mathcal{C}^{\cdot}$. Let $P \in \mathbf{P}$. Any section of $\sigma^* \mathcal{C}^i = \mathcal{O}_{\mathbf{P}} \otimes_{\sigma^{-1}\mathcal{O}_{\mathbf{P}}} \sigma^{-1} \mathcal{C}^i$ near P is a sum of sections of the form $b \otimes a^{-1}\alpha$, with b a section of $\mathcal{O}_{\mathbf{P}}$ near P, with $a \in A_l$ not vanishing at $\sigma(P)$, and with $\alpha \in (\Omega^i)_l = \Omega^{i, l-i}$. Define $\epsilon_{\mathcal{C}}(b \otimes a^{-1}\alpha) := b\gamma(a)^{-1}\gamma(\alpha)$, which is a section of $\mathcal{C}^i$ near P. This map is well-defined, and is an isomorphism of sheaves. Furthermore, since e and γ are determined only up to a scalar in $k^\times$, this indeterminacy is cancelled out, so $\epsilon_{\mathcal{C}}$ is canonical. On $\mathcal{C}^0 = \mathcal{O}_{\mathbf{P}}$ the equivariance $\epsilon_{\mathcal{C}}$ is the identity, so $\Gamma^\sigma_*(\mathcal{C}^0, \epsilon_{\mathcal{C}}) = \Gamma^\sigma_* \mathcal{O}_{\mathbf{P}} = A_\sigma$.

Definition 3.1.2. The canonical deformation of (Ω^*, η) at $\sigma \in \operatorname{Aut}_k(\mathbf{P}^n_k)$ is the pair (Ω^*_σ, η), where Ω^*_σ is the $\mathbb{Z}^2$-graded k-algebra $\Gamma^\sigma_*(\mathcal{C}^{\cdot}, \epsilon_{\mathcal{C}})$, and where η is the odd derivation $\Gamma^\sigma_*(\tilde{\eta}) : \Omega^*_\sigma \to \Omega^{*-1}_\sigma$.

The Euler sequence on $\mathbf{P}^n_k$ is the exact sequence of sheaves

$$0 \to \Omega^1_{\mathbf{P}/k} \to \mathcal{C}^1 \xrightarrow{\tilde{\eta}} \mathcal{O}_{\mathbf{P}} \to 0 \tag{3.1.2}$$

where $\Omega^1_{\mathbf{P}/k} \to \mathcal{C}^1$ comes from the derivation $\mathcal{O}_{\mathbf{P}} \to \mathcal{C}^1$, which is the sheafification of the de Rham derivative $\mathrm{d} : A \to \Omega^1$ (cf. [Li] §8). Denoting by ϵ the canonical

σ-equivariance on $\Omega^*_{\mathbf{P}/k}$, we find that the sequence (3.1.2) is σ-equivariant. Taking top exterior powers there is a canonical isomorphism of σ-equivariant $\mathcal{O}_{\mathbf{P}}$-modules

$$(\mathcal{C}^{n+1}, \epsilon_{\mathcal{C}}) \cong (\Omega^n_{\mathbf{P}/k}, \epsilon) \ , \tag{3.1.3}$$

and an application of the functor Γ^σ_* gives an isomorphism of A_σ-bimodules

$$\Omega^{n+1}_\sigma = \Gamma^\sigma_*(\mathcal{C}^{n+1}, \epsilon_{\mathcal{C}}) \cong \Gamma^\sigma_*(\Omega^n_{\mathbf{P}/k}, \epsilon) \ . \tag{3.1.4}$$

3.2. The Sheaf $\underline{\Omega}^*$ on PGL_n. In this subsection we indicate a global approach to the deformations A_σ and Ω^*_σ. Let $G := \mathrm{PGL}_n$, $n \geq 1$, which as a scheme is the affine open subscheme of $\mathrm{Proj}\, k[a_{00}, \ldots, a_{nn}]$ defined by $\det[a_{ij}] \neq 0$. The action of G on $\mathbf{P} = \mathbf{P}^n_k$ is a k-morphism $\mu : G \times \mathbf{P} \to \mathbf{P}$, determined by

$$\begin{array}{rcl} \mu^* \mathcal{O}_{\mathbf{P}}(1) & \xrightarrow{\simeq} & p_1^* \mathcal{O}_G(1) \otimes p_2^* \mathcal{O}_{\mathbf{P}}(1) \\ \mu^*(x_i) & \mapsto & \sum_{j=0}^n a_{ij} \otimes x_j \end{array} \tag{3.2.1}$$

where p_1 and p_2 are the projections from $G \times \mathbf{P}$ to its factors (see [GIT] §1.3).

For $l \in \mathbb{N}$ let $\mu_l : G \times \mathbf{P} \to \mathbf{P}$ correspond to $\sigma^l : \mathbf{P} \to \mathbf{P}$, $\sigma \in G(k)$. Thus $\mu_0 = p_2$ and $\mu_1 = \mu$. Define an $\mathcal{O}_G$-module

$$\underline{A}_l := p_{1*} \left[\mu_0^* \mathcal{O}_{\mathbf{P}}(1) \times \cdots \otimes \mu_{l-1}^* \mathcal{O}_{\mathbf{P}}(1) \right] \ .$$

There is a product $\underline{A}_i \times \underline{A}_j \to \underline{A}_{i+j}$ given by applying $(p_1 \times \mu_i)^*$ to the second factor and then tensoring. According to the Cohomology and Base Change Theorem, $\underline{A} := \bigoplus_{l \in \mathbb{N}} \underline{A}_l$ is a flat $\mathcal{O}_G$-algebra. For each $\sigma \in G(k)$ one has $\underline{A} \otimes k(\sigma) = A_\sigma$.

Next define the $\mathcal{O}_G$-module

$$\underline{\Omega}^{(i,j)} := p_{1*} \left[\mu_0^* \mathcal{O}_{\mathbf{P}}(1) \times \cdots \otimes \mu_{i+j-1}^* \mathcal{O}_{\mathbf{P}}(1) \otimes \mu_{i+j}^* \mathcal{C}^i \right] \ .$$

Set $\underline{\Omega}^i := \bigoplus_{j \geq 0} \underline{\Omega}^{(i,j)}$ and $\underline{\Omega}^* := \bigoplus_{i=0}^n \underline{\Omega}^i$. Again $\underline{\Omega}^*$ is a flat $\mathcal{O}_G$-module. To define the product on $\underline{\Omega}^*$ we need to specify an equivariance (a *linearization* in the language of [GIT]):

$$\epsilon_{\mathcal{C}} : (p_1 \times \mu)^* p_2^* \mathcal{C}^{\cdot} = \mu^* \mathcal{C}^{\cdot} \xrightarrow{\simeq} p_2^* \mathcal{C}^{\cdot} \ .$$

We shall do this for $\mathcal{C}^1$.

Let $V := \bigoplus_{i=0}^n k \cdot v_i$ be pure of total degree 0, so $\Omega^1 \cong A(-1) \otimes_k V$ as graded A-modules, and $\mathcal{C}^1 \cong \mathcal{O}_{\mathbf{P}}(-1) \otimes_k V$. Define

$$\phi : \begin{cases} \mu^*[\mathcal{O}_{\mathbf{P}} \otimes_k V] = \mathcal{O}_{G \times \mathbf{P}} \otimes_k V \to p_1^* \mathcal{O}_G(1) \otimes_k V \\ 1 \otimes v_i \mapsto \sum_{j=0}^n a_{ij} \otimes v_j \end{cases}$$

and let

$$\psi : \mu^* \mathcal{O}_{\mathbf{P}}(-1) \xrightarrow{\simeq} p_1^* \mathcal{O}_G(-1) \otimes p_2^* \mathcal{O}_{\mathbf{P}}(-1)$$

be the inverse of the dual of (3.2.1). Then

$$\begin{aligned} \epsilon_{\mathcal{C}} := \psi \otimes \phi : \mu^* \mathcal{C}^1 &\cong \mu^* \mathcal{O}_{\mathbf{P}}(-1) \otimes \mu^* \left[\mathcal{O}_{\mathbf{P}} \otimes_k V \right] \\ &\xrightarrow{\simeq} p_2^* \mathcal{O}_{\mathbf{P}}(-1) \otimes_k V \cong p_2^* \mathcal{C}^1 \end{aligned} \tag{3.2.2}$$

is the desired equivariance. For each $\sigma \in G(k)$, the σ-equivariance $(\mathcal{C}^{\cdot}, \epsilon_{\mathcal{C}})$ of §3.1 is recovered in the fibre.

3.3. Twisting by a Linear Action. Let $\langle\gamma\rangle$ be an infinite cyclic group, and let $k\langle\gamma\rangle$ be the group ring. Consider $k\langle\gamma\rangle$ as a $\mathbb{Z}$-graded ring, pure of degree 0. Let M, M' and N be $\mathbb{Z}$-graded $k\langle\gamma\rangle$-modules, and let dot : $M \times M' \to$, $(m, m') \mapsto m \cdot m'$, be a k-bilinear, γ-invariant, degree 0 pairing. Thus for $m \in M_i$ and $m' \in M'_j$, one has $m \cdot m' \in N_{i+j}$ and $\gamma m \cdot \gamma m' = \gamma(m \cdot m')$. Define a new pairing

$$\odot : \begin{cases} M \times M' \to N \\ (m, m') \mapsto m \cdot \gamma^i m' \ , \text{ for } m \in M_i \ . \end{cases} \tag{3.3.1}$$

The pairing dot : $M \times M' \to N$ corresponds to a $k\langle\gamma\rangle$-linear homomorphism $M \otimes_k M' \to N$, where as usual $\gamma(m \otimes m') = \gamma m \otimes \gamma m'$. Thus taking $N := M \otimes_k M'$ and $m \cdot m' := m \otimes m'$, we get the universal γ-twisted pairing $\odot : M \times M' \to M \otimes_k M'$, $(m, m') \mapsto m \odot m' = m \otimes \gamma^i m'$, for $m \in M_i$. The twisted pairing is associative: given graded $k\langle\gamma\rangle$-modules M, M' and M'', and given $(m, m', m'') \in M_i \times M'_{i'} \times M''_{i''}$, we have

$$\begin{aligned} (m \odot m') \odot m'' &= (m \otimes \gamma^i m') \otimes \gamma^{i+i'} m'' \\ &= m \otimes \gamma^i (m' \otimes \gamma^{i'} m'') \\ &= m \odot (m' \odot m'') \in (M \otimes_k M' \otimes_k M'')_{i+i'+i''} \ . \end{aligned}$$

Let A be a $\mathbb{Z}$-graded k-algebra, with multiplication dot : $A \times A \to A$, and let $\gamma : A \to A$ be an automorphism of graded k-algebras. We may view $\gamma : A \to A$ as an action by $k\langle\gamma\rangle$, and view dot : $A \times A \to A$ as an invariant pairing. Define the twist of A by γ to be the k-algebra $A^{\sim\gamma}$, whose underlying k-module is A, and whose multiplication is $\odot : A \times A \to A$, the twisted pairing of (3.3.1) (cf. [ATV2] §8).

Suppose M is a $\mathbb{Z}$-graded left A-module, which is also a $k\langle\gamma\rangle$-module, and such that the multiplication pairing dot : $A \times M \to M$ is γ-invariant. Denote by $\mathsf{GrMod}^{\langle\gamma\rangle}(A)$ the category consisting of such modules, where a morphism $M \to N$ is a degree 0, A-linear, γ-invariant homomorphism.

Lemma 3.3.1. *Let M be an object of $\mathsf{GrMod}^{\langle\gamma\rangle}(A)$. Then M can be twisted into an $A^{\sim\gamma}$-module $M^{\sim\gamma}$ by modifying the multiplication as in (3.3.1). The assignment $M \mapsto M^{\sim\gamma}$ is an equivalence of categories $\mathsf{GrMod}^{\langle\gamma\rangle}(A) \to \mathsf{GrMod}^{\langle\gamma\rangle}(A^{\sim\gamma})$. Similarly for right A-modules and for A-bimodules.*

Observe that as graded $k\langle\gamma\rangle$-modules, $M = M^{\sim\gamma}$.

Our application is as follows. Let $(X, \sigma, \mathcal{L})$ be a triple as in §3.1. Fix some equivariance $e : \sigma^* \mathcal{L} \xrightarrow{\simeq} \mathcal{L}$ (if one exists...). Then for any σ-equivariant $\mathcal{O}_X$-module $(\mathcal{M}, e_{\mathcal{M}})$ the tensor product $(\mathcal{L}^{\otimes i} \otimes \mathcal{M}, e^{\otimes i} \otimes e_{\mathcal{M}})$ is equivariant. This gives a γ-action on $\Gamma(X, \mathcal{L}^{\otimes i} \otimes \mathcal{M})$, namely $\gamma = (e^{\otimes i} \otimes e_{\mathcal{M}})\sigma^*$. Setting $A = \Gamma_* \mathcal{O}_X := \bigoplus_i \Gamma(X, \mathcal{L}^{\otimes i})$ and $M = \Gamma_* \mathcal{M} := \bigoplus_i \Gamma(X, \mathcal{L}^{\otimes i} \otimes \mathcal{M})$ we find that $M \in \mathsf{GrMod}^{\langle\gamma\rangle}(A^{\mathrm{e}})$. In [Ye1] §6 an untwisting map is defined for all $\mathcal{O}_X$-modules. We shall define another untwisting map, somewhat better behaved, but which works only for equivariant sheaves (cf.

[ATV1], proof of Prop. 7.4). Given an equivariant $\mathcal{O}_X$-module $(\mathcal{M}, e_{\mathcal{M}})$ and an integer i, set

$$U : (\Gamma^{\sigma}_{*}\mathcal{M})_i = \Gamma(X, \mathcal{L} \otimes \mathcal{L}^{\sigma} \otimes \cdots \otimes \mathcal{L}^{\sigma^{i-1}} \otimes \mathcal{M}^{\sigma^i}) \xrightarrow{1 \otimes e \otimes \cdots \otimes e^{i-1} \otimes e^{i}_{\mathcal{M}}} \Gamma(X, \mathcal{L} \otimes \mathcal{L} \otimes \cdots \otimes \mathcal{L} \otimes \mathcal{M}) = (\Gamma_{*}\mathcal{M})_i \ . \tag{3.3.2}$$

The map $U : A_{\sigma} = \Gamma^{\sigma}_{*}\mathcal{O}_X \to A = \Gamma_{*}\mathcal{O}_X$ is an isomorphism of graded k-modules. As for multiplication:

Proposition 3.3.2. *The map $U : A_{\sigma} \to A^{\sim\gamma}$ is an isomorphism of graded k-algebras. Given a σ-equivariant $\mathcal{O}_X$-module (resp. algebra) $(\mathcal{M}, e_{\mathcal{M}})$, the map $U : \Gamma^{\sigma}_{*}\mathcal{M} \to (\Gamma_{*}\mathcal{M})^{\sim\gamma}$ is an isomorphism of graded A_{σ}-bimodules (resp. graded k-algebras). U is functorial in $(\mathcal{M}, e_{\mathcal{M}})$.*

Proof. It suffices to check for a bimodule. Say $a \in (A_{\sigma})_l$, $b \in (\Gamma^{\sigma}_{*}\mathcal{M})_m$ and $c \in (A_{\sigma})_n$. Then one has

$$\begin{aligned}
a \cdot b \cdot c \ &= a \otimes (1 \otimes e^{-n})(\sigma^*)^l(b) \otimes (\sigma^*)^{l+m}(c) \ ; \\
U(a \cdot b \cdot c) \ &= (1 \otimes e \otimes \cdots \otimes e^{l-1})(a) \\
&\quad \otimes (e^l \otimes \cdots \otimes e^{l+m-1} \otimes e^{l+m+n}_{\mathcal{M}})(1^{\otimes m} \otimes e^{-n}_{\mathcal{M}})(\sigma^*)^l(b) \\
&\quad \otimes (e^{l+m} \otimes \cdots \otimes e^{l+m+n-1})(\sigma^*)^{l+m}(c) \ ; \\
U(a) \ &= (1 \otimes e \otimes \cdots \otimes e^{l-1})(a) \ ; \\
\gamma^l U(b) \ &= \gamma^l((1 \otimes e \otimes \cdots \otimes e^{m-1} \otimes e^m_{\mathcal{M}})(b)) \ ; \\
\gamma^{l+m} U(c) \ &= \gamma^{l+m}((1 \otimes e \otimes \cdots \otimes e^{n-1})(c)) \ ;
\end{aligned}$$

so that

$$U(a \cdot b \cdot c) = U(a) \odot U(b) \odot U(c) \ .$$

□

3.4. Proofs of Theorems 1.2.1 and 1.3.3. Let $X = \mathbf{P} := \mathbf{P}^n_k$, let $\mathcal{L} := \mathcal{O}_{\mathbf{P}}(1)$ and let σ be some k-automorphism of $\mathbf{P}$. First let us prove Thm. 1.3.3, for $A = k[x_0, \ldots, x_n]$. The isomorphism of equivariant sheaves (3.1.3) induces an isomorphism of A_{σ}-bimodules $\Omega^{n+1}_{\sigma} = \Gamma^{\sigma}_{*}(\mathcal{C}^{n+1}, \epsilon_{\mathcal{C}}) \cong \Gamma^{\sigma}_{*}(\Omega^n_{\mathbf{P}/k}, \epsilon)$. According to [Ye1] Example 7.14, $\Gamma^{\sigma}_{*}(\Omega^n_{\mathbf{P}/k}, \epsilon)[n+1]$ is a balanced dualizing complex over A_{σ}.

Now let us prove Thm. 1.2.1. Fix some equivariance $e : \sigma^*\mathcal{O}_{\mathbf{P}}(1) \xrightarrow{\simeq} \mathcal{O}_{\mathbf{P}}(1)$; this determines an automorphism $\gamma = e\sigma^* : A \to A$. According to Prop. 3.3.2 we may work with $A^{\sim\gamma}$ and $(\Omega^*)^{\sim\gamma}$, instead of with A_{σ} and Ω^*_{σ}. The odd derivation η is already defined. Since $\mathrm{d} : \Omega^* \to \Omega^{*+1}$ has total degree 0, it is also an odd derivation of $(\Omega^*)^{\sim\gamma}$ (cf. §2.2). Property (3) is also easily verified.

Observe that $U : (A_{\sigma})_1 = \Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(1)) \to A_1 = \Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(1))$ is the identity. Therefore $x_0, \ldots, x_n$ are elements of A_{σ}. Set $\xi_i := \mathrm{d}\, x_i \in \Omega^{(1,0)}_{\sigma}$. Let $[a_{ij}]$ be the matrix of γ: $\gamma x_i = \sum_{j=0}^n a_{ij} x_j$. Since d and γ commute, $[a_{ij}]$ is also the matrix of γ acting on the ξ_i. From Prop. 3.3.2 it follows that there is an isomorphism of $\mathbb{Z}^2$-graded k-algebras

$$\Omega^*_{\sigma} \cong k\langle x_0, \ldots, x_n, \xi_0, \ldots, \xi_n \rangle / (\text{relations}) \tag{3.4.1}$$

with

$$\text{relations} = \begin{cases} (\sum_l a_{jl} x_l) x_i - (\sum_l a_{il} x_l) x_j \\ (\sum_l a_{jl} \xi_l) \xi_i + (\sum_l a_{il} \xi_l) \xi_j \\ (\sum_l a_{il} \xi_l) \xi_i \\ (\sum_l a_{jl} x_l) \xi_i - (\sum_l a_{il} \xi_l) x_j \end{cases} \tag{3.4.2}$$

for all $i, j = 0, \ldots, n$. The ideal $I = (\text{relations}) \subset k[a_{00}, \ldots]\langle x_0, \ldots, \xi_0, \ldots \rangle$ is independent of the equivariance e, and is $\mathbb{Z}$-homogeneous in the a_{ij}. This means that I descends to a sheaf $\underline{I}$ on $G = \mathrm{PGL}_n$. In fact $\underline{I}$ is the kernel of the surjection of sheaves of $\mathcal{O}_G$-algebras $\mathcal{O}_G\langle x_0, \ldots, \xi_0, \ldots \rangle \to \underline{\Omega}^*$, which sends $x_i \mapsto p_2^*(x_i) \in \Gamma(G, \underline{A}_1)$ and $\xi_i \mapsto (1 \otimes \epsilon_{\mathcal{C}}^{-1}) p_2^*(\xi_i) \in \Gamma(G, \underline{\Omega}^{1,0})$.

Finally, if $[a_{ij}]$ is a triangular matrix, then conditions (3.4.2) conform to the hypothesis of Thm. B of [Su2]: there is an ordering algorithm for Ω_σ^*. This implies the existence of a DGA bialgebra $\Omega^*(\mathrm{M}_{n,\sigma})$, which coacts on $\Omega_\sigma^* = \Omega^*(\mathbf{A}_\sigma^n)$, and is a flat deformation of $\Omega^*(\mathrm{M}_n)$.

4. Some Remarks and Questions

4.1. More on the Integral. The objects of derived categories are cochain complexes. Accordingly, we shall consider (Ω_q^*, η) as a cochain complex, with Ω_q^i in dimension $-i$. The augmentation map $A_q \to k = A_q/\mathfrak{m}_q$ induces an isomorphism $(\Omega_q^*, \eta) \xrightarrow{\simeq} k$ in $\mathsf{D}(A_q^{\mathrm{e}})$. This isomorphism defines not only the "left" integral $\int$ of (1.2.2), but also a "right" integral $\int^\circ : \mathrm{RHom}_{A^\circ}(k, \Omega_q^n[n]) \xrightarrow{\simeq} k$. Denote by D and D° the dualizing functors associated to $\Omega_q^n[n]$. Then $\int : Dk \to k$ is an isomorphism in $\mathsf{D}(A_q^{\mathrm{e}})$, and it is natural to try to compare $D^\circ(\int) : D^\circ k \to D^\circ D k = k$ with $\int^\circ : D^\circ k \to k$. It turns out that $D^\circ(\int) = \int^\circ$, and we leave this to the reader as an exercise. (Hint: use the DGA structure of (Ω_q^*, η).)

In the projective linear case there is another integral, this one coming from Serre Duality on projective space. Assume for simplicity that the field k is algebraically closed. As usual we set $A := k[x_0, \ldots, x_n]$, $n \geq 1$, and $\mathbf{P} := \mathbf{P}_k^n = \operatorname{Proj} A$. Let σ be some k-automorphism of $\mathbf{P}$. Let $\mathcal{K}_{\mathbf{P}}^{\cdot}$ be the *Grothendieck residue complex* on $\mathbf{P}$. For all i, the sheaves $\mathcal{K}_{\mathbf{P}}^i$ are injective $\mathcal{O}_{\mathbf{P}}$-modules, and they are 0 unless $-n \leq i \leq 0$. At the extreme ends one has $\mathcal{K}_{\mathbf{P}}^{-n} = \Omega_{\mathbf{P}/k}^n \otimes k(\mathbf{P})$ and $\mathcal{K}_{\mathbf{P}}^0 = \bigoplus_P \mathrm{Hom}_k^{\mathrm{cont}}(\hat{\mathcal{O}}_{\mathbf{P},P}, k)$, the sum running over all closed points P of $\mathbf{P}$. The inclusion $\Omega_{\mathbf{P}/k}^n \hookrightarrow \mathcal{K}_{\mathbf{P}}^{-n}$ induces a quasi-isomorphism $\Omega_{\mathbf{P}/k}^n[n] \to \mathcal{K}_{\mathbf{P}}^{\cdot}$. At the other end, $\mathrm{Tr} : \Gamma(\mathbf{P}, \mathcal{K}_{\mathbf{P}}^0) \to k$, $\mathrm{Tr}(\sum_P \phi_P) = \sum_P \phi_P(1)$, induces an isomorphism $\mathrm{Tr} : \mathrm{H}^0(\mathbf{P}, \mathcal{K}_{\mathbf{P}}^{\cdot}) \xrightarrow{\simeq} k$. Together we get Serre's isomorphism $\mathrm{H}^n(\mathbf{P}, \Omega_{\mathbf{P}/k}^n) \xrightarrow{\simeq} k$. See [Ye2] for details.

The residue complex $\mathcal{K}_{\mathbf{P}}^{\cdot}$ has a canonical σ-equivariance ϵ, compatible with the equivariance ϵ on $\Omega_{\mathbf{P}/k}^n$. Consider the complex of graded A_σ-bimodules $\Gamma_*^\sigma(\mathcal{K}_{\mathbf{P}}^{\cdot}, \epsilon)$. There is a canonical isomorphism of bimodules $\mathrm{H}^0 \Gamma_*^\sigma(\mathcal{K}_{\mathbf{P}}^{\cdot}, \epsilon) \xrightarrow{\simeq} A'_\sigma$, where $A'_\sigma = \mathrm{Hom}_k(A_\sigma, k)$. This isomorphism is induced by Tr, and is a manifestation of Serre

Duality (see [Ye1] Lemma 7.9). Let $\mathcal{K}^{\cdot}_{A_\sigma}$ be the mapping cone over the homomorphism of complexes $\Gamma^\sigma_*(\mathcal{K}^{\cdot}_{\mathbf{P}}, \epsilon) \to A'_\sigma$, so $\mathcal{K}^0_{A_\sigma} = A'_\sigma$, and

$$\Gamma^\sigma_*(\mathcal{K}^{\cdot}_{\mathbf{P}}, \epsilon) \to A'_\sigma \to \mathcal{K}^{\cdot}_{A_\sigma} \to \Gamma^\sigma_*(\mathcal{K}^{\cdot}_{\mathbf{P}}, \epsilon)[1]$$

is a triangle in $\mathsf{D}(A^{\mathrm{e}}_\sigma)$. In [Ye1] §7 it is proved that all the $\mathcal{K}^i_{A_\sigma}$ are graded-injective over A_σ, and for $i < 0$ they have no $\mathfrak{m}_\sigma$-torsion. Since the cohomology of an invertible sheaf on $\mathbf{P}$ vanishes in intermediate dimensions, and using (3.1.4), we see that $\Omega^{n+1}_\sigma[n+1] \to \mathcal{K}^{\cdot}_{A_\sigma}$ is an isomorphism in $\mathsf{D}(A^{\mathrm{e}}_\sigma)$. Define

$$\begin{aligned} \textstyle\int_{\mathbf{P}} : \operatorname{Ext}^{n+1}_{A_\sigma}(k, \Omega^{n+1}_\sigma) &= \operatorname{Ext}^0_{A_\sigma}(k, \Omega^{n+1}_\sigma[n+1]) \\ &\cong \mathrm{H}^0 \operatorname{Hom}^{\cdot}_{A_\sigma}(k, \mathcal{K}^{\cdot}_{A_\sigma}) \\ &\cong (\mathcal{K}^0_{A_\sigma})_0 \xrightarrow{\mathrm{Tr}} k \ . \end{aligned} \tag{4.1.1}$$

One can show that

$$\int_{\mathbf{P}} = (-1)^{\lfloor (i+1)/2 \rfloor} \int \tag{4.1.2}$$

where $\lfloor r \rfloor$ is the integer satisfying $r - 1 < \lfloor r \rfloor \le r$. Let us sketch a proof of this claim. First reduce the problem to the commutative case, i.e. $\sigma = 1$. This is done using untwisting. Next observe that both integrals correspond to 0-cocycles in the complex $\operatorname{Hom}^{\cdot}_A(\Omega^*, \mathcal{K}^{\cdot}_A)$. It remains to write down a coboundary which equals their difference; this is done using the immersions $\mathbf{P}^0 \subset \mathbf{P}^1 \subset \cdots \subset \mathbf{P}$ as the hyperplanes at infinity, and induction. Note that this is similar to the calculations in [Li] §8, and to Exercise 7.4 in [Ha], Ch. III.7.

Going back to $A = k[x_1, \ldots, x_n]$, let Q be either of the parameter spaces, Q_n or PGL_{n-1}. From the global description of $(\underline{\Omega}^*, \eta)$ on Q (at the end of §2.2 for Q_n; in §3.2 for PGL_{n-1}) we see that each $\underline{\Omega}^i$ is a free $\mathbb{Z}$-graded, left $\underline{A}$-module, of rank $\binom{n}{i}$. For PGL_{n-1} the Künneth Formula is used. Since Q is an affine scheme, the category of $\mathbb{Z}$-graded, $\mathcal{O}_Q$-quasi-coherent $\underline{A}$-modules is equivalent to the category of $\mathbb{Z}$-graded $\Gamma(Q, \underline{A})$-modules. Hence $(\underline{\Omega}^*, \eta)$ may be used to calculate the sheaves $\underline{\operatorname{Ext}}^{\cdot}_{\underline{A}}(\mathcal{O}_Q, \underline{\Omega}^n)$, giving rise to the global integral (1.2.2).

4.2. Uniqueness of the Deformations. Is it possible to characterize the deformation Ω^*_q in some meaningful way ? For instance, is it universal for some property ? Another problem is to characterize the algebra $B_q := k[\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_n] \subset \Omega^*_q$. In the multiparameter quantum case, this algebra is isomorphic to $((A_q)^!)^\circ$, the opposite of the quadratic dual of A_q (see [Ma1] and [AST] §1). But this isomorphism is not canonical, and seems to be a "coincidence". There is no such isomorphism for A_σ, if $\sigma \in \mathrm{PGL}_{n-1}$ is not diagonalizable.

The knowledge of B_q does determine Ω^*_q, given the data of Thm. 1.2.1, and assuming that $\operatorname{char} k \neq 2$. Writing $\xi_i := \mathrm{d}\, x_i$, one has

$$\begin{aligned} x_j \xi_i - q_{ji} \xi_i x_j &= \tfrac{1}{2}[\mathrm{d}, \eta]_{\mathrm{super}}(x_j \xi_i - q_{ji} \xi_i x_j) \\ &= \tfrac{1}{2}\, \mathrm{d}(x_j x_i - q_{ji} x_i x_j) + \tfrac{1}{2} \eta(\xi_j \xi_i + q_{ji} \xi_i \xi_j) = 0 \ , \end{aligned}$$

so the fourth type of relation in (2.2.1) is a consequence of the first three. Similarly for the relations (3.4.2). But the first three types of relations are the relations of A_q and B_q.

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