THE RIGID DUALIZING COMPLEX OF A UNIVERSAL ENVELOPING ALGEBRA

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ABSTRACT. Let k be a field and A a noetherian (noncommutative) k-algebra. The rigid dualizing complex of A was introduced by Van den Bergh. When $A = \mathrm{U}(\mathfrak{g})$, the enveloping algebra of a finite dimensional Lie algebra $\mathfrak{g}$, Van den Bergh conjectured that the rigid dualizing complex is $(\mathrm{U}(\mathfrak{g}) \otimes \bigwedge^n \mathfrak{g})[n]$, where $n = \dim \mathfrak{g}$. We prove this conjecture, and give a few applications in representation theory and Hochschild cohomology.

0. INTRODUCTION

Dualizing complexes were introduced as part of Grothendieck Duality Theory on schemes, in [RD], and the noncommutative version was first studied in [Ye]. The basic change is that a dualizing complex over a noncommutative ring is a complex of bimodules. For technical reasons we work with noetherian algebras over a base field k, and abbreviate $\otimes := \otimes_k$. Given an algebra A, we write A° for the opposite algebra, and $A^{\mathrm{e}} := A \otimes A^{\circ}$. We consider left modules by default. A dualizing complex R is an object in the bounded derived category of bimodules $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A^{\mathrm{e}})$, of finite injective dimension on both sides, such that the functors $\mathrm{R}\operatorname{Hom}_A(-, R)$ and $\mathrm{R}\operatorname{Hom}_{A^{\circ}}(-, R)$ induce a duality (i.e. a contravariant equivalence) between $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$ and $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A^{\circ})$. The subscript f denotes complexes with finitely generated cohomologies. See [Ye] and [YZ] for details on noncommutative Grothendieck duality.

In the fundamental paper [VdB1], Van den Bergh defined the *rigid dualizing complex* of a k-algebra A. A dualizing complex R is rigid if there exists an isomorphism

$$\rho : R \xrightarrow{\simeq} \mathrm{R}\operatorname{Hom}_{A^{\mathrm{e}}}(A, R \otimes R) \tag{0.1}$$

in $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$, which we shall call a *rigidifying isomorphism*. According to [VdB1], a rigid dualizing complex R, if it exists, is unique up to isomorphism. Moreover it turns out that rigid dualizing complexes are functorial with respect to finite homomorphisms of k-algebras (under some technical restrictions; cf. Theorem 1.2).

For instance, if A is a commutative finite type k-algebra, $\pi : X = \operatorname{Spec} A \to \operatorname{Spec} k$ is the structural morphism and $\pi^! : \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, k) \to \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$ is the twisted inverse image of [RD], then $R := \pi^! k$ is a rigid dualizing complex, and ρ is the fundamental class of the diagonal $X \hookrightarrow X \times X$.

Regarding existence of rigid dualizing complexes, Van den Bergh proved the following result: if A is filtered such that $B := \operatorname{gr} A$ is a connected graded noetherian k-algebra, and B has a *balanced dualizing complex* in the sense of [Ye], then A has

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a rigid dualizing complex. In particular this holds for $A = \mathrm{U}(\mathfrak{g})$, the universal enveloping algebra of a finite dimensional Lie algebra $\mathfrak{g}$.

Our main result verifies a conjecture of Van den Bergh (private communication, 1996):

Theorem 0.2. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over k. Then the rigid dualizing complex of the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ is*

$$R = \big(\mathrm{U}(\mathfrak{g}) \otimes \bigwedge\nolimits^n \mathfrak{g}\big)[n],$$

where $n = \dim \mathfrak{g}$, and we consider $\bigwedge^n \mathfrak{g}$ as a $\mathrm{U}(\mathfrak{g})$-bimodule with trivial action from the left and adjoint action from the right.

Observe that in the two extreme cases – $\mathfrak{g}$ abelian or semisimple – the adjoint representation on $\bigwedge^n \mathfrak{g}$ is trivial. But for a solvable Lie algebra we can get something nontrivial, as shown in Example 2.5. The semisimple case was already known to Van den Bergh (cf. [VdB2] Corollary 6).

An indication that Theorem 0.2 should be true can be seen by deforming $\mathfrak{g}$ to an abelian Lie algebra. In the abelian case $A = \mathrm{U}(\mathfrak{g})$ is a commutative polynomial algebra, and there is a canonical isomorphism $\mathrm{U}(\mathfrak{g}) \otimes \bigwedge^n \mathfrak{g} \cong \Omega^n_{A/k}$. As mentioned before, the complex $\Omega^n_{A/k}[n] = \pi^! k$ is the rigid dualizing complex of A (cf. Remark 2.8).

The proof of Theorem 0.2 is at the end of Section 1. In Section 2 we give a few corollaries of Theorem 0.2, and also an analogous result for a ring $\mathcal{D}(C)$ of differential operators over a smooth commutative k-algebra C.

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1. Proof of Main Result

Let us start with some general facts about rigid dualizing complexes of filtered k-algebras.

If γ is an automorphism of a ring A then the twist of a right module M by γ is M_γ, where the new action is via γ. In particular the twisted bimodule A_γ has basis 1_γ, and $1_\gamma \cdot a = \gamma(a) \cdot 1_\gamma$ for $a \in A$. The shift by $i \in \mathbb{Z}$ of a graded module M is denoted by $M(i)$, whereas the shift of a complex $M^{\cdot}$ is $M^{\cdot}[i]$.

Proposition 1.1. *Let A be a filtered k-algebra, and assume $\operatorname{gr} A$ is a connected graded, noetherian, Artin-Schelter Gorenstein algebra.*

1. *A has a rigid dualizing complex $R_A = \omega_A[n]$ for some integer n and invertible bimodule ω_A. Furthermore $\omega_A \cong A_\gamma$ where γ is a filtered k-algebra automorphism of A.*
2. *The balanced dualizing complex of $\operatorname{gr} A$ is $R_{\operatorname{gr} A} = \omega_{\operatorname{gr} A}[n]$, and $\omega_{\operatorname{gr} A} \cong (\operatorname{gr} A)_{\operatorname{gr}(\gamma)}(m)$ for some integer m.*

Proof. (Cf. [YZ] Proposition 6.18.) Let $\tilde{A} := \operatorname{Rees} A \subset A[t, t^{-1}]$ denote the Rees algebra. Recall that t is a central variable and $(\operatorname{Rees} A)_i = F_i A \cdot t^i$. Since $\tilde{A}$ is also AS-Gorenstein its balanced dualizing complex is $R_{\tilde{A}} = \tilde{A}_{\tilde{\gamma}}(m-1)[n+1]$ where $\tilde{\gamma}$ is a graded k-algebra automorphism and $m, n \in \mathbb{Z}$. Because $\tilde{A}_{\tilde{\gamma}}$ is $k[t]$-central, $\tilde{\gamma}$ is in fact a $k[t]$-algebra automorphism. Now by [YZ] Theorem 6.2, $R_A \cong (\tilde{A}_{\tilde{\gamma}} \otimes_{\tilde{A}} A)[n]$. On the other hand, using the exact sequence $0 \to \tilde{A}(-1) \xrightarrow{t} \tilde{A} \to \operatorname{gr} A \to 0$ we get

$$R_{\operatorname{gr} A} \cong \mathrm{R}\operatorname{Hom}_{\tilde{A}}(\operatorname{gr} A, \tilde{A}_{\tilde{\gamma}}(m-1)[n+1]) \cong (\tilde{A}_{\tilde{\gamma}} \otimes_{\tilde{A}} \operatorname{gr} A)(m)[n].$$

□

We call ω_A the *dualizing bimodule* of A and γ is the *dualizing automorphism.*

Next let us quote a result from [YZ]. A filtration $\{F_i A\}$ is said to be noetherian connected if $\operatorname{gr}^F A$ is a noetherian connected graded k-algebra. A ring homomorphism $A \to B$ is finite centralizing if $B = \sum_{i=1}^{l} A \cdot b_i$ for some elements $b_1, \ldots, b_l \in B$ that commute with A.

Theorem 1.2 ([YZ] Theorem 6.17). *Let $A \to B$ be a finite centralizing homomorphism of k-algebras. Suppose A has a noetherian connected filtration $\{F_i A\}$ and $\operatorname{gr}^F A$ has a balanced dualizing complex. Then the algebras A and B have rigid dualizing complexes R_A and R_B respectively, and the trace morphism $\operatorname{Tr}_{B/A} : R_B \to R_A$ in $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$ exists. The trace induces isomorphisms*

$$R_B \cong \mathrm{R}\operatorname{Hom}_A(B, R_A) \cong \mathrm{R}\operatorname{Hom}_{A^\circ}(B, R_A)$$

in $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$.

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over the field k, let $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra, and denote by $\mathbf{K}_{\cdot}(\mathfrak{h})$ the Chevalley-Eilenberg complex of $\mathrm{U}(\mathfrak{h})$, namely the free resolution of the trivial $\mathfrak{h}$-module k (cf. [CE] Section XIII.7 or [Lo] Section 10.1.3). Recall that for any i one has $\mathbf{K}_i(\mathfrak{h}) := \mathrm{U}(\mathfrak{h}) \otimes \bigwedge^i \mathfrak{h}$, a free left $\mathrm{U}(\mathfrak{h})$-module (the action on the exterior power $\bigwedge^i \mathfrak{h}$ is trivial). The boundary operator $\delta : \mathbf{K}_i(\mathfrak{h}) \to \mathbf{K}_{i-1}(\mathfrak{h})$ is

$$\begin{aligned}\delta(1 \otimes x_1 \wedge \cdots \wedge x_i) = &\sum_{p=1}^{i} (-1)^{p+1} x_p \otimes x_1 \wedge \cdots \widehat{x}_p \cdots \wedge x_i \\ &+ \sum_{1 \leq p < q \leq i} (-1)^{p+q} \otimes [x_p, x_q] \wedge x_1 \wedge \cdots \widehat{x}_p \cdots \widehat{x}_q \cdots \wedge x_i\end{aligned}$$

for $x_1, \ldots, x_i \in \mathfrak{h}$. Define

$$\mathbf{K}_i(\mathfrak{g}; \mathfrak{h}) := \mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{h})} \mathbf{K}_i(\mathfrak{h}) \cong \mathrm{U}(\mathfrak{g}) \otimes \bigwedge^i \mathfrak{h},$$

so that $(\mathbf{K}_{\cdot}(\mathfrak{g}; \mathfrak{h}), \delta)$ is a complex of free left $\mathrm{U}(\mathfrak{g})$-modules. As usual for any two $\mathrm{U}(\mathfrak{g})$-modules M, N the tensor product $M \otimes N$ is also a $\mathrm{U}(\mathfrak{g})$-module by the coproduct.

Lemma 1.3. *Suppose $\mathfrak{h} \subset \mathfrak{g}$ is an ideal, and consider $\bigwedge^i \mathfrak{h}$ as a right $\mathrm{U}(\mathfrak{g})$-module by the adjoint action, so that $\mathbf{K}_i(\mathfrak{g}; \mathfrak{h})$ becomes a $\mathrm{U}(\mathfrak{g})$-bimodule.*

1. *The boundary operator $\delta : \mathbf{K}_i(\mathfrak{g}; \mathfrak{h}) \to \mathbf{K}_{i-1}(\mathfrak{g}; \mathfrak{h})$ commutes with the right $\mathrm{U}(\mathfrak{g})$-action.*
2. *There is a quasi-isomorphism of complexes of $\mathrm{U}(\mathfrak{g})$-bimodules $\mathbf{K}^{\cdot}(\mathfrak{g}; \mathfrak{h}) \to \mathrm{U}(\mathfrak{g}/\mathfrak{h})$.*

Proof. 1. Since $\bigwedge^i \mathfrak{h} \subset \bigwedge^i \mathfrak{g}$ is a $\mathrm{U}(\mathfrak{g})$-submodule for the adjoint action, it follows that $\mathbf{K}_i(\mathfrak{g}; \mathfrak{h}) \subset \mathbf{K}_i(\mathfrak{g})$ is a sub $\mathrm{U}(\mathfrak{g})$-bimodule. Hence we may assume that $\mathfrak{h} = \mathfrak{g}$

and $\mathbf{K}_{\cdot}(\mathfrak{g};\mathfrak{h}) = \mathbf{K}_{\cdot}(\mathfrak{g})$. But then the assertion is [Lo] Proposition 10.1.7. (I wish to thank P. Smith for referring me to [Lo].)

2. As usual we let $\mathbf{K}^i(\mathfrak{g};\mathfrak{h}) := \mathbf{K}_{-i}(\mathfrak{g};\mathfrak{h})$, and the coboundary operator is $(-1)^{i+1}\delta : \mathbf{K}^i(\mathfrak{g};\mathfrak{h}) \to \mathbf{K}^{i+1}(\mathfrak{g};\mathfrak{h})$. Since $\mathrm{U}(\mathfrak{h}) \to \mathrm{U}(\mathfrak{g})$ is flat we get $\mathrm{H}^i\mathbf{K}^{\cdot}(\mathfrak{g};\mathfrak{h}) = 0$ if $i < 0$. For $i = 0$ we note that $\mathrm{U}(\mathfrak{g}) \cdot \mathfrak{h} = \mathfrak{h} \cdot \mathrm{U}(\mathfrak{g})$ is a two-sided ideal, and

$$\mathrm{U}(\mathfrak{g}/\mathfrak{h}) \cong \mathrm{U}(\mathfrak{g}) / \mathrm{U}(\mathfrak{g}) \cdot \mathfrak{h} \cong \mathrm{H}^0\mathbf{K}^{\cdot}(\mathfrak{g};\mathfrak{h})$$

as $\mathrm{U}(\mathfrak{g})$-bimodules. □

For any k-module M let $M^* := \mathrm{Hom}_k(M, k)$. We consider $\bigwedge^n \mathfrak{g}^*$ as a right $\mathrm{U}(\mathfrak{g})$-module with the coadjoint action, and a left $\mathrm{U}(\mathfrak{g})$-module with the trivial action.

Lemma 1.4. *Let $\mathfrak{h} \subset \mathfrak{g}$ be an ideal, with $\dim_k \mathfrak{h} = m$. Assume that $\gamma(\mathrm{U}(\mathfrak{g}) \cdot \mathfrak{h}) = \mathrm{U}(\mathfrak{g}) \cdot \mathfrak{h}$, where γ is the dualizing automorphism of $\mathrm{U}(\mathfrak{g})$. Then*

$$\mathrm{Ext}^q_{\mathrm{U}(\mathfrak{g})}\big(\mathrm{U}(\mathfrak{g}/\mathfrak{h}), \mathrm{U}(\mathfrak{g})\big) \cong \begin{cases} \mathrm{U}(\mathfrak{g}/\mathfrak{h}) \otimes \bigwedge^m \mathfrak{h}^* & \text{if } q = m \\ 0 & \text{if } q \neq m \end{cases}$$

as $\mathrm{U}(\mathfrak{g})$-bimodules.

Proof. Since $\mathrm{gr}\,\mathrm{U}(\mathfrak{g})$ is a commutative polynomial algebra in n variables we know that its balanced dualizing complex is $R_{\mathrm{gr}\,\mathrm{U}(\mathfrak{g})} \cong (\mathrm{gr}\,\mathrm{U}(\mathfrak{g})(-n)[n]$. Therefore by Proposition 1.1 the rigid dualizing complexes of $\mathrm{U}(\mathfrak{g})$ and $\mathrm{U}(\mathfrak{g}/\mathfrak{h})$ are $R_{\mathrm{U}(\mathfrak{g})} \cong \mathrm{U}(\mathfrak{g})_{\gamma}[n]$ and $R_{\mathrm{U}(\mathfrak{g}/\mathfrak{h})} \cong \mathrm{U}(\mathfrak{g}/\mathfrak{h})_{\tau}[n-m]$, respectively, where τ is the dualizing automorphism of $\mathrm{U}(\mathfrak{g}/\mathfrak{h})$. According to Theorem 1.2 we get the vanishing of all Ext^q, $q \neq m$, and

$$M := \mathrm{Ext}^m_{\mathrm{U}(\mathfrak{g})}\big(\mathrm{U}(\mathfrak{g}/\mathfrak{h}), \mathrm{U}(\mathfrak{g})\big) \cong \mathrm{U}(\mathfrak{g}/\mathfrak{h})_{\tau\gamma^{-1}}$$

as $\mathrm{U}(\mathfrak{g})$-bimodules.

According to Lemma 1.3 we get

$$M = \mathrm{H}^m\,\mathrm{Hom}_{\mathrm{U}(\mathfrak{g})}\big(\mathbf{K}^{\cdot}(\mathfrak{g};\mathfrak{h}), \mathrm{U}(\mathfrak{g})\big),$$

so the bimodule M is a quotient of $\mathrm{U}(\mathfrak{g}) \otimes \bigwedge^m \mathfrak{h}^*$. Let α be any k-basis of $\bigwedge^m \mathfrak{h}^*$, and let β be the image of $1 \otimes \alpha \in \mathrm{U}(\mathfrak{g}) \otimes \bigwedge^m \mathfrak{h}^*$ in the $\mathrm{U}(\mathfrak{g}/\mathfrak{h})$-bimodule M. Hence for any $x \in \mathfrak{g}$ we have

$$\beta \cdot x = (x - \mathrm{tr}(\mathrm{ad}_{\bigwedge^m \mathfrak{h}^*} x)) \cdot \beta.$$

Since M is free of rank 1 on either side as $\mathrm{U}(\mathfrak{g}/\mathfrak{h})$-module, and since $\mathrm{U}(\mathfrak{g}/\mathfrak{h})$ is an integral domain, it follows that the generator β is a basis of M. Sending $\beta \mapsto 1 \otimes \alpha \in \mathrm{U}(\mathfrak{g}/\mathfrak{h}) \otimes \bigwedge^m \mathfrak{h}^*$ is the desired isomorphism of $\mathrm{U}(\mathfrak{g})$-bimodules. □

Here is another result of Van den Bergh (cf. [VdB2], proof of Corollary 6).

Lemma 1.5. *Let A be a positively filtered k-algebra such that $\mathrm{gr}\,A$ is commutative and $\mathrm{gr}_0 A = k$. Let $\mathfrak{g} := \mathrm{gr}_1 A$, so $\mathfrak{g}$ is a Lie algebra over k. Let γ be a filtered k-algebra automorphism of A such that $\mathrm{gr}(\gamma)$ is the identity. Then there is a Lie homomorphism $\lambda : \mathfrak{g} \to k$ such that $\gamma(a) = a + \lambda(\bar{a})$ for all $a \in F_1 A$, where $\bar{a} \in \mathfrak{g}$ is the symbol of a.*

Proof. Define $\lambda(a) := \gamma(a) - a$ for $a \in F_1 A$. It factors through $F_1 A \twoheadrightarrow \mathfrak{g} \to F_0 A \hookrightarrow F_1 A$, is easily seen to be k-linear, and $\lambda([a, b]) = 0$. □

At last here is the proof of our main result.

Proof of Theorem 0.2. According to Proposition 1.1, the rigid dualizing complex of $\mathrm{U}(\mathfrak{g})$ is $R_{\mathrm{U}(\mathfrak{g})} \cong \mathrm{U}(\mathfrak{g})_\gamma[n]$; and $\mathrm{gr}(\gamma)$ is the identity. In view of Lemma 1.5, it remains to prove that $\lambda = -\operatorname{tr}\operatorname{ad}_{\bigwedge^n \mathfrak{g}}$. Since λ is a Lie homomorphism it has to vanish on the commutator ideal $\mathfrak{h} := [\mathfrak{g}, \mathfrak{g}]$, and so it factors through $\mathfrak{a} := \mathfrak{g}/\mathfrak{h}$. Therefore it suffices to prove that the induced automorphism $\bar{\gamma}$ of $\mathrm{U}(\mathfrak{a})$ satisfies $\bar{\gamma}(y) = y - \operatorname{tr}(\operatorname{ad}_{\bigwedge^n \mathfrak{g}} y)$ for $y \in \mathfrak{a}$.

The algebra $\mathrm{U}(\mathfrak{a})$ is a commutative polynomial algebra in $l = n - m$ variables, where $m = \dim_k \mathfrak{h}$, so its rigid dualizing complex is $\mathrm{U}(\mathfrak{a})[l]$. According to Lemma 1.4 and Theorem 1.2 we get

$$\mathrm{U}(\mathfrak{a}) \cong \operatorname{Ext}^m_{\mathrm{U}(\mathfrak{g})}\big(\mathrm{U}(\mathfrak{a}), \mathrm{U}(\mathfrak{g})_\gamma\big) \cong \mathrm{U}(\mathfrak{a})_\gamma \otimes \bigwedge^m \mathfrak{h}^*$$

as $\mathrm{U}(\mathfrak{g})$-bimodules. Therefore $\mathrm{U}(\mathfrak{a})_{\bar{\gamma}} \cong \mathrm{U}(\mathfrak{a}) \otimes \bigwedge^m \mathfrak{h}$, so $\bar{\gamma}(y) = y - \operatorname{tr}(\operatorname{ad}_{\bigwedge^m \mathfrak{h}} y)$ for all $y \in \mathfrak{a}$. Finally, since $\bigwedge^{n-m} \mathfrak{a}$ is a trivial representation of $\mathfrak{g}$, one has $\bigwedge^m \mathfrak{h} \cong \bigwedge^n \mathfrak{g}$. $\square$

Question 1.6. Suppose $\mathfrak{g}$ is semisimple and $\operatorname{char} k = 0$. Does the quantum enveloping algebra $\mathrm{U}_q(\mathfrak{g})$ admit a rigid dualizing complex? If so, what is it?

2. Some Corollaries and Complements

Corollary 2.1. *Let M be any finitely generated $\mathrm{U}(\mathfrak{g})$-module, pure of* GKdim $= m$, *and let $I := \operatorname{Ann}_{\mathrm{U}(\mathfrak{g})} M$. Then*

$$\operatorname{Ann}_{\mathrm{U}(\mathfrak{g})^\circ} \operatorname{Ext}^{n-m}_{\mathrm{U}(\mathfrak{g})}\big(M, \mathrm{U}(\mathfrak{g})\big) = \gamma(I) \subset \mathrm{U}(\mathfrak{g})^\circ,$$

where γ is the dualizing automorphism.

Proof. Let us view γ as an anti-isomorphism $\gamma : \mathrm{U}(\mathfrak{g}) \to \mathrm{U}(\mathfrak{g})^\circ$. Define $M' := \operatorname{Ext}^{n-m}_{\mathrm{U}(\mathfrak{g})}\big(M, \mathrm{U}(\mathfrak{g})\big)$ and $I' := \operatorname{Ann}_{\mathrm{U}(\mathfrak{g})^\circ} M'$. By [YZ] Proposition 6.18(4) one has $\gamma(I) \subset I'$. Since M is pure, $M \subset M'' := \operatorname{Ext}^{n-m}_{\mathrm{U}(\mathfrak{g})^\circ}\big(M', \mathrm{U}(\mathfrak{g})\big)$. Hence $\gamma^{-1}(I') \subset \operatorname{Ann}_{\mathrm{U}(\mathfrak{g})} M'' \subset I$. $\square$

It is a standard fact that if M is a finite dimensional representation of $\mathfrak{g}$, then $\operatorname{Ext}^q_{\mathrm{U}(\mathfrak{g})}\big(M, \mathrm{U}(\mathfrak{g})\big) = 0$ for $q < n$. The group $\operatorname{Ext}^n_{\mathrm{U}(\mathfrak{g})}\big(M, \mathrm{U}(\mathfrak{g})\big)$ is a right $\mathrm{U}(\mathfrak{g})$-module, but the structure is not obvious[1]. Since we can make M into a $\mathrm{U}(\mathfrak{g})$-bimodule with trivial right action, the next corollary gives the answer.

Corollary 2.2. *Suppose M is a finite dimensional k-central $\mathrm{U}(\mathfrak{g})$-bimodule. Then there is an isomorphism of $\mathrm{U}(\mathfrak{g})$-bimodules*

$$\operatorname{Ext}^n_{\mathrm{U}(\mathfrak{g})}\big(M, \mathrm{U}(\mathfrak{g})\big) \cong M^* \otimes \bigwedge^n \mathfrak{g}^*,$$

which is functorial in M.

Proof. Let $I := \operatorname{Ann}_{\mathrm{U}(\mathfrak{g})} M$ and $B := \mathrm{U}(\mathfrak{g})/I$. Since $k \to B$ is a finite homomorphism the rigid dualizing complex of B is $B^* = \operatorname{Hom}_k(B, k)$. By [YZ] Proposition 3.9,

$$\operatorname{Ext}^n_{\mathrm{U}(\mathfrak{g})}\big(M, \mathrm{U}(\mathfrak{g}) \otimes \bigwedge^n \mathfrak{g}\big) \cong \operatorname{Hom}_B(M, B^*) \cong M^*$$

as $\mathrm{U}(\mathfrak{g})$-bimodules. Now twist by $\bigwedge^n \mathfrak{g}^*$. $\square$

[1] The right module structure was calculated by S. Chemla [Bull. Soc. Math. France **122** (1994)].

Theorem 0.2 has an interpretation in terms of Hochschild cohomology. For a $\mathrm{U}(\mathfrak{g})$-bimodule M denote by $\mathrm{H}^q(\mathrm{U}(\mathfrak{g}), M)$ and $\mathrm{H}_q(\mathrm{U}(\mathfrak{g}), M)$ the Hochschild cohomology and homology, respectively.

Corollary 2.3. *There are* $\mathrm{U}(\mathfrak{g})$*-bimodule isomorphisms*

$$\mathrm{H}^q\big(\mathrm{U}(\mathfrak{g}), \mathrm{U}(\mathfrak{g})^{\mathrm{e}}\big) \cong \begin{cases} \mathrm{U}(\mathfrak{g}) \otimes \bigwedge^n \mathfrak{g}^* & \text{if } q = n \\ 0 & \text{if } q \neq n. \end{cases}$$

Proof. Let's write $\omega := \omega_{\mathrm{U}(\mathfrak{g})}$ and $\omega^\vee := \mathrm{Hom}_{\mathrm{U}(\mathfrak{g})}(\omega, \mathrm{U}(\mathfrak{g}))$. By formula (0.1), $\omega \cong \mathrm{Ext}^n_{\mathrm{U}(\mathfrak{g})}\big(\mathrm{U}(\mathfrak{g}), \omega \otimes \omega\big)$ as bimodules, so applying the twist $- \otimes_{\mathrm{U}(\mathfrak{g})^{\mathrm{e}}} (\omega^\vee \otimes \omega^\vee)$ we get $\omega^\vee \cong \mathrm{Ext}^n_{\mathrm{U}(\mathfrak{g})}\big(\mathrm{U}(\mathfrak{g}), \mathrm{U}(\mathfrak{g})^{\mathrm{e}}\big)$. But by Theorem 0.2, $\omega^\vee \cong \mathrm{U}(\mathfrak{g}) \otimes \bigwedge^n \mathfrak{g}^*$. □

In [VdB2], Van den Bergh proves a Poincaré duality between the Hochschild cohomology and homology of certain Gorenstein algebras A. We obtain the following variation of his result.

Corollary 2.4. *Let M be any k-central* $\mathrm{U}(\mathfrak{g})$*-bimodule. Then*

$$\mathrm{H}^q\big(\mathrm{U}(\mathfrak{g}), M\big) \cong \mathrm{H}_{n-q}\big(\mathrm{U}(\mathfrak{g}), M \otimes \bigwedge^n \mathfrak{g}^*\big).$$

Proof. Corollary 2.3 says that

$$\mathrm{R}\,\mathrm{Hom}_{\mathrm{U}(\mathfrak{g})^{\mathrm{e}}}\big(\mathrm{U}(\mathfrak{g}), \mathrm{U}(\mathfrak{g})^{\mathrm{e}}\big)[n] \cong \omega^\vee \cong \mathrm{U}(\mathfrak{g}) \otimes \bigwedge^n \mathfrak{g}^*$$

in $\mathsf{D}(\mathsf{Mod}\,\mathrm{U}(\mathfrak{g})^{\mathrm{e}})$. Copying the proof of [VdB2] Theorem 1 we obtain

$$\begin{aligned} \mathrm{H}^q\big(\mathrm{U}(\mathfrak{g}), M\big) &\cong \mathrm{H}^q\,\mathrm{R}\,\mathrm{Hom}_{\mathrm{U}(\mathfrak{g})^{\mathrm{e}}}\big(\mathrm{U}(\mathfrak{g}), M\big) \\ &\cong \mathrm{H}^q\Big(\mathrm{R}\,\mathrm{Hom}_{\mathrm{U}(\mathfrak{g})^{\mathrm{e}}}\big(\mathrm{U}(\mathfrak{g}), \mathrm{U}(\mathfrak{g})^{\mathrm{e}}\big) \otimes^{\mathrm{L}}_{\mathrm{U}(\mathfrak{g})^{\mathrm{e}}} M\Big) \\ &\cong \mathrm{H}^{q-n}\big(\omega^\vee \otimes^{\mathrm{L}}_{\mathrm{U}(\mathfrak{g})^{\mathrm{e}}} M\big) \\ &\cong \mathrm{H}^{q-n}\big(\mathrm{U}(\mathfrak{g}) \otimes^{\mathrm{L}}_{\mathrm{U}(\mathfrak{g})^{\mathrm{e}}} (M \otimes_{\mathrm{U}(\mathfrak{g})} \omega^\vee)\big) \\ &\cong \mathrm{H}_{n-q}\big(\mathrm{U}(\mathfrak{g}), M \otimes \bigwedge^n \mathfrak{g}^*\big). \end{aligned}$$

□

Here is an easy example where the dualizing bimodule ω is not trivial.

Example 2.5. Let $\mathfrak{g}$ be the nonabelian 2-dimensional Lie algebra, with basis x, y such that $[x, y] = y$. Then $\mathrm{tr}(\mathrm{ad}_{\bigwedge^2 \mathfrak{g}}\, x) = 1$.

If $\mathrm{char}\, k = 0$ and C is a smooth, integral, commutative k-algebra then the ring of differential operators $\mathcal{D}(C)$ is noetherian and has finite global dimension. Since $\mathcal{D}(C)$ can be deformed to a smooth commutative k-algebra (namely the algebra of functions on the cotangent bundle of $\mathrm{Spec}\, C$), one could expect $\mathcal{D}(C)$ to have a rigid dualizing complex. This is indeed true, and follows from results in $\mathcal{D}$-module theory.

Theorem 2.6. *Let C be a smooth, integral, commutative k-algebra of dimension n, and assume* $\mathrm{char}\, k = 0$*. Let $\mathcal{D}(C)$ be the ring of differential operators. Then the rigid dualizing complex of $\mathcal{D}(C)$ is $\mathcal{D}(C)[2n]$.*

Proof. Let $X := \mathrm{Spec}\, C$ and $X^{\mathrm{e}} := X \times X \cong \mathrm{Spec}\, C^{\mathrm{e}}$. Then $\Gamma(X, \mathcal{D}_X) \cong \mathcal{D}(C)$, $\Gamma(X^{\mathrm{e}}, \mathcal{D}_{X^{\mathrm{e}}}) \cong \mathcal{D}(C) \otimes \mathcal{D}(C)$ and $\mathcal{D}(C)^\circ \cong \omega_C \otimes_C \mathcal{D}(C) \otimes_C \omega_C^\vee$.

The sheaf $\mathcal{D}_X \otimes_{\mathcal{O}_X} \omega_X^\vee$ is filtered, and has two commuting left $\mathcal{D}_X$-module structures. The two structures coincide on $\mathrm{gr}(\mathcal{D}_X \otimes_{\mathcal{O}_X} \omega_X^\vee) \cong (\mathrm{gr}\, \mathcal{D}_X) \otimes_{\mathcal{O}_X} \omega_X^\vee$.

Hence there is an involution of $\mathcal{D}_X \otimes_{\mathcal{O}_X} \omega_X^\vee$, which is the identity on the subsheaf $\omega_X^\vee = F_0(\mathcal{D}_X \otimes_{\mathcal{O}_X} \omega_X^\vee)$, and exchanges the two $\mathcal{D}_X$-module structures.

Denote by $\mathbf{D}_X$ the duality functor on left $\mathcal{D}_X$-modules, namely $\mathbf{D}_X \mathcal{M} := \mathrm{R}\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X \otimes_{\mathcal{O}_X} \omega_X^\vee)[n]$; cf. [Bo] VI.3.6. Let $f : X \hookrightarrow X^{\mathrm{e}}$ be the diagonal embedding. According to [Bo] Proposition VII.9.6 there is a functorial isomorphism $\mathbf{D}_{X^{\mathrm{e}}}\, f_+ \cong f_+\, \mathbf{D}_X$. We shall apply this isomorphism with the $\mathcal{D}_X$-module $\mathcal{O}_X$.

First note that $\mathbf{D}_X\, \mathcal{O}_X \cong \mathcal{O}_X$, as can be checked using the quasi-isomorphism $\Omega^{\cdot}_X(\mathcal{D}_X)[n] \otimes_{\mathcal{O}_X} \omega_X^\vee \to \mathcal{O}_X$ in $\mathsf{Mod}\, \mathcal{D}_X$; cf. [Bo] VI.3.5. Next, by [Bo] Theorem VI.7.4(ii) and Theorem VI.7.11 (Kashiwara's Theorem) we see that $f_+\, \mathcal{O}_X \cong \mathcal{D}_X \otimes_{\mathcal{O}_X} \omega_X^\vee$ in $\mathsf{Mod}\, \mathcal{D}_{X^{\mathrm{e}}}$. Thus we have an isomorphism of $\mathcal{D}_{X^{\mathrm{e}}}$-modules

$$\mathcal{D}_X \otimes_{\mathcal{O}_X} \omega_X^\vee \cong \mathcal{E}xt^{2n}_{\mathcal{D}_{X^{\mathrm{e}}}}\big(\mathcal{D}_X \otimes_{\mathcal{O}_X} \omega_X^\vee, \mathcal{D}_{X^{\mathrm{e}}} \otimes_{\mathcal{O}_{X^{\mathrm{e}}}} \omega_{X^{\mathrm{e}}}^\vee\big).$$

Passing to global sections, replacing $\mathcal{D}(C)$ by $\mathcal{D}(C)^\circ$ and using the involution of $\mathcal{D}(C) \otimes_C \omega_C^\vee$, we get

$$\begin{aligned}
&\mathcal{D}(C) \otimes_C \omega_C^\vee \\
&\quad \cong \mathrm{Ext}^{2n}_{\mathcal{D}(C)\otimes\mathcal{D}(C)}\big(\mathcal{D}(C) \otimes_C \omega_C^\vee, (\mathcal{D}(C) \otimes_C \omega_C^\vee) \otimes (\mathcal{D}(C) \otimes_C \omega_C^\vee)\big) \\
&\quad \cong \mathrm{Ext}^{2n}_{\mathcal{D}(C)\otimes\mathcal{D}(C)^\circ}\big(\mathcal{D}(C), (\mathcal{D}(C) \otimes_C \omega_C^\vee) \otimes \mathcal{D}(C)\big) \\
&\quad \cong \mathrm{Ext}^{2n}_{\mathcal{D}(C)^{\mathrm{e}}}\big(\mathcal{D}(C), \mathcal{D}(C) \otimes \mathcal{D}(C)\big) \otimes_C \omega_C^\vee.
\end{aligned}$$

Twisting by ω_C and shifting degrees we obtain an isomorphism

$$\mathcal{D}(C)[2n] \cong \mathrm{R}\,\mathrm{Hom}_{\mathcal{D}(C)^{\mathrm{e}}}\big(\mathcal{D}(C), \mathcal{D}(C)[2n] \otimes \mathcal{D}(C)[2n]\big)$$

in $\mathsf{D}(\mathsf{Mod}\, \mathcal{D}(C)^{\mathrm{e}})$. □

By the same arguments given for Corollaries 2.3 and 2.4, one has:

Corollary 2.7. *Let $\mathcal{D}(C)$ be as above. Then there are $\mathcal{D}(C)$-bimodule isomorphisms*

$$\mathrm{H}^q\big(\mathcal{D}(C), \mathcal{D}(C)^{\mathrm{e}}\big) \cong \begin{cases} \mathcal{D}(C) & \text{if } q = 2n \\ 0 & \text{if } q \neq 2n. \end{cases}$$

For any k-central $\mathcal{D}(C)$-bimodule M one has

$$\mathrm{H}^q\big(\mathcal{D}(C), M\big) \cong \mathrm{H}_{2n-q}\big(\mathcal{D}(C), M\big).$$

Remark 2.8. One can show that there is a canonical choice for the rigidifying isomorphism ρ of the complex $R = \omega[n]$, $\omega = \mathrm{U}(\mathfrak{g}) \otimes \bigwedge^n \mathfrak{g}$. This amounts to choosing an isomorphism of bimodules $\rho : \omega \xrightarrow{\simeq} E^n(\mathrm{U}(\mathfrak{g}))$, where $E^n(\mathrm{U}(\mathfrak{g})) := \mathrm{Ext}^n_{\mathrm{U}(\mathfrak{g})^{\mathrm{e}}}\big(\mathrm{U}(\mathfrak{g}), \omega \otimes \omega\big)$. Here is a sketch of the proof. Let $A := \mathrm{gr}\, \mathrm{U}(\mathfrak{g}) = \mathrm{S}(\mathfrak{g})$. The bimodule ω is filtered, and there is a canonical isomorphism $\mathrm{gr}\, \omega \cong \Omega^n_{A/k}$. The standard spectral sequence of the filtration identifies $\mathrm{gr}\, E^n(\mathrm{U}(\mathfrak{g}))$ with $E^n(A) := \mathrm{Ext}^n_{A^{\mathrm{e}}}(A, \Omega^{2n}_{A^{\mathrm{e}}/k})$. But as mentioned in the Introduction, $\Omega^n_{A/k}[n]$ is the rigid dualizing complex of A, and it comes equipped with a canonical isomorphism $\Omega^n_{A/k} \xrightarrow{\simeq} E^n(A)$. This isomorphism determines ρ. A similar statement holds for Theorem 2.6. As a consequence the isomorphisms of Corollaries 2.3, 2.4 and 2.7 are canonical. (I thank Van den Bergh for mentioning this idea to me.)

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