DUALIZING COMPLEXES AND TILTING COMPLEXES OVER SIMPLE RINGS

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0. Introduction

Simple rings, like fields, are literally 'simple' in many ways. Hence quite a few invariants of rings become trivial for simple rings. We show that this principle applies to the derived Picard group, which classifies dualizing complexes over a ring.

In this paper all rings are algebras over a base field k, ring homomorphisms are all over k, and bimodules are all k-central. The symbol $\otimes$ denotes $\otimes_k$. For a ring B, B° denotes the opposite ring.

We shall write $\mathsf{Mod}\, A$ for the category of left A-modules, and $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)$ will stand for the bounded derived category. A brief review of key definitions such as dualizing complexes, two-sided tilting complexes and the derived Picard group $\mathrm{DPic}(A)$ is included in the body of the paper.

Theorem 0.1. *Let A and B be rings and let $T \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}(A \otimes B^\circ))$ be a two-sided tilting complex. Suppose either A or B is a Goldie simple ring.*

1. *$T \cong P[n]$ for some integer n and some invertible A-B-bimodule P. Therefore A and B are Morita equivalent, and in particular both are Goldie simple rings.*
2. *The structure of the derived Picard group of A is $\mathrm{DPic}(A) = \mathbb{Z} \times \mathrm{Pic}(A)$.*

An algebra is called *Gorenstein* if it has finite left and right injective dimension.

Theorem 0.2. *Let A be a left noetherian ring and let B be a right noetherian ring. Let R be a dualizing complex over (A, B). Assume either of the two conditions below hold.*

- (i) *A and B are both Goldie simple rings.*
- (ii) *Either A or B is a Goldie simple ring, and either A or B is noetherian and admits some dualizing complex.*

Then $R \cong P[n]$ for some integer n and some invertible A-B-bimodule P, the rings A and B are Morita equivalent, and both are noetherian Gorenstein simple rings.

One motivating question is to classify all dualizing and tilting complexes over the Weyl algebras. When the base field has characteristic zero, this question is answered by Theorems 0.1 and 0.2. When the base field has positive characteristic, the same answer is given in Section 5.

Theorem 0.2 also has a surprising consequence.

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Corollary 0.3. *Let A be a filtered ring such that the associated graded ring* $\operatorname{gr} A$ *is connected graded and noetherian. Suppose either one of the following conditions holds:*

(i) $\operatorname{gr} A$ *is commutative.*
(ii) $\operatorname{gr} A$ *is PI.*
(iii) $\operatorname{gr} A$ *is FBN.*
(iv) $\operatorname{gr} A$ *has enough normal elements in the sense of* [YZ, p.36].
(v) $\operatorname{gr} A$ *is a factor ring of a graded AS-Gorenstein ring.*

If A is simple, then A is Gorenstein. In cases (i-iv)*, A is also Auslander-Gorenstein and Cohen-Macaulay.*

For example, every simple factor ring A of the enveloping algebra $U(L)$ of a finite dimensional Lie algebra L is Auslander-Gorenstein and Cohen-Macaulay. This is also true for simple factor rings of many quantum algebras listed in [GL].

In Section 1 we review some basic facts about bimodules over simple rings. Theorem 0.1 is proved at the end of Section 2. Theorem 0.2 is proved in Section 3, and Corollary 0.3 is proved in Section 4. In Section 5 we prove statements analogous to Theorems 0.1 and 0.2 when A is a Weyl algebra over a base field k of positive characteristic. In Section 6 we discuss an example of Goodearl and Warfield which shows that not every noetherian simple ring is Gorenstein.

1. Preliminaries

Let A be a ring (i.e. a k-algebra). By an A-module we mean a left A-module. With this convention an A°-module means a right A-module. A finitely generated A-module is called *finite*.

Our reference for derived categories is [RD]. As for derived categories and derived functors of bimodules, such as $\mathrm{R}\operatorname{Hom}$ and $\otimes^{\mathrm{L}}$, the reader is referred to [Ye1] and [Ye2].

The following elementary facts will be used later.

Lemma 1.1. *Let A be a ring and let B be a* (*left and right*) *Goldie simple ring. Let M be a nonzero A-B-bimodule finite on both sides. Then:*

(1) *M is a generator of* $\mathsf{Mod}\, B^\circ$.
(2) *If the canonical homomorphism $A \to \operatorname{End}_{B^\circ}(M)$ is bijective, then M is projective as A-module.*
(3) *Suppose that A is also a Goldie simple ring, and that both $A \to \operatorname{End}_{B^\circ}(M)$ and $B^\circ \to \operatorname{End}_A(M)$ are bijective. Then M is an invertible bimodule.*

Proof. (1) Suppose $M = \sum_{i=1}^p A \cdot m_i$ and let $N_i := \operatorname{Ann}_{B^\circ}(m_i)$. Then $\operatorname{Ann}_{B^\circ} M = \bigcap_{i=1}^p N_i$. Since B is a simple ring and $M \neq 0$ we must have $\operatorname{Ann}_{B^\circ} M = 0$. Hence for some i the right ideal $N_i \subset B$ is not essential. This implies the element m_i is not torsion, and so the B°-module M is not torsion.

At this point we can forget the A-module structure on M. So let M be a finite B°-module that is not torsion. We will show that $\operatorname{Hom}_{B^\circ}(M, B) \neq 0$. Replacing M by a quotient of it we may assume M is a finite uniform torsion-free B°-module. In this case we have injections $M \to M \otimes_B Q \to Q$ where Q is the total ring of fractions of B.

Without loss of generality we can assume M is a finite B°-submodule of Q. Thus $M = \sum_{i=1}^q s_i^{-1} x_i \cdot B^\circ$ for certain $s_i, x_i \in B$ with s_i regular elements. Passing to a left common denominator we have $s_i^{-1}x_i = s^{-1}y_i$ for suitable $s, y_i \in B$. Therefore left multiplication by s is a nonzero B^{o}-linear map $\lambda_s : M \to B$.

Finally we reduce to the case of a finite B^{o}-module M such that $\mathrm{Hom}_{B^{\mathrm{o}}}(M, B) \neq 0$. Let $I \subset B$ be the union of the images of all B^{o}-linear homomorphisms $M \to B$. This is a nonzero two-sided ideal, and hence $I = B$. So there are some homomorphisms $\phi_i : M \to B$ such that $1 \in \sum_{i=1}^{r} \phi_i(M) \subset B$. Thus $\sum \phi_i : M^r \to B$ is surjective, proving that M is a generator of $\mathsf{Mod}\, B^{\mathrm{o}}$.

(2) Since M is a generator of $\mathsf{Mod}\, B^{\circ}$ and $A \to \mathrm{End}_{B^{\circ}}(M)$ is bijective, a theorem of Morita [Mo] (see [AF, 17.8]) says that M is a finite projective A-module.

(3) By parts (1) and (2), the A-module M is also a finite projective generator. By Morita's theorem the bimodule M is invertible. □

Lemma 1.2. *Let M be a bounded complex of B°-modules with nonzero cohomology such that $\mathrm{Ext}^i_{B^{\circ}}(M, M) = 0$ for all $i < 0$. Let $i_0 := \min\{i \mid \mathrm{H}^i M \neq 0\}$ and $j_0 := \max\{j \mid \mathrm{H}^j M \neq 0\}$. If $i_0 \neq j_0$ (i.e. $i_0 < j_0$), then $\mathrm{Hom}_{B^{\circ}}(\mathrm{H}^{j_0} M, \mathrm{H}^{i_0} M) = 0$.*

Proof. This is true because a nonzero morphism from $\mathrm{H}^{j_0} M$ to $\mathrm{H}^{i_0} M$ gives rise to a nonzero element in $\mathrm{Ext}^{i_0 - j_0}_{B^{\circ}}(M, M)$. □

Lemma 1.3. *Let M be a bounded complex of A-B-bimodules with nonzero cohomology. Suppose the following conditions hold:*

(i) *B is Goldie and simple.*
(ii) *$\mathrm{Ext}^i_{B^{\circ}}(M, M) = 0$ for all $i \neq 0$.*
(iii) *$\mathrm{H}^{j_0} M$ is finite on both sides, where j_0 is as in Lemma 1.2.*

Then $M \cong (\mathrm{H}^{j_0} M)[-j_0]$ in $\mathsf{D}(\mathsf{Mod}(A \otimes B^{\circ}))$.

Proof. By Lemma 1.1(1), $\mathrm{H}^{j_0} M$ is a generator of $\mathsf{Mod}\, B^{\circ}$. Let i_0 be as in Lemma 1.2. If $i_0 < j_0$ then the conclusion of Lemma 1.2 contradicts the fact that $\mathrm{H}^{j_0} M$ is a generator of $\mathsf{Mod}\, B^{\circ}$. Therefore $i_0 = j_0$ and the assertion follows. □

2. Two-Sided Tilting Complexes

The following definition is due to Rickard [Ri1, Ri2] and Keller [Ke]. Recall that "ring" means "k-algebra".

Definition 2.1. Let A and B be rings and let $T \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}(A \otimes B^{\circ}))$ be a complex. We say T is a *two-sided tilting complex* over (A, B) if there exists a complex $T^{\vee} \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}(B \otimes A^{\circ}))$ such that $T \otimes^{\mathrm{L}}_B T^{\vee} \cong A$ in $\mathsf{D}(\mathsf{Mod}(A \otimes A^{\circ}))$, and $T^{\vee} \otimes^{\mathrm{L}}_A T \cong B$ in $\mathsf{D}(\mathsf{Mod}(B \otimes B^{\circ}))$.

The complex T, when considered as a complex of left A-modules, is perfect, and the set $\mathsf{add}\, T \subset \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)$, namely the direct summands of finite direct sums of T, generates the category $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)_{\mathrm{perf}}$ of perfect complexes. The formula for $T^{\vee}$ is $T^{\vee} \cong \mathrm{R}\,\mathrm{Hom}_A(T, A)$. The canonical morphism $B \mapsto \mathrm{R}\,\mathrm{Hom}_A(T, T)$ in $\mathsf{D}(\mathsf{Mod}(B \otimes B^{\circ}))$ is an isomorphism. The functor $M \mapsto T \otimes^{\mathrm{L}}_B M$ is an equivalence $\mathsf{D}(\mathsf{Mod}\, B) \to \mathsf{D}(\mathsf{Mod}\, A)$ preserving boundedness. By symmetry there are three more variations of all these assertions (e.g. $T^{\vee}$ is a perfect complex of A°-modules). See [Ye2] for proofs.

The next definition is due to the first author [Ye2]. When $B = A$ we write $A^{\mathrm{e}} := A \otimes A^{\circ}$.

Definition 2.2. Let A be ring. The *derived Picard group* of A is defined to be

$$\mathrm{DPic}(A) := \frac{\{ \text{ two-sided tilting complexes } T \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A^{\mathrm{e}}) \}}{\text{isomorphism}},$$

with operation $(T, S) \mapsto T \otimes^{\mathrm{L}}_A S$.

Clearly the definition of the group $\mathrm{DPic}(A)$ is relative to the base field k. For instance if $A = K$ is a field extension of k then $\mathrm{DPic}(K) = \mathbb{Z} \times \mathrm{Gal}(K/k)$, where $\mathrm{Gal}(K/k)$ is the Galois group (cf. [Ye2, 3.4]).

The derived Picard group was computed in various cases, see [Ye2] and [MY]. As shown in [Ye2], the derived Picard group classifies the isomorphism classes of dualizing complexes (cf. next section).

There are some obvious tilting complexes. If P is an invertible A-bimodule and n is an integer, then $T := P[n]$ is a two-sided tilting complex. Recall that the (noncommutative) Picard group $\mathrm{Pic}(A)$ of A is the group of isomorphism classes of invertible bimodules. It follows that $\mathrm{DPic}(A)$ contains a subgroup $\mathbb{Z} \times \mathrm{Pic}(A)$.

Proof of Theorem 0.1. (1) Assume that B is simple and Goldie. Let

$$j_0 := \max\{i \mid \mathrm{H}^i(T) \neq 0\}.$$

Without loss of generality we may assume that $j_0 = 0$ (after a complex shift). As in [Ye2, 1.1], $\mathrm{H}^0(T)$ is finite on both sides. By Lemma 1.3 it follows that $T \cong P$ where $P := \mathrm{H}^0(T)$.

Since P is a two-sided tilting complex we have

$$\mathrm{End}_{B^\circ}(P) \cong \mathrm{H}^0 \mathrm{R}\,\mathrm{Hom}_{B^\circ}(T, T) \cong A.$$

By Lemma 1.1(2), P is a projective A-module. According to [Ye2, 2.2], P is an invertible A-B-bimodule. The functor $M \mapsto P \otimes_B M$ is then an equivalence $\mathsf{Mod}\, B \to \mathsf{Mod}\, A$.

(2) Take $A = B$. By part (1) every tilting complex is isomorphic to $P[n]$. The assertion follows. □

3. Dualizing Complexes

The definition of a dualizing complex over a noncommutative graded ring is due to the first author [Ye1]. The following more general definition appeared in [YZ].

Definition 3.1. Assume A is a left noetherian ring and B is a right noetherian ring. A complex $R \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}(A \otimes B^\circ))$ is called a *dualizing complex over* (A, B) if it satisfies the following conditions:

(i) R has finite injective dimension over A and over B°.
(ii) R has finite cohomology modules over A and over B°.
(iii) The canonical morphisms $B \to \mathrm{R}\,\mathrm{Hom}_A(R, R)$ in $\mathsf{D}(\mathsf{Mod}(B \otimes B^\circ))$ and $A \to \mathrm{R}\,\mathrm{Hom}_{B^\circ}(R, R)$ in $\mathsf{D}(\mathsf{Mod}(A \otimes A^\circ))$ are both isomorphisms.

If moreover $A = B$, we say R is a *dualizing complex over* A.

Whenever we say R is a dualizing complex over (A, B) we are tacitly assuming that A is left noetherian and B is right noetherian.

Recall that an algebra A is Gorenstein if it has finite left and right injective dimension. Hence a noetherian ring A is Gorenstein if and only if the bimodule $R := A$ is a dualizing complex. Existence of dualizing complexes for non-Gorenstein rings is studied in [VdB, YZ].

If A is noetherian and has at least one dualizing complex then the derived Picard group $\operatorname{DPic}(A)$ classifies the isomorphism classes of dualizing complexes. Indeed, given a dualizing complex R, any other dualizing complex R' is isomorphic to $R \otimes^{\mathrm{L}}_A T$ for some two-sided tilting complex T, and T is unique up to isomorphism.

Proof of Theorem 0.2. By Lemma 1.3, $R \cong P[n]$ for some bimodule P and some integer n.

Since R is dualizing the canonical homomorphisms $A \to \operatorname{End}_{B^{\circ}}(P)$ and $B^{\circ} \to \operatorname{End}_A(P)$ are isomorphisms. When both A and B are Goldie and simple (condition (i)), Lemma 1.1(3) implies that P is invertible.

Now assume A is noetherian, and it has some dualizing complex R_1 (condition (ii)). Then by the proof of [Ye2, 4.5] – suitably modified to fit our situation – the complex $T := \mathrm{R}\operatorname{Hom}_A(R_1, R) \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}(A \otimes B^{\circ}))$ is a two-sided tilting complex. Since either A or B is a Goldie simple ring, it follows from Theorem 0.1 that both A and B are Goldie simple rings. As above we deduce that P is an invertible bimodule.

Under both conditions the rings A and B are Morita equivalent. Since the bimodule P is a dualizing complex over (A, B) it has finite injective dimension on both sides. But on the other hand P is a progenerator on both sides, and hence A has finite injective dimension on the left, and B has finite injective dimension on the right. By Morita equivalence, both A and B are (two-sided) noetherian and have finite left and right injective dimensions. $\square$

Remark 3.2. One can define dualizing complexes in a slightly more general situation, by replacing the noetherian condition with the weaker coherence condition (see [Ye1, 3.3]). Thus in Definition 3.1 A is a left coherent ring, B is a right coherent ring, and in condition (ii) of the word 'finite' is replaced with 'coherent'. It is not hard to check that a "coherent" version of Theorem 0.2 holds.

Example 3.3. Let $A := \bigcup A_n$ where A_n is the nth Weyl algebra over the field k. Using the method of faithful flatness (see [MR, Sect. 7.2] and [RSW]) we see that A has the following properties:

(i) A is neither left nor right noetherian.
(ii) A has infinite Krull, Gelfand-Kirillov, injective, and global dimensions.
(iii) A is a Goldie domain (i.e. a left and right Ore domain).
(iv) A is a coherent ring.

Suppose now $\operatorname{char} k = 0$. Then A is a simple ring. Therefore Theorem 0.1 holds. For instance, the derived Picard group of A is $\mathbb{Z} \times \operatorname{Pic}(A)$. By the "coherent" version of Theorem 0.2 (see Remark 3.2) A does not admit a dualizing complex, because A is not Gorenstein.

Examples of noetherian simple rings with infinite Krull dimension were given by Shamsuddin [Sh] and Goodearl-Warfield [GW]. It is not hard to show that these simple rings also have infinite injective dimension (see Section 6).

4. The Auslander Condition

Let R be a dualizing complex over (A, B) and let M be an A-module. The *grade* of M with respect to R is defined to be

$$j_R(M) = \min\{q \mid \operatorname{Ext}^q_A(M, R) \neq 0\}.$$

The grade of a B°-module is defined similarly.

We recall the definitions of the Auslander condition and the Cohen-Macaulay condition. Gelfand-Kirillov dimension is denoted by GKdim.

Definition 4.1. Let R be a dualizing complex over (A, B).

(1) R is called *Auslander* if the two conditions below hold.
  (i) For every finite A-module M, every q, and every B°-submodule $N \subset \operatorname{Ext}^q_A(M, R)$ one has $j_R(N) \geq q$.
  (ii) The same holds after exchanging A and B°.

(2) If there is a constant s such that

$$j_R(M) + \operatorname{GKdim} M = s$$

for all finite A-modules or finite B°-modules M, then R is called *Cohen-Macaulay.*

The *canonical dimension* with respect to an Auslander dualizing complex R is defined to be

$$\operatorname{Cdim}_R M = -j_R(M)$$

for all finite A-modules or B°-modules M. By [YZ, 2.10], Cdim_R is a finitely partitive, exact dimension function. See [MR, 6.8.4] for the definition of dimension function.

When A is a Gorenstein ring and the bimodule $R := A$ is an Auslander dualizing complex then A is called an *Auslander-Gorenstein* ring. If A is an Auslander-Gorenstein ring such that $R := A$ is also Cohen-Macaulay, then A is called an *Auslander-Gorenstein Cohen-Macaulay ring.* This is the usage in [Le] and [Bj]. We remind the critical reader that unlike commutative rings, a noncommutative Gorenstein ring need not be either Auslander or Cohen-Macaulay.

The following is easy.

Lemma 4.2. *Let A, B, C be rings. Let L, M, N be bounded complexes over $B^\circ, A, A \otimes B^\circ$ respectively, and let P be a invertible B-C-bimodule.*

(1) *For every i there is an isomorphism of A-modules*

$$\operatorname{Ext}^i_{B^\circ}(L, N) \cong \operatorname{Ext}^i_{C^\circ}(L \otimes_B P, N \otimes_B P).$$

(2) *Suppose A is left noetherian and $H^j M$ is finite over A for all j. Then, for every i there is an isomorphism of C°-modules*

$$\operatorname{Ext}^i_A(M, N \otimes_B P) \cong \operatorname{Ext}^i_A(M, N) \otimes_B P.$$

Proof. (1) This follows from the fact that $- \otimes_B P$ induces a Morita equivalence.

(2) This is obvious when $M = A[i]$. Then the assertion follows from the facts that M has a bounded above resolution by finite free A-modules and P is a flat B°-module. □

Proposition 4.3. *Let A, B, C be rings. Let P be an invertible B-C-bimodule and n an integer. Suppose R is a dualizing complex over (A, B), and let $R_1 = R \otimes_B P[n]$, which is a dualizing complex over (A, C). Then R is Auslander (resp. Cohen-Macaulay) if and only if R_1 is.*

Proof. Without loss of generality we may assume $n = 0$.

Let us assume R_1 is Auslander; we will prove that R is also Auslander. Given a finite A-module M and an integer i, let N be a B°-submodule of $\operatorname{Ext}^i_A(M,R)$. Then $N \otimes_B P$ is a C°-submodule of

$$\operatorname{Ext}^i_A(M,R) \otimes_B P \cong \operatorname{Ext}^i_A(M, R \otimes_B P) = \operatorname{Ext}^i_A(M, R_1).$$

By the Auslander condition for R_1, we have $\operatorname{Ext}^j_{C^\circ}(N \otimes_B P, R_1) = 0$ for all $j < i$. Hence $\operatorname{Ext}^j_{B^\circ}(N,R) = 0$ by Lemma 4.2(1). This is the Auslander condition for R. The converse follows from the fact $R = R_1 \otimes_C P^\vee$.

The argument above also shows that

$$\operatorname{Cdim}_R(M) = \operatorname{Cdim}_{R_1}(M) \quad \text{and} \quad \operatorname{Cdim}_R(N) = \operatorname{Cdim}_{R_1}(N \otimes_B P)$$

for all finite A-modules M and finite B°-modules N. Since GKdim is preserved by Morita equivalence, R is Cohen-Macaulay if and only if R_1 is. □

Proof of Corollary 0.3. By [YZ], A has a dualizing complex R in all cases. Furthermore in cases (i-iv), R is Auslander and Cohen-Macaulay. By Theorem 0.2, $R \cong P[n]$ for some invertible A-bimodule P and some integer n, and A is a Gorenstein ring. In cases (i-iv) the Auslander-Gorenstein and Cohen-Macaulay properties of A follow from Proposition 4.3. □

5. Weyl Algebras in Positive Characteristics

In this section we study dualizing complexes and two-sided tilting complexes over the Weyl algebras A_n when $\operatorname{char} k > 0$.

Proposition 5.1. *Let B be an Azumaya algebra over its center $\mathrm{Z}(B)$, and suppose $\operatorname{Spec} \mathrm{Z}(B)$ is connected. Let A be another ring and $T \in \mathsf{D}(\mathsf{Mod}(A \otimes B^\circ))$ a two-sided tilting complex. Then $T \cong P[n]$ for some integer n and some invertible A-B-bimodule P.*

Proof. Use the proof of [Ye2, 2.7], noting that for a prime ideal $\mathfrak{p} \subset \mathrm{Z}(B)$, the localization $B \otimes_{\mathrm{Z}(B)} \mathrm{Z}(B)_{\mathfrak{p}}$ is a local ring. □

The following lemma takes care of dualizing complexes.

Lemma 5.2. *Let A be a left noetherian ring and B a right noetherian ring.*

1. *If A has finite injective dimension as left module, and B has finite injective dimension as right module, then every two-sided tilting complex T over (A,B) is also a dualizing complex over (A,B).*
2. *If A or B is noetherian and Gorenstein, then every dualizing complex over (A,B) is also a two-sided tilting complex.*

Proof. (1) By [Ye2, 1.6 and 1.7] the cohomologies of T are finite modules on both sides and the morphisms $B \to \mathrm{R}\operatorname{Hom}_A(T,T)$ and $A \to \mathrm{R}\operatorname{Hom}_{B^\circ}(T,T)$ are isomorphisms. Since T has finite projective dimension over A and the left module A has finite injective dimension it follows that T also has finite injective dimension over A. Likewise on the right.

(2) If A is a noetherian Gorenstein ring then the bimodule A is a dualizing complex over A. Let R be any dualizing complex over (A,B). As mentioned earlier, the proof of [Ye2, 4.5] – suitably modified to fit our situation – shows that the complex $\mathrm{R}\operatorname{Hom}_A(A,R)$ is a two-sided tilting complex over (A,B). But $R \cong \mathrm{R}\operatorname{Hom}_A(A,R)$. □

Proposition 5.3. *Let B be an Azumaya algebra over its center $\mathrm{Z}(B)$. Suppose $\mathrm{Z}(B)$ is a noetherian Gorenstein ring and $\operatorname{Spec}\mathrm{Z}(B)$ is connected. Let A be a left noetherian ring and R a dualizing complex over (A,B). Then $R \cong P[n]$ for some integer n and some invertible A-B-bimodule P.*

Proof. First we show that B is also Gorenstein. Let d be the injective dimension of $C := \mathrm{Z}(B)$. For any prime ideal $\mathfrak{p}$ of C the local ring $C_{\mathfrak{p}}$ is Gorenstein, of injective dimension $\leq d$, and hence also the completion $\widehat{C}_{\mathfrak{p}}$. Now the completion $\widehat{B}_{\mathfrak{p}} := B \otimes_C \widehat{C}_{\mathfrak{p}}$ is isomorphic to a matrix ring $\mathrm{M}_r(\widehat{C}_{\mathfrak{p}})$, so by Morita equivalence $\widehat{B}_{\mathfrak{p}}$ is Gorenstein. Faithful flatness (going over all primes $\mathfrak{p}$) shows the vanishing of $\mathrm{Ext}^i_B(M,B)$ and $\mathrm{Ext}^i_{B^\circ}(N,B)$ for all finite modules M and N and all $i > d$, so we deduce that B is Gorenstein.

Now we may use Proposition 5.1 and Lemma 5.2(2). □

Corollary 5.4. *Let B be the nth Weyl algebra over k, with $\operatorname{char} k > 0$. Let A be any left noetherian k-algebra, and let R be any dualizing complex, or any two-sided tilting complex, over (A,B). Then $R \cong P[n]$ for some invertible A-B-bimodule P and integer n.*

Proof. By a result of Revoy [Re], the Weyl algebra B is Azumaya with center a polynomial algebra over k. Now use the Propositions 5.1 and 5.3. □

6. Goodearl-Warfield's example

We use an example of Goodearl-Warfield [GW, 4.6] to show that not every noetherian simple ring has finite injective dimension. This can also be done for the example of Shamsuddin [Sh].

Example 6.1. Let $R[\theta;\delta]$ be the noetherian simple domain of infinite Krull dimension constructed in [GW, 4.6]. In this example the base field k is an infinite extension of $\mathbb{Q}$. The ring R is a noetherian regular commutative k-algebra of infinite Krull dimension obtained by localizing a polynomial ring of countably many variables, which is essentially the example of Nagata [Na, Example 1, p.203].

Let d be any positive integer. By the construction of R, there is a prime ideal $\mathfrak{p} \subset R$ such that the height of $\mathfrak{p}$ is at least d. Then $R_{\mathfrak{p}}$ has finite global dimension $\geq d$. Hence $R_{\mathfrak{p}}[\theta;\delta]$ has finite global dimension $\geq d$ and has finite injective dimension $\geq d$ over itself. Since $R_{\mathfrak{p}}[\theta;\delta]$ is a localization of $R[\theta;\delta]$ [GW, 1.1], the injective dimension of $R[\theta;\delta]$ is at least d. Since d is arbitrarily chosen, the injective dimension of $R[\theta;\delta]$ is infinite.

By Theorem 0.2 there is no dualizing complex over $R[\theta;\delta]$.

We conclude this paper by the following question.

Question 6.2. Does every noetherian finitely generated simple ring of finite Krull (or Gelfand-Kirillov) dimension have finite left and right injective dimension?

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