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Fundamentals of Analysis for EE

Homework 1

The weight of Questions 3,5,7,9,13 is 10 points, other questions – 5 points.

Question 1. Let A be any infinite set. Prove that for every countable set B the following equality holds: $|A| = |A \cup B|$.

Question 2. Find an explicit formula for a bijective mapping $f : [0,1] \to R$.
Is it possible to find such a mapping continuous?

Question 3. For every set $A \subset R$ and every number $r \in R$ define the set

$$A + r = \{x + r : x \in A\} \subset R.$$

Assume that A is a countable set. Prove that there exists a number $r \in R$ such that

$$A \cap (A + r) = \varnothing.$$

Question 4. Let $A \subseteq R$ be a set with the following property: $|a_1 + a_2 + ... + a_n| \le 1$ for every finite subset $\{a_1, a_2, ..., a_n\} \subseteq A.$ Prove that A is a finite or a countable set.

Question 5. Let $f : R \to R$ be a monotone real function. Denote by A the set of all points of discontinuity of f. Prove that A is a finite or a countable set.
Hint: Which kind of discontinuity may have any monotone function?

Question 6. Denote by $C[a,b]$ the collection of all continuous real-valued functions defined on $[a,b]$. What is the cardinality of $C[a,b]$?

Question 7. Let $A \subseteq R$ be an uncountable set. Prove that there exists $t \in R$ such that both sets $A \cap (-\infty, t)$ and $A \cap (t, \infty)$ are uncountable.

Question 8. Let (X, d) be a metric space. Prove that for any 3 points $x, y, z \in X$ the following inequality holds: $d(x,y) \ge |d(x,z) - d(z,y)|$.

Question 9. Let E be a finite set. Denote the power set of E by X : $X = \{A \subseteq E\}$.
For every two subsets $A \subseteq E$, $B \subseteq E$ define the number $d(A,B) = |A \triangle B|$.
(Here $A \triangle B = (A \setminus B) \cup (B \setminus A)$ denotes the symmetric difference).
Prove that (X, d) is a metric space.

Question 10. In the set N of natural numbers define $d(m,n)=\frac{|m-n|}{mn}$. Is d a metric?

Question 11. Let (X,d) be a metric space. Find all values of constant numbers C such that

(a) $C\cdot d$ defines a metric on the set X.

(b) $C+d$ defines a metric on the set X.

Question 12. Let d_1 and d_2 be two metrics defined on a set X. Find which formulas below necessarily define a metric on the same set X:

(a) d_1+d_2.

(b) $\max\{d_1,d_2\}$.

(c) $\min\{d_1,d_2\}$.

Question 13. Let (X,d) be a metric space. Suppose that a sequence $\{x_n\}_{n=1}^{\infty}\subset X$ converges to $x\in X$ according to metric d. Prove that $\lim_{n\to\infty} d(x_n,y)=d(x,y)$ for each $y\in X$.

Question 14. Definition: The sets $B(P_0,r)=\{P\in X: d(P,P_0)<r\}$ and $\overline{B}(P_0,r)=\{P\in X: d(P,P_0)\le r\}$ are called a ball (a closed ball, respectively) with the radius $r>0$ in the metric space (X,d).

Let (X,d) be a metric space. Assume that for some two balls there is an inclusion $B(P_1,r_1)\subseteq B(P_2,r_2)$. Does it imply that necessarily $r_1\le r_2$?

Question 15. Assume that d_1 and d_2 are two metrics in the space R^3 defined as follows: If $P_1=P_1(x_1,y_1,z_1);\ P_2=P_2(x_2,y_2,z_2)$ are any two points in R^3, then

$$d_1(P_1,P_2)=|z_2-z_1|+|y_2-y_1|+|x_2-x_1|$$

$$d_2(P_1,P_2)=\max\{|z_2-z_1|,|y_2-y_1|,|x_2-x_1|\}$$

Describe the geometric form of the closed balls $\overline{B}(P_0,r)$, where $P_0=(0,0,0), r=1$ in the metric spaces (R^3,d_1) and (R^3,d_2).