Prof. Arkady Leiderman

Fundamentals of Analysis for EE

Homework 2

The weight of Questions 3,4,8,14,15 is 10 points, other questions – 5 points.

Question 1. Consider the following subset of the plane:

$A = \{(x, \sin\left(\frac{1}{x}\right) : x > 0\} \subset R^2$. What is the closure $\mathrm{cl}(A)$ in R^2?

Question 2. Let $f(x): R \to R$ be a function.

(a) Prove that if $f(x)$ is continuous at every point x, then the graph of the function

$\Gamma = \{(x, f(x) : x \in R\}$ is a closed subset of R^2.

(b) Is the converse also true: If the graph of a function is closed in R^2, then the function $f(x)$ is continuous at every point?

Question 3. Let (X, d) be a metric space and A be a non-empty subset of X.

Define a number $\rho(x, A) = \inf\{d(x, y) : y \in A\}$ for every $x \in X$.

Prove that the closure $\mathrm{cl}(A) = \{x \in X : \rho(x, A) = 0\}$.

Question 4. A metric space (X, d) is called separable if there is a countable set $A \subseteq X$ such that $\mathrm{cl}(A) = X$. A family of open sets $B = \{U_\alpha\}$, where every $U_\alpha \subseteq X$, is called a base for X if for every point $x \in X$ and every open set V containing x there is $U_\alpha \in B$ such that $x \in U_\alpha \subseteq V$.

(a) Prove that any Euclidean space R^n is separable.

(b) Prove that any metric space (X, d) is separable if and only if it has a countable base.

Question 5. In $X = R$ define $d(x, y) = |arctg(x) - arctg(y)|$ for every $x, y \in X$.

(a) Prove that $d(x, y)$ defines a metric in X.

(b) Is (X, d) a complete metric space?

Question 6. Let $C[-1,1]$ be a metric space of continuous functions with the metric

$d(f,g)=\int_{-1}^{1}|f(x)-g(x)|\,dx$. Consider the following sequence $\{f_n(x)\}_{n=1}^{\infty}$:

$$f_n(x)=\begin{cases} 1 & \text{if } x\in[1/n,1] \\ nx & \text{if } x\in[-1/n,1/n] \\ -1 & \text{if } x\in[-1,-1/n] \end{cases}$$

Prove that $\{f_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence. Is the metric d complete?

Question 7. Let (X,d_X) and (Y,d_Y) be two metric spaces.

(a) Assume that a mapping $f:(X,d_X)\to(Y,d_Y)$ satisfies the Lipschitz condition:

$\exists K>0\ \forall x_1,x_1\in X\ d_Y(f(x_1),f(x_2))\le K\,d_X(x_1,x_2)$.

Prove that $f(x)$ is a uniformly continuous mapping on (X,d_X).

(b) Fix a point $p\in X$ and define a function $f:X\to R$ by the formula: $f(x)=d_X(x,p)$.

Prove that $f(x)$ is a uniformly continuous function on (X,d_X).

Question 8. Consider a closed ball in the space $C[0,1]$:

$\overline{B}=\{f\in C[0,1]:\max\limits_{x\in[0,1]}|f(x)|\le 1\}$. Is the set $\overline{B}$ compact?

Question 9. Prove that if (X,d) is a compact metric space, then d is a complete metric.

Question 10. Let (X,d_X) and (Y,d_Y) be two metric spaces and $f:(X,d_X)\to(Y,d_Y)$ be a continuous mapping.

(a) Prove that if $K\subset X$ is a compact set, then its image $f(K)\subset Y$ is also a compact set.

(b) Let (X,d_X) be the Euclidean space R^n. Prove that if $A\subset R^n$ is a bounded set, then its image $f(A)\subset Y$ is also a bounded set.

Question 11. Let $f:(X,d)\to R$ be a continuous function defined on a metric space (X,d). Prove that for any $c\in R$ the set $A_c(f)=\{x\in X : f(x)\geq c\}$ is closed, and the set $D_c(f)=\{x\in X : f(x)>c\}$ is open.

Question 12. Let $f:(X,d)\to R$ be a function defined on a metric space (X,d). Assume that the sets $A_c(f)=\{x\in X : f(x)\geq c\}$ and $B_c(f)=\{x\in X : f(x)\leq c\}$ are closed for any $c\in R$. Prove that the function f is continuous on X.

Question 13. *Definition*: set A is called a G_δ -set in a metric space (X,d) if A is the countable intersection of open sets. Set A is called a F_σ -set in a metric space (X,d) if A is the countable union of closed sets.

Prove that every closed set in a metric space is G_δ, and every open set in a metric space is F_σ.

Question 14. Prove that the set Q of rational numbers is not a G_δ -set in R.

Hint: Use the Baire Category Theorem.

Question 15. Construct a subset of R which is not G_δ and is not F_σ.