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Fundamentals of Analysis for EE

Homework 3

The weight of Questions 5,8,10,12,15 is 10 points, other questions – 5 points

<u>Question 1</u>. Let X be any non-empty set and $\{\mathcal{B}_\alpha : \alpha \in A\}$ a family of $\sigma-$algebras of subsets of X. Prove that the intersection $\bigcap\{\mathcal{B}_\alpha : \alpha \in A\}$ is also a $\sigma-$algebra.

<u>Question 2</u>. Let X be any set with the cardinality bigger than $\aleph_0$.
Define a family $\mathcal{F}$ of subsets of X:

$\mathcal{F} = \{A \subset X : A$ is at most countable or $X \setminus A$ is at most countable$\}$

and define $\mu(A) = \begin{cases} 1 & \text{if } |X \setminus A| \le \aleph_0 \\ 0 & \text{if } |A| \le \aleph_0 \end{cases}$. Prove that $(X, \mathcal{F}, \mu)$ is a measure space.

<u>Question 3</u>. Give an example of a set $X \neq \varnothing$ such that μ defined by the following rule:
for any subset $A \subseteq X$, $\mu(A) = \begin{cases} \infty & \text{if } |A| \ge \aleph_0 \\ 0 & \text{if } |A| < \aleph_0 \end{cases}$, is not a measure.

<u>Question 4</u>. Let X be any non-empty set. Fix a point $p \in X$. Define μ on the $\sigma-$algebra of all subsets $\mathcal{P}(X)$ by the following rule: if $p \in A$ then $\mu(A) = 1$; if $p \notin A$ then $\mu(A) = 0$.

Prove that $(X, \mathcal{P}(X), \mu)$ is a measure space. (Such μ is called an atomic measure).

<u>Question 5</u>. Let $(X, \mathcal{B}, \mu)$ be any measure space. Assume that $A_i \in \mathcal{B}$ for every $i = 1, 2, 3, ...$

(a) Prove that if $A_1 \subseteq A_2 \subseteq A_3 \subseteq ... \subseteq A_n \subseteq ...$, then $\mu(\bigcup_{i=1}^{\infty} A_i) = \lim_{n\to\infty} \mu(A_n)$.

(b) Prove that if $A_1 \supseteq A_2 \supseteq A_3 \supseteq ... \supseteq A_n \supseteq ...$, and $\mu(A_1) < \infty$, then
$\mu(\bigcap_{i=1}^{\infty} A_i) = \lim_{n\to\infty} \mu(A_n)$.

Does the claim (b) hold without the assumption that $\mu(A_1) < \infty$?

Question 6. Let X be a non-empty set, $\mathcal{B}$ a $\sigma-$algebra of subsets of X and $\varphi : \mathcal{B} \to [0, \infty)$ a positive finitely additive function defined on elements of $\mathcal{B}$. Prove that φ is a measure, i.e. φ is $\sigma-$additive if and only if the following condition holds:

Let $A_i \in \mathcal{B}$ for every $i = 1, 2, 3, \ldots$. If $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots \supseteq A_n \supseteq \ldots$, and $\bigcap_{i=1}^{\infty} A_i = \varnothing$ then $\lim_{n\to\infty} \varphi(A_n) = 0$.

Question 7. Let $(X, \mathcal{B}, \mu)$ be any measure space with $\mu(X) < \infty$. Assume that $\{A_r : r \in (0,1]\}$ is a family of measurable subsets of X satisfying $A_r \subseteq A_s$ for $r \le s$.

Prove that the intersection $A = \bigcap_{r>0} A_r$ is measurable and $\mu(A) = \lim_{r\to 0} \mu(A_r)$.

Question 8. Denote by $\mathcal{B}(\mathbb{R})$ the $\sigma-$algebra of Borel sets of the real line $\mathbb{R}$.

If $\mathcal{K}$ is any family of subsets of $\mathbb{R}$, then $\sigma(\mathcal{K})$ denotes the minimal (with respect to inclusion) $\sigma-$algebra, which contains $\mathcal{K}$. Prove that $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{K}_1) = \sigma(\mathcal{K}_2) = \sigma(\mathcal{K}_3)$, where

$\mathcal{K}_1$- collection of all open intervals (a,b) in $\mathbb{R}$;

$\mathcal{K}_2$- collection of all closed intervals $[a,b]$ in $\mathbb{R}$;

$\mathcal{K}_3$- collection of all half-open half-closed intervals $[a,b)$ in $\mathbb{R}$.

Question 9. Let μ denote the Lebesgue measure in the real line $\mathbb{R}$.
(a) Give an example of an open set $A \subset \mathbb{R}$ such that A is not bounded and $\mu(A) = 1$;
(b) Give an example of a compact set $A \subset \mathbb{R}$ such that $|A| = 2^{\aleph_0}$ and $\mu(A) = 0$;
(c) Give an example of a compact set $A \subset \mathbb{R}$ such that $|A| = 2^{\aleph_0}$, the set A does not contain any interval but $\mu(A) > 0$.

Question 10. Let μ denote the Lebesgue measure in the real line $\mathbb{R}$.
Prove that the following properties are equivalent
(a) $A \subset \mathbb{R}$ is a Lebesgue measurable set;
(b) For every $\varepsilon > 0$ there exists an open set $A \subset U$ such that $\mu(U \setminus A) < \varepsilon$;
(c) For every $\varepsilon > 0$ there exists a closed set $F \subset A$ such that $\mu(A \setminus F) < \varepsilon$.

Question 11. Let μ denote the Lebesgue measure in the real line $\mathbb{R}$.
Remind that a set A is called G_δ if $A = \bigcap\{U_n : n \in \mathbb{N}\}$, where each set U_n is open, and a set A is called F_σ if $A = \bigcup\{F_n : n \in \mathbb{N}\}$, where each set F_n is closed.
Prove that the following properties are equivalent
(a) $A \subset \mathbb{R}$ is a Lebesgue measurable set ;
(b) $A = G \setminus M$, where a set G is G_δ and $\mu(M) = 0$;
(c) $A = F \cup M$, where a set F is F_σ and $\mu(M) = 0$.

Question 12. Let μ denote the Lebesgue measure in the real line $\mathbb{R}$ and $A \subset \mathbb{R}$ be a Lebesgue measurable set. Denote by $A + x = \{a + x : a \in A\} \subset \mathbb{R}$.
Prove that the set $A + x$ is also Lebesgue measurable and $\mu(A) = \mu(A + x)$ for every $x \in \mathbb{R}$, which means that the Lebesgue measure in the real line $\mathbb{R}$ is translation-invariant.

Question 13. Let $V \subset \mathbb{R}$ be a Vitali set which is not Lebesgue measurable.
Assume that $A \subset V$ is a Lebesgue measurable set. Prove that $\mu(A) = 0$.

Question 14. Let $(X, \mathcal{B}, p)$ be any probability measure space.

(a) Let $A_1, A_2 \in \mathcal{B}$ and assume that $p(A_1) + p(A_2) > 1$. Is it possible that $\bigcap_{i=1}^{2} A_i = \varnothing$?

(b) Let $A_1, A_2, A_3 \in \mathcal{B}$ and assume that $p(A_1) + p(A_2) + p(A_3) > 2$.

Is it possible that $\bigcap_{i=1}^{3} A_i = \varnothing$?

Question 15. Let $A \subset [-1,1]$ be a Lebesgue measurable set with $\mu(A) > 1$.
Prove that the point $t = 1$ belongs to the set $A - A$.

Hint: Consider sets $A_0 = A \cap [-1, 0]$, $A_1 = A \cap [0,1]$, $A_2 = A_0 + 1$