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Fundamentals of Analysis for EE

Homework 4

The weight of Questions 5,7,10,13,15 is 10 points, other questions – 5 points

Question 1. Let $\mathcal{F}$ be the $\sigma-$algebra of subsets of $\mathbb{R}$ defined as follows:

$\mathcal{F} = \{\varnothing, (-\infty, 0], (0, \infty), (-\infty, \infty)\}$. Describe all measurable functions $f(x): \mathbb{R} \to \mathbb{R}$.

Question 2. Let X be any non-empty set and $\mathcal{F}$ be a $\sigma-$algebra of its subsets.

Prove that a function $f(x): X \to \mathbb{R}$ is measurable if and only if $f^{-1}(-\infty, q) \in \mathcal{F}$ for every rational number $q \in Q$.

Question 3. Let X be any non-empty set and $\mathcal{F}$ be a $\sigma-$algebra of its subsets.

Assume that 2 functions $f(x): X \to \mathbb{R}$ and $g(x): X \to \mathbb{R}$ are measurable.
Let $\varphi(x): \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function of two real variables.
Prove that the function $h(x) = \varphi(f(x), g(x)): X \to \mathbb{R}$ is measurable.

Question 4. Let $f(x): \mathbb{R} \to \mathbb{R}$ be a monotone real function. Prove that $f(x)$ is a Borel measurable function, i.e. $f^{-1}(U)$ is a Borel set for every Borel set $U \subseteq \mathbb{R}$.

Question 5. Let X be any non-empty set and $\mathcal{F}$ be a $\sigma-$algebra of its subsets.

Assume that $f_n(x): X \to \mathbb{R}$ is a measurable function for every natural n.
Denote by A the set of those $x \in X$ such that there exists a finite limit $\lim_{n\to\infty} f_n(x)$.

Prove that A is a measurable set, i.e. $A \in \mathcal{F}$.

Question 6. Let $f(x): \mathbb{R} \to \mathbb{R}$ be a measurable function. Assume that $f^{-1}(N)$ has Lebesgue measure zero whenever N has Lebesgue measure zero.
Prove that $f^{-1}(A)$ is a Lebesgue measurable set for every Lebesgue measurable set A.

Question 7. Let $f(x): [a,b] \to \mathbb{R}$ be a Lebesgue measurable function. Assume that $\int_{[a,c]} f(x)\, d\mu = 0$ for every $c \in [a,b]$. Prove that $f(x) \sim 0$, i.e.
if $A = \{x \in X : f(x) \neq 0\}$, then $\mu(A) = 0$.

Question 8. Let $E \subset \mathbb{R}$ be a Lebesgue measurable set and $\mu(E) < \infty$. Assume that $f(x): E \to \mathbb{R}$ is a measurable function. Prove that there exists a measurable function $\varphi(x): E \to \mathbb{R}$ such that $\varphi(x) > 0$ for each $x \in E$ and $\int_E \varphi(x) f(x)\, d\mu < \infty$.

Question 9. Let $E \subset \mathbb{R}$ be a Lebesgue measurable set and $\mu(E) < \infty$. Assume that $f(x): E \to \mathbb{R}$ is a measurable function. Define sets $B_n = \{n \le |f| < n+1\}$ for each integer $n \ge 0$. Prove that the following statements are equivalent:
(a) $\int_E |f(x)|\, d\mu < \infty$; (b) $\Sigma_n n\mu(B_n) < \infty$.

Question 10. Let $f(x):[0,1] \to \mathbb{R}$ be a Lebesgue measurable function and $\int_{[0,1]} f(x) d\mu = C$. Prove that there exists $t \in [0, \frac{1}{2}]$ such that $\int_{[t, t+\frac{1}{2}]} f(x) d\mu = \frac{1}{2} C$.

Question 11. Let $f_n(x):[a,b] \to \mathbb{R}$ be a Lebesgue measurable function for every natural n. Assume that $\lim_{n\to\infty} f_n(x) = f(x)$ almost for all $x \in [a,b]$.

(a) Prove that $\lim_{n\to\infty} \int_{[a,b]} \cos(f_n(x)) d\mu = \int_{[a,b]} \cos(f(x)) d\mu$.

(b) Does it hold necessarily that $\lim_{n\to\infty} \int_{[a,b]} f_n(x) d\mu = \int_{[a,b]} f(x) d\mu$?

Question 12. Let $f_n(x):[0,\infty) \to \mathbb{R}$ defined by

$$f_n(x) = \begin{cases} 1, & x \in [n, n+1/n] \\ 0, & otherwise \end{cases}$$ for every natural n.

(a) Prove that $\lim_{n\to\infty} f_n(x) = 0$ for all x.

(b) Prove that $\lim_{n\to\infty} \int_{[0,\infty)} f_n(x) d\mu = 0$.

(c) Nevertheless, there is no dominating function $g(x)$, i.e. $g(x) \ge |f_n(x)|$ for all x and n, and such that $\int_{[0,\infty)} g(x) d\mu < \infty$.

Question 13. Let $C \subset [0,1]$ denote the standard ternary Cantor set.
Define the following function $f(x):[0,1] \to \mathbb{R}$: if $x \in C$, then $f(x) = x$; and for every $x \notin C$, if x belongs to a removed interval of the length $\frac{1}{3^n}$, then $f(x) = \frac{1}{2^n}$.

(For instance, $f|_{\left(\frac{1}{3},\frac{2}{3}\right)}=\frac{1}{2}$, $f|_{\left(\frac{1}{9},\frac{2}{9}\right)\cup\left(\frac{7}{9},\frac{8}{9}\right)}=\frac{1}{4}$).

(a) Find the value of $\int_{[0,1]} f(x)\,d\mu$ in the sense of Lebesgue.

(b) Is $f(x)$ a Riemann integrable function? If yes what is $\int_0^1 f(x)dx$?

<u>Question 14</u>. Let μ denote the Lebesgue measure in the real line $\mathbb{R}$ and $1 \le p < \infty$ is a number. Let $E \subset \mathbb{R}$ be a Lebesgue measurable set with $0 < \mu(E) < \infty$. Denote by $L_p(E)$ the linear space of measurable functions $f(x): E \to \mathbb{R}$ such that $\int_E |f(x)|^p\,d\mu < \infty$.

(Recall that we identify two functions $f(x)$ and $g(x)$ if $E(f \ne g)$ is a null-set).

Prove that $L_p(E)$ is a normed space with the norm $\|f\|_p = [\int_E |f(x)|^p\,d\mu]^{\frac{1}{p}}$.

<u>Question 15</u>. Let $E \subset \mathbb{R}$ be a Lebesgue measurable set with $0 < \mu(E) < \infty$.

(a) Prove that if $f(x) \in L_2(E)$, then also $f(x) \in L_1(E)$.

(b) Is it true that if $[f(x)]^2$ is a bounded Lebesgue integrable function, then always $f(x)$ is also a Lebesgue integrable function?