המחלקה למתמטיקה, בן-גוריון

אשנב למתמטיקה

ביום *שלישי, 3 בינואר, 2017*

בשעה *18:30 – 20:00*

באולם *101-*

ההרצאה

פונקציות הרמוניות וספירת מסלולים פשוטים

תינתן על-ידי

אריאל ידין

תקציר: המושג של "פונקציה הרמונית" חוזר לעבודות של לפלס ופוריה (במאה ה-18!) ויש לו חשיבות עצומה בפיסיקה, הנדסה, ומדעים בכללי. בעזרת "ההרמוניות" השונות ניתן לתאר את כל הגלים האפשריים. זה, למשל, מהווה את הבסיס לקידוד mp3. במאות ה-19 וה-20 הכלילו את המושג גם לאוביקטים מתמטיים מודרניים יותר, וההחשיבות הגיאומטרית שלו הובנה יותר. אנחנו נשוחח על הגדרה גיאומטרית של פונקציה הרמונית, שהיא כללית למדי. נסביר איך ניתן להשתמש בפונקציות כאלה כדי לספור מסלולים פשוטים - שזו בעיה קשה בפני עצמה.

אני אשתדל להסביר את כל המושגים המופיעים בתקציר במהלך ההרצאה.

Harmonic Functions

Ariel Yadin

Ben Gurion University

Eshnav, Jan 2017

classical notions

random walks

gambler's ruin

counting simple paths

heat equation

Jean-Baptiste
Joseph Fourier
(1768–1830)

$u(x, t)$ = heat at time t, point $x \in [0, L]$
Fourier's law + conservation of energy:

$$\frac{\partial}{\partial t} u = \alpha \cdot \frac{\partial^2}{\partial x^2} u$$

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heat equation

$$\frac{\partial}{\partial t} u = \Delta u \qquad \Delta = \frac{\partial^2}{\partial x^2}$$

$$f(x+\varepsilon) = f(x) + f'(x)\varepsilon + \tfrac{1}{2} f''(x)\varepsilon^2 + O(\varepsilon^3)$$
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$$u(x, t+\delta) - u(x,t) = \delta \cdot \frac{2}{\varepsilon^2} \cdot \left(\frac{u(x+\varepsilon, t) + u(x-\varepsilon, t)}{2} - u(x,t) \right)$$

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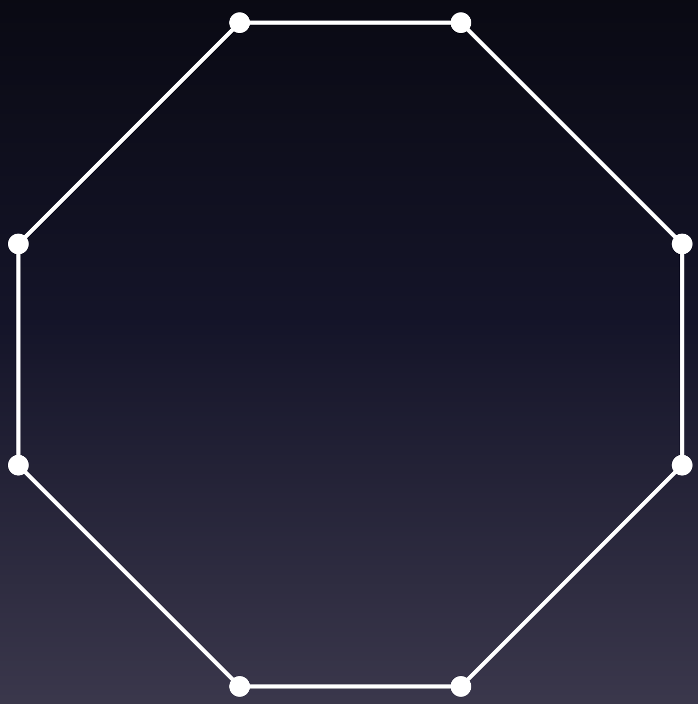
discrete geometry - graphs

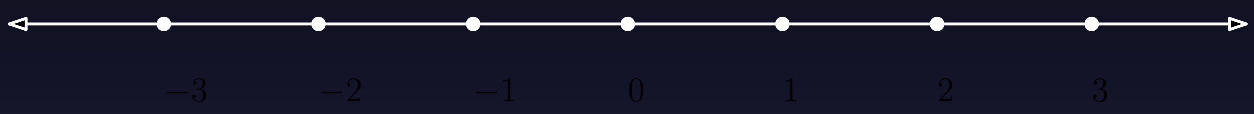
A graph is a collection of vertices (nodes) and edges (connections).

graphs: cycle

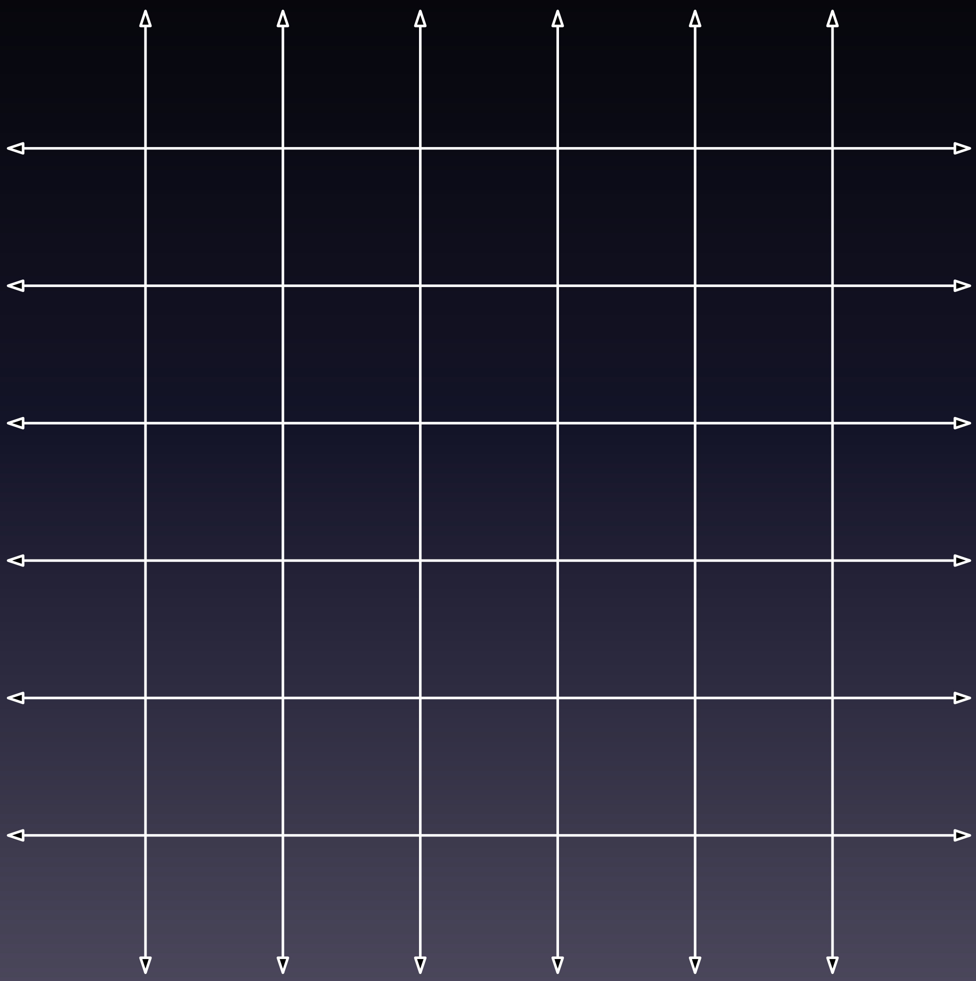


graphs: finite line

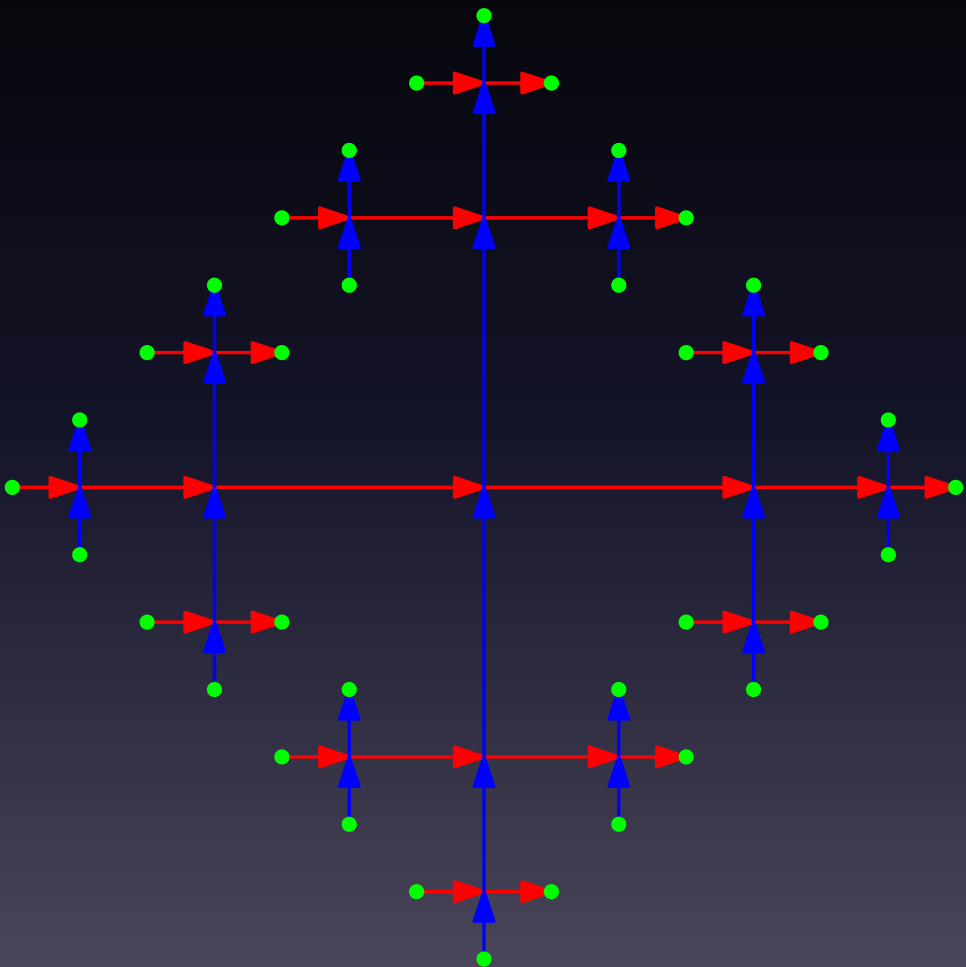
graphs: $\mathbb{Z}$



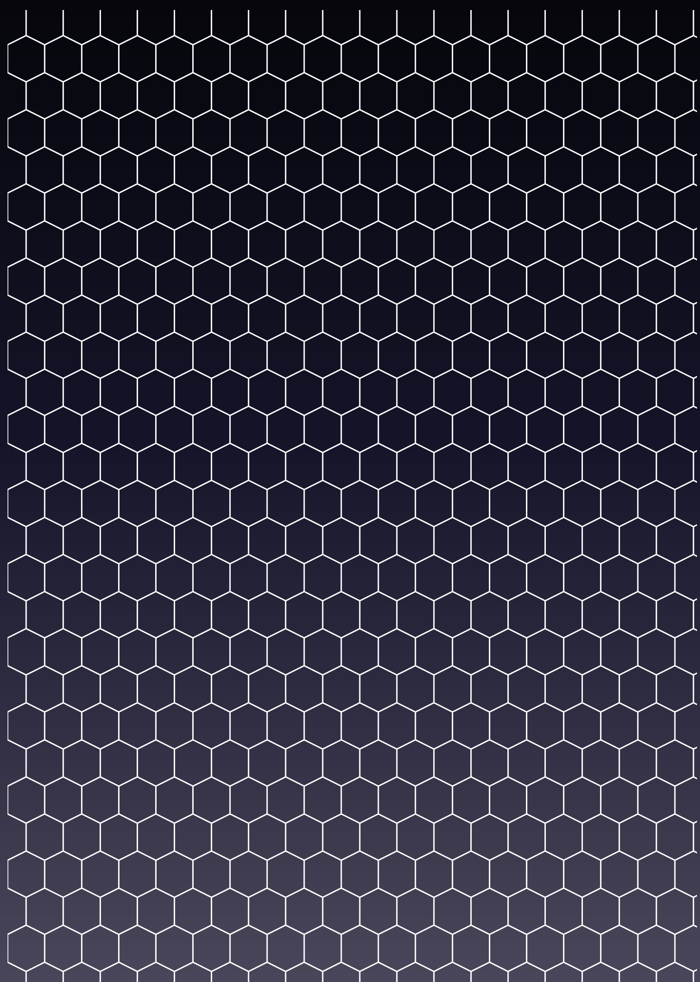
graphs: $\mathbb{Z}^2$



graphs: regular tree



graphs: hexagonal lattice



graphs: notation

- $x \sim y \iff x, y$ are neigbors
- $\deg(x)$ = the number of neighbors of x
- paths give a metric (geometry)
- boundaries

$$\partial D = \{x \notin D \ : \ \exists\, y \sim x \ , \ y \in D\}$$

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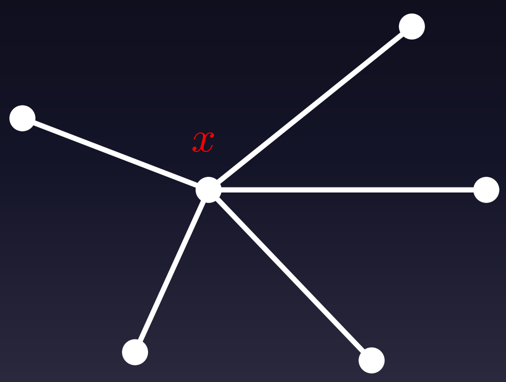

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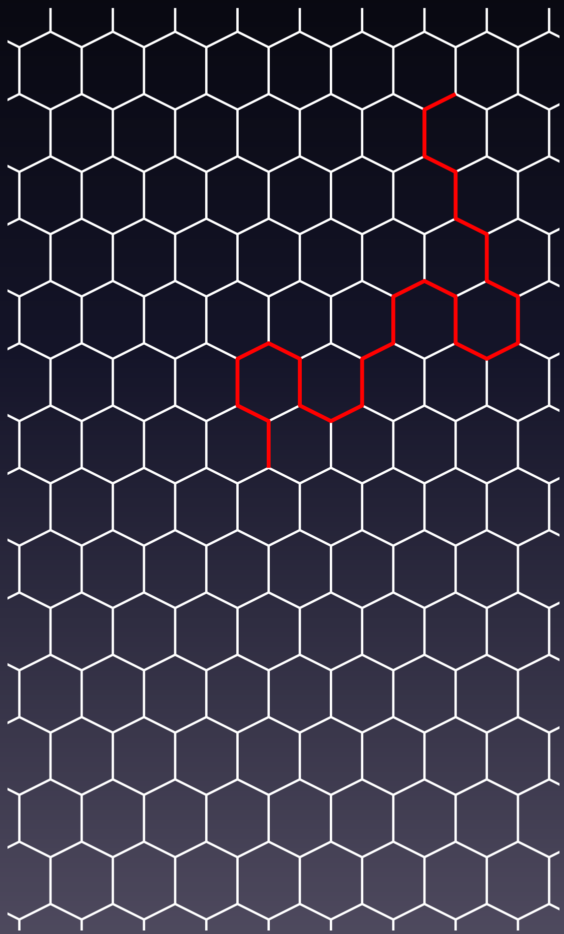

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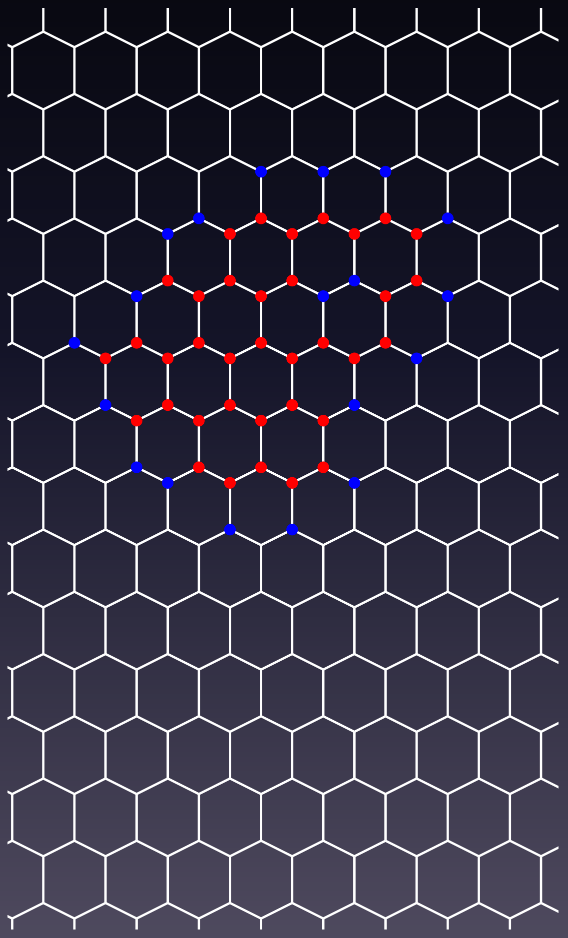
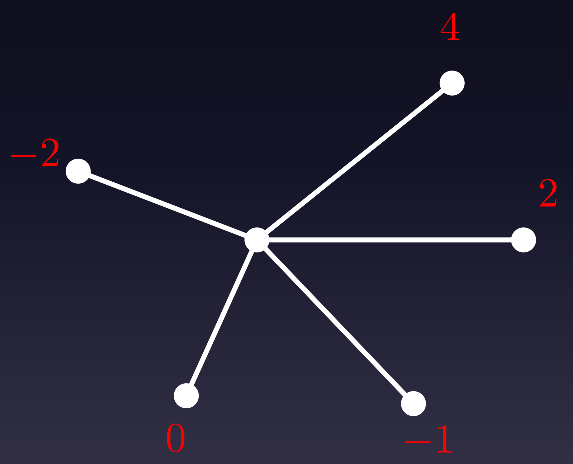


harmonic function

Definition

A function is harmonic at x if

$$f(x) = \frac{1}{\deg(x)} \sum_{y \sim x} f(y)$$



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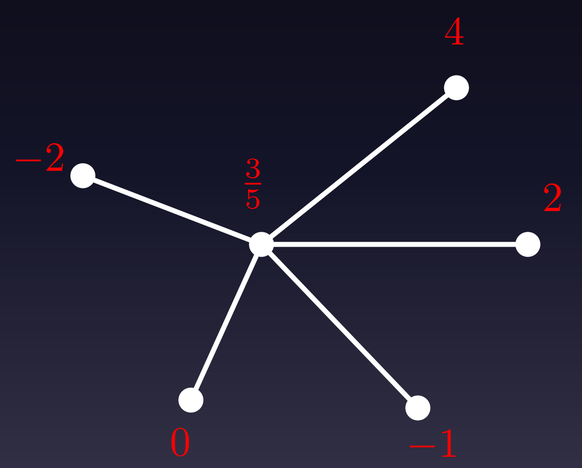

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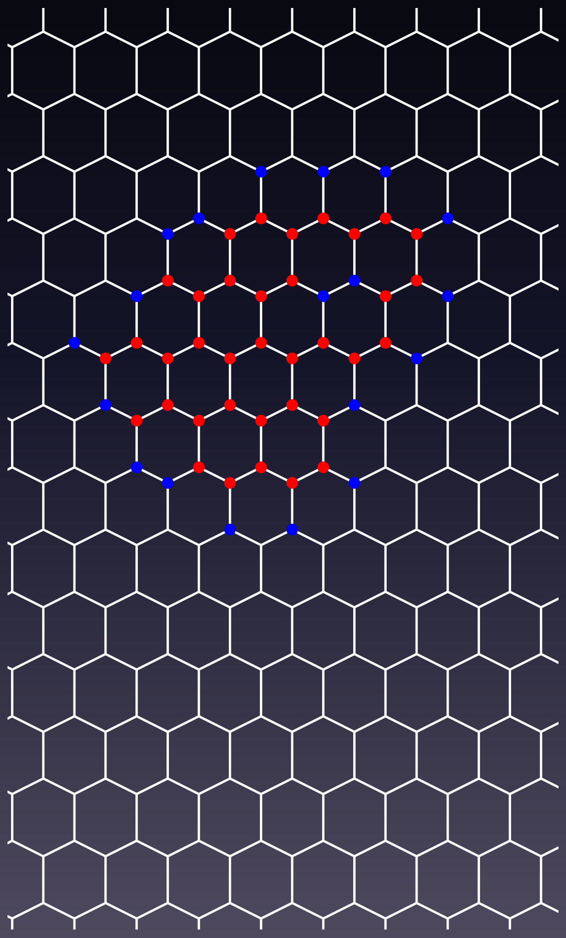
A function is harmonic at x if

$$\Delta f(x) := \frac{1}{\deg(x)} \sum_{y \sim x} (f(y) - f(x)) = 0$$

harmonic functions

- maximum (minimum) principle
- boundary conditions uniquely define harmonic functions
- Liouville's Theorem: A bounded function harmonic on all of $\mathbb{Z}^d$ is constant.



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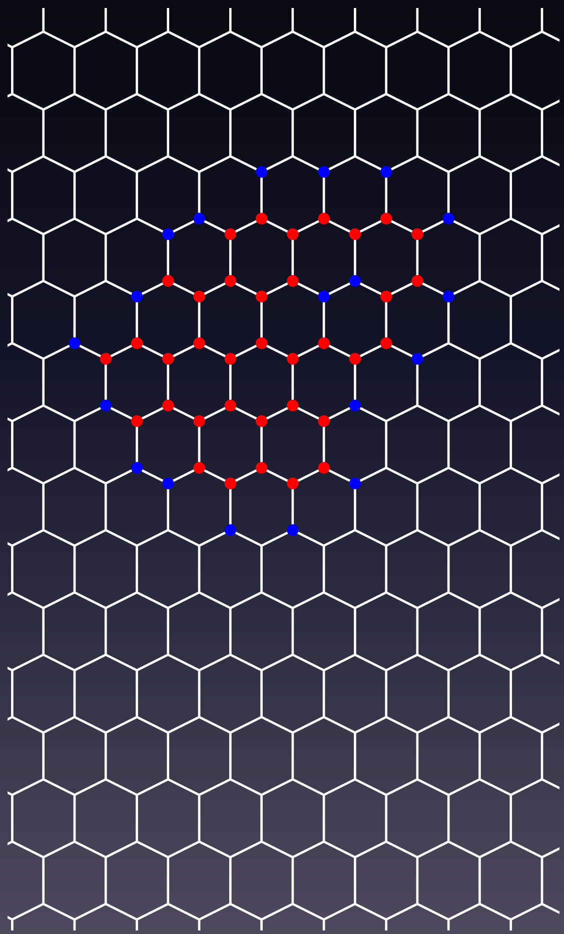

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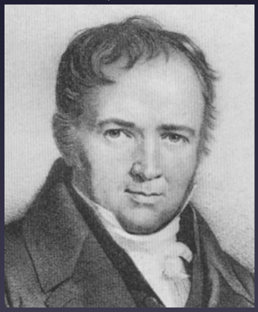
Peter Gustav Lejeune Dirichlet (1805–1859)



Siméon Denis Poisson (1781–1840)

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Joseph Liouville
(1809–1882)

heat evolution

$$u(x,t+\delta) - u(x,t) = \delta \cdot \frac{2}{\varepsilon^2} \cdot \left(\frac{u(x+\varepsilon,t)+u(x-\varepsilon,t)}{2} - u(x,t) \right)$$

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harmonic functions are stable under heat evolution

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classical notions

random walks

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counting simple paths

random walk

Given a graph G, walk randomly on a graph: at every time step, independently of the past, choose a uniform neighbor and move to it

Dirichlet problem solution

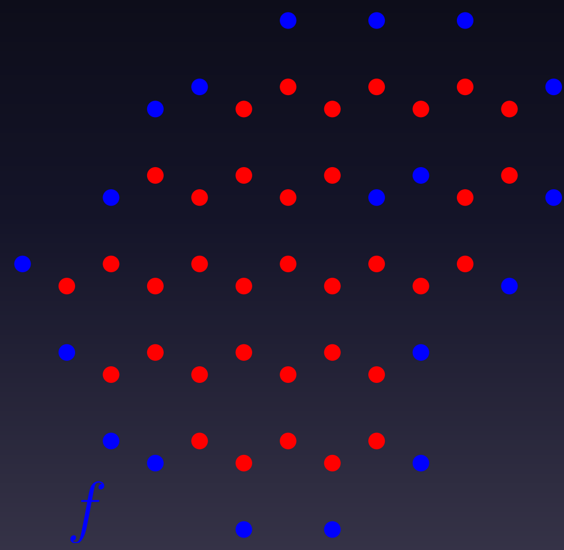
- in a graph G let D be some finite domain with a boundary

  $$\partial D = \{x \notin D \ : \ \exists \ y \sim x \ , \ y \in D\}$$

- $f : \partial D \to \mathbb{R}$ boundary conditions
- the function

  $$u(x) = \mathbb{E}[f(X_T) \mid X_0 = x]$$

  is harmonic in D and coincides with the boundary conditions f



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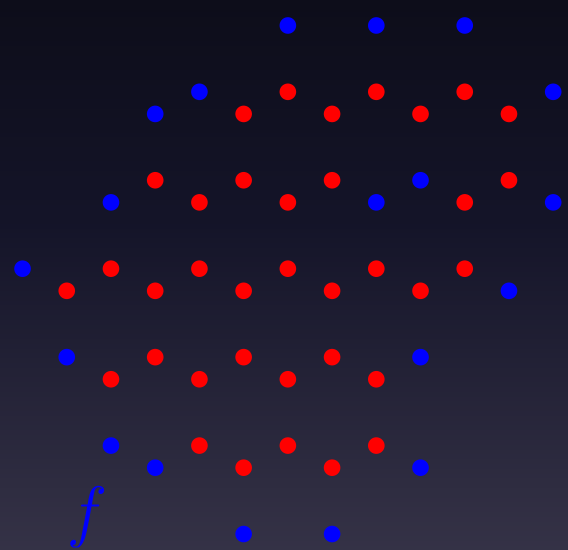

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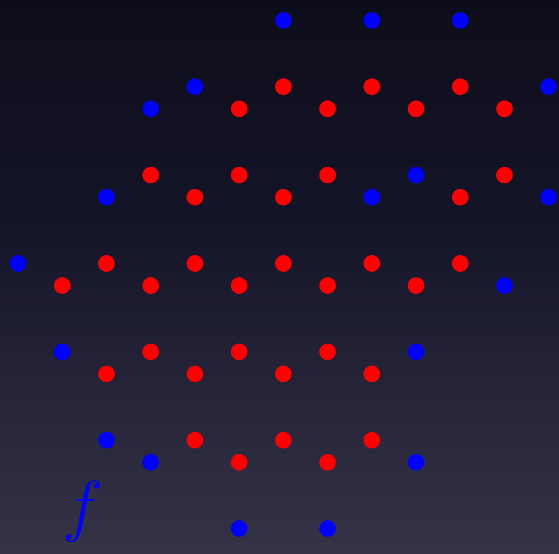

heat equation solution

- In a graph G let D be some finite domain with a boundary

$$\partial D = \{x \notin D \ : \ \exists\, y \sim x \ , \ y \in D\}.$$

- Let $f : \partial D \cup D \to \mathbb{R}$ be some initial / boundary conditions.
- The function

$$u(x,t) = \mathbb{E}[f(X_{T \wedge t}) \mid X_0 = x]$$

is harmonic in D and coincides with the initial conditions f, and solves the heat equation

$$u(x,t+1) - u(x,t) = \Delta u(x,t).$$

classical notions

random walks

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fair game

- play the following game:
  at each step, a coin is tossed
  you gain one coin, or lose one coin, each with probability $\frac{1}{2}$
- let $X_0, X_1, \ldots,$ be the number of coins at step t
- question: starting with x coins, what is the probability to win N coins without going bankrupt ?

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gambler's ruin

- $f(x)$ = probability to reach N before 0 started at x
- $f(0) = 0, f(N) = 1$
- $0 < x < N \quad \Rightarrow \quad f(x) = \frac{1}{2} f(x+1) + \frac{1}{2} f(x-1)$
- f is harmonic in $(0, N)$
- solution: $f(x) = \frac{x}{N}$
  (unique by maximum principle)

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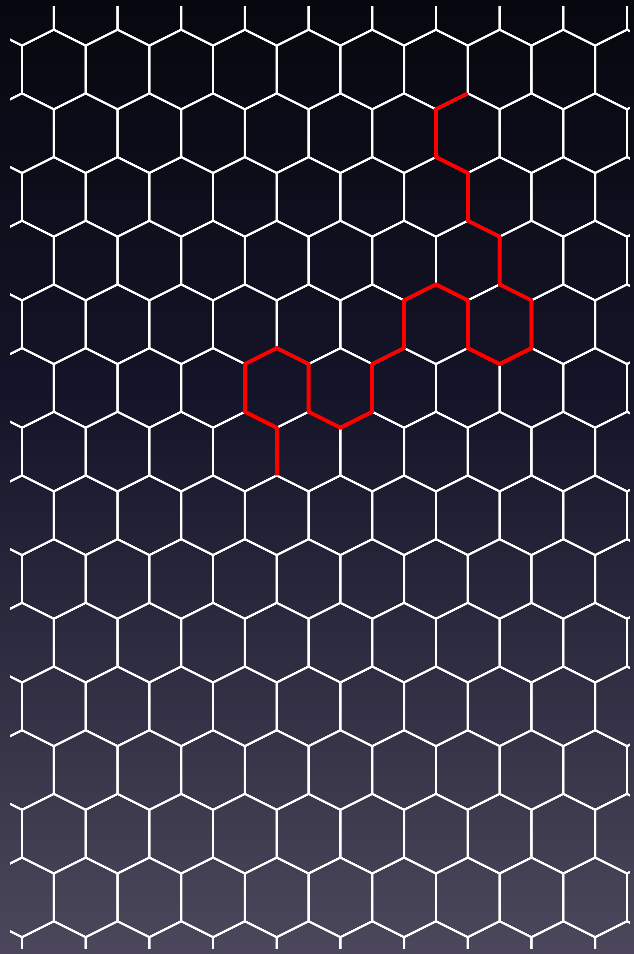
SAW

- in a graph G fix some root vertex o
- let SAW_n be the set of all simple paths of length n started at o
- Problem: count $|\mathsf{SAW}_n| =?$

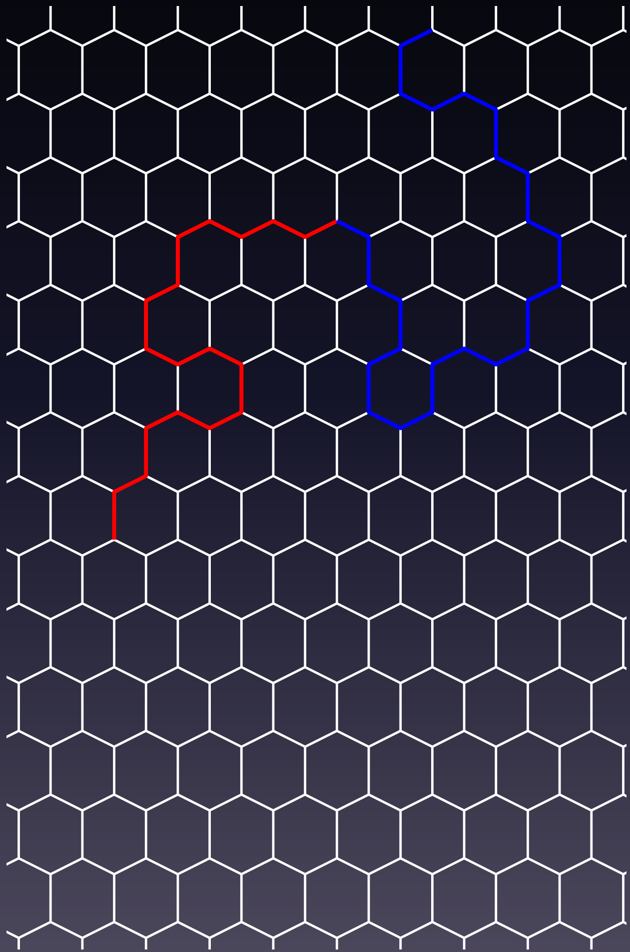


SAW

- easy using Fekete's Lemma: the limit exists:

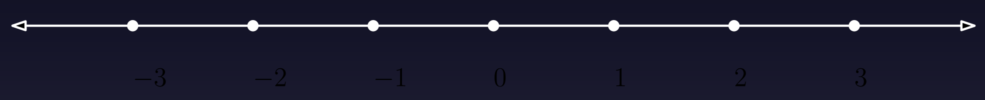
  $$\mu = \mu(G) := \lim_{n \to \infty} |\mathsf{SAW}_n|^{1/n}$$

- μ is called the connective constant



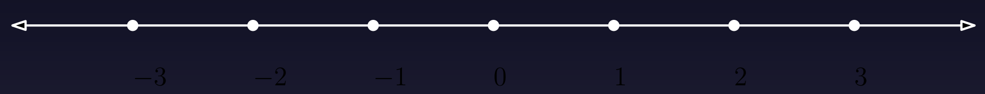
connective constant

Example: $\mu(\mathbb{Z}) =$



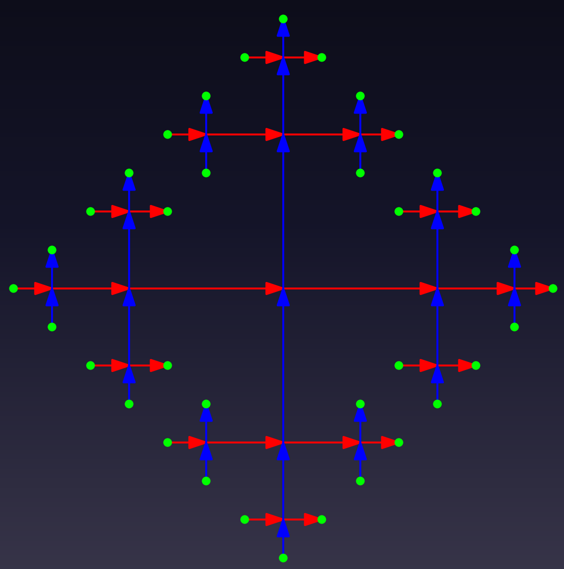
connective constant

Example: $\mu(\mathbb{Z}) = 1$



connective constant

Example: regular tree, $\mu(\mathbb{T}_d) =$



connective constant

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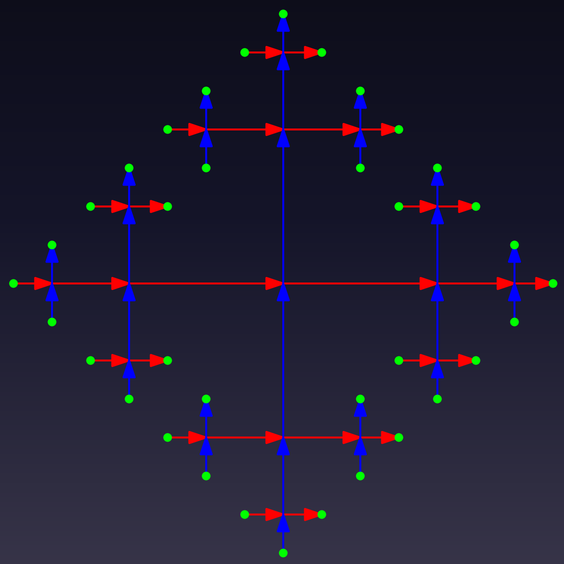
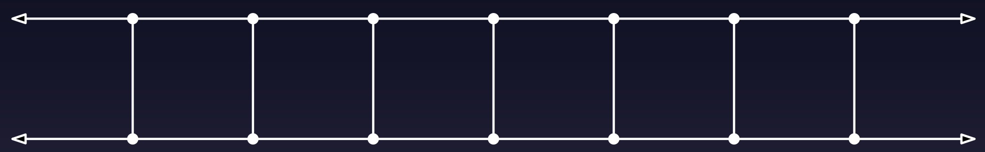


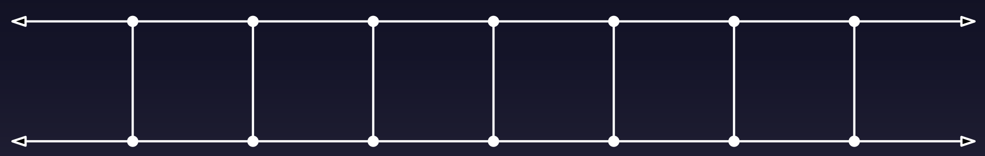
connective constant

Example: the ladder $\mu(L) =$



connective constant

Example: the ladder $\mu(L) = \frac{1+\sqrt{5}}{2}$



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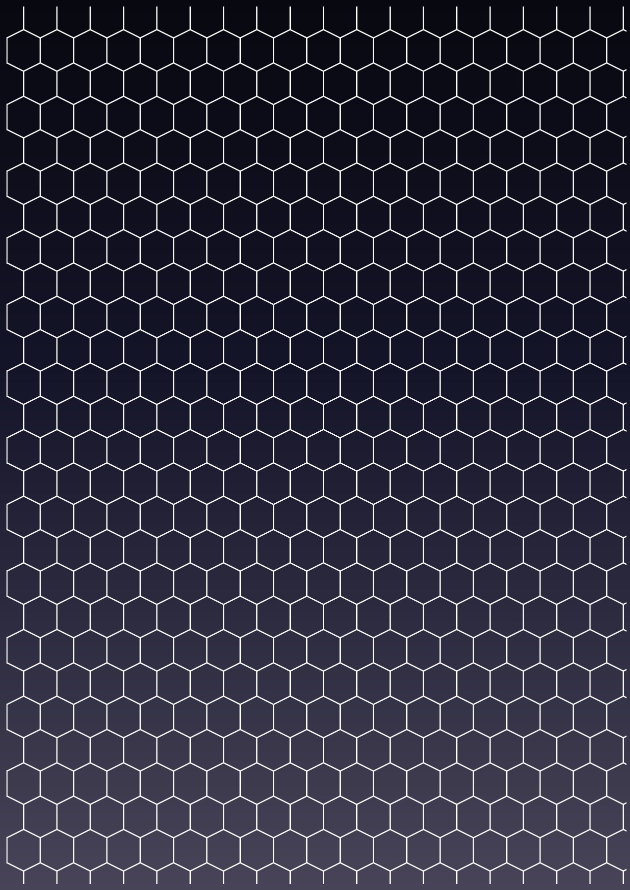

Theorem (Duminil-Copin & Smirnov 2010)

$\mu(\mathbb{H}) =$

connective constant

Theorem (Duminil-Copin & Smirnov 2010)

$\mu(\mathbb{H}) = \sqrt{2+\sqrt{2}}$

Hugo Duminil-Copin

Stas Smirnov

calculating $\mu(\mathbb{H})$

- $\frac{1}{\mu}$ is the radius of convergence for the generating function

$$P(z) = \sum_{n=0}^{\infty} |\mathsf{SAW}_n| z^n = \sum_{v \in \mathbb{H}} \sum_{\omega : o \to v} z^{|\omega|}$$

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reductions

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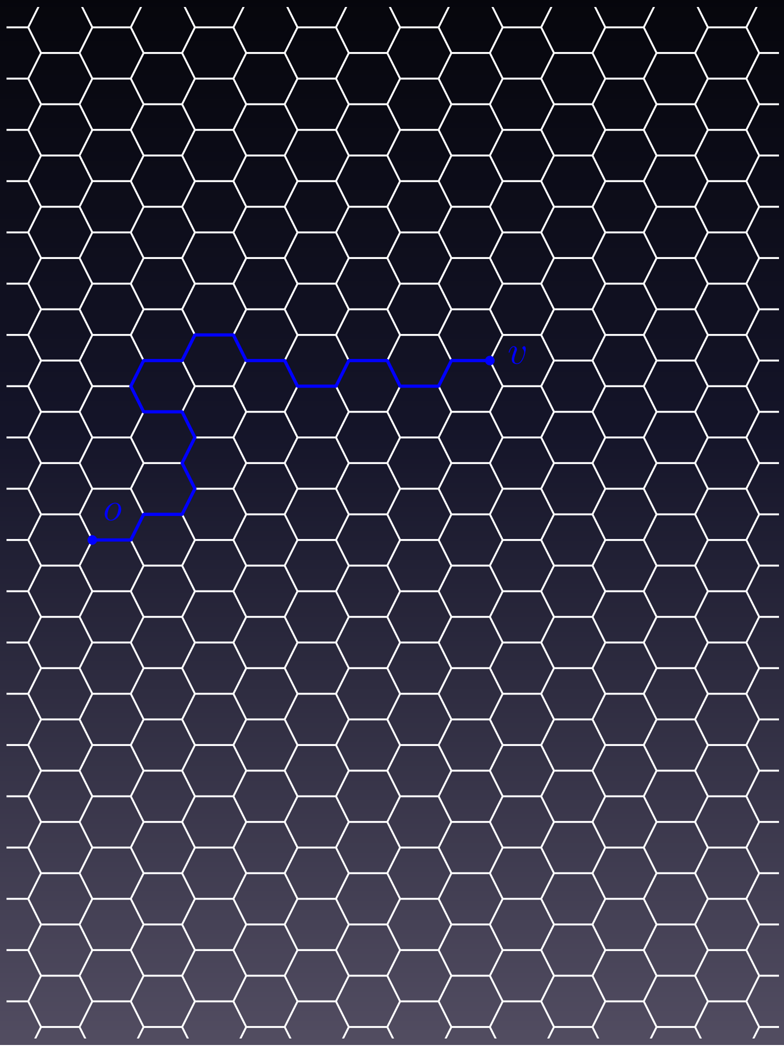

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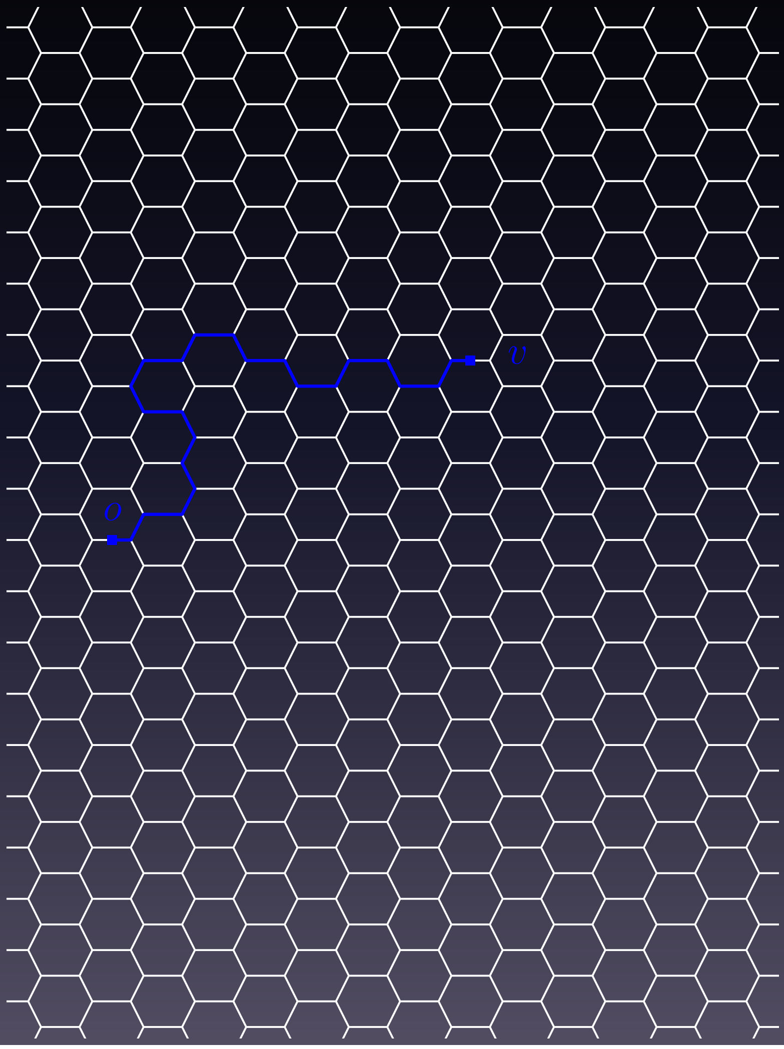
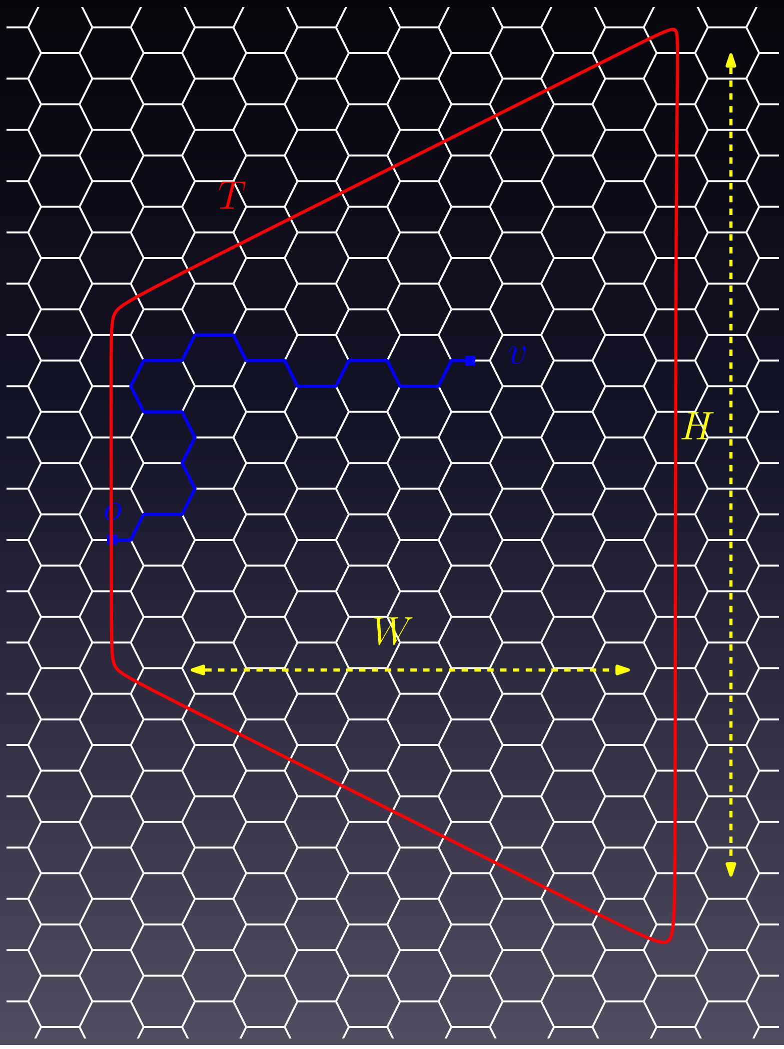


reductions

$$P(z) = \sum_{v \in T} \sum_{\omega : o \to v} z^{|\omega|}$$



- define a function on mid-edges in T:

$$F(p) = \sum_{\omega : o \to p} z^{|\omega|} e^{i\alpha\theta(\omega)}$$

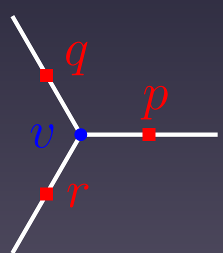
  where $\theta(\omega)$ is the winding of ω

- extend F to the vertices by averaging F on the mid edges around each vertex:

$$F(v) = (p - v)F(p) + (q - v)F(q) + (r - v)F(r)$$

  where p, q, r are the mid-edges adjacent to v

  Question: can we find z, α so that F is “holomorphic” ? that is, so that F is 0 on each vertex?



- define a function on mid-edges in T:

$$F(p) = \sum_{\omega : o \to p} z^{|\omega|} e^{i\alpha\theta(\omega)}$$

  where $\theta(\omega)$ is the winding of ω

- extend F to the vertices by averaging F on the mid edges around each vertex:

$$F(v) = (p - v)F(p) + (q - v)F(q) + (r - v)F(r)$$

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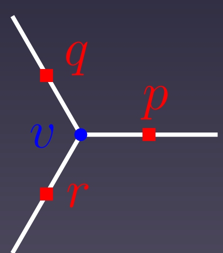

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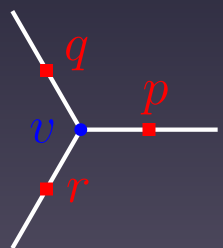
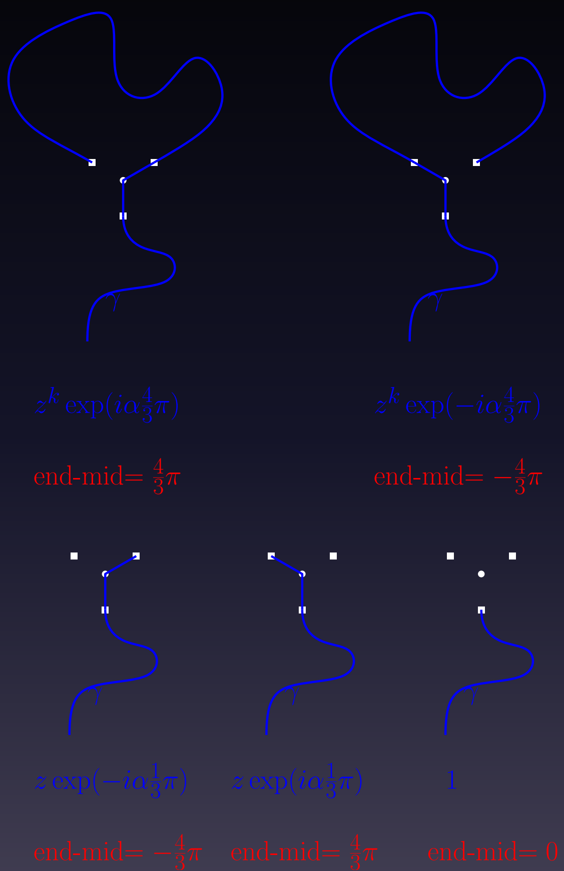


$$0 = e^{i\frac{4}{3}\pi} \cdot z^k e^{i\alpha\frac{4}{3}\pi} + e^{-i\frac{4}{3}\pi} \cdot z^k e^{-i\alpha\frac{4}{3}\pi}$$
$$+ e^{-i\frac{4}{3}\pi} \cdot z e^{-i\alpha\frac{1}{3}\pi} + e^{i\frac{4}{3}\pi} \cdot z e^{i\alpha\frac{1}{3}\pi} + 1 \cdot 1$$



Equations:

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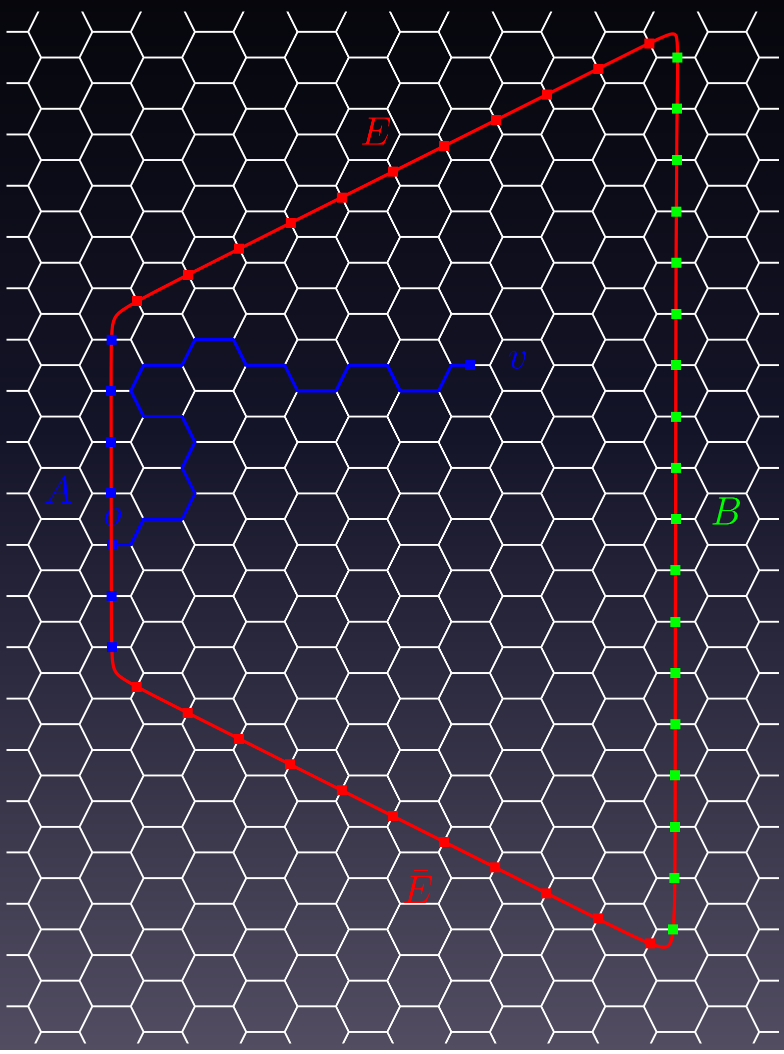
e.g. $k = 0$ and $\alpha = -\frac{5}{8}$ and

$$z^{-1} = 2\cos\tfrac{\pi}{8} = \sqrt{2+\sqrt{2}}$$

summary so far

$$P(z) = \sum_{v \in T} \sum_{\omega : o \to v} z^{|\omega|}$$



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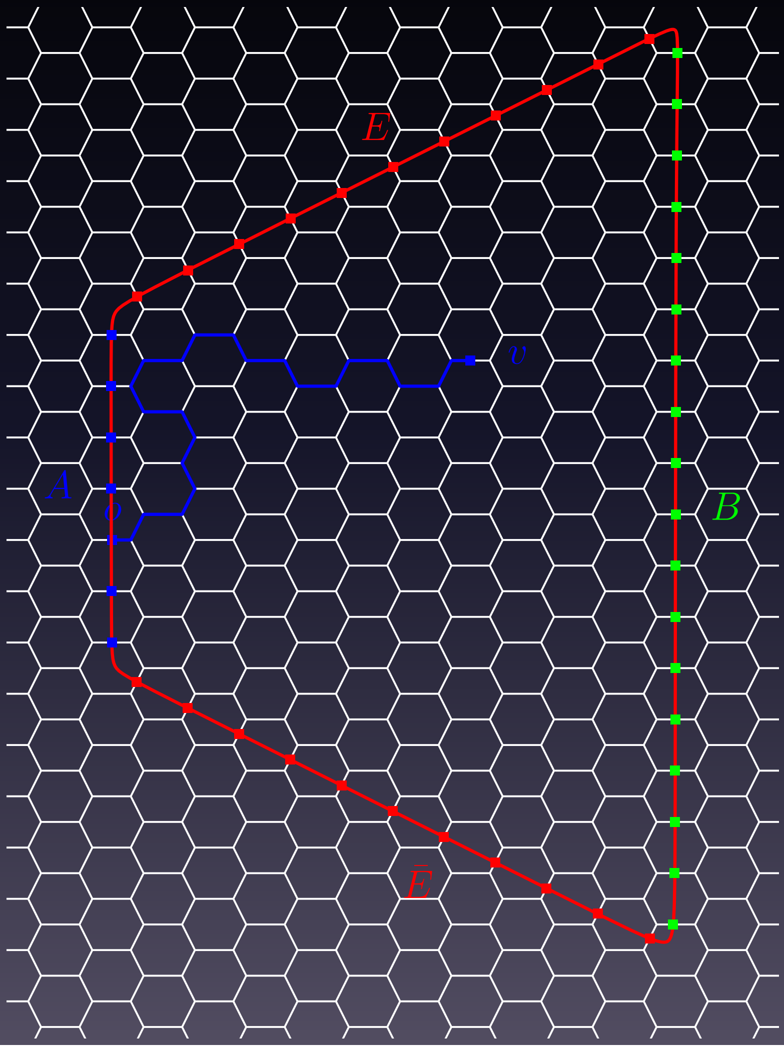

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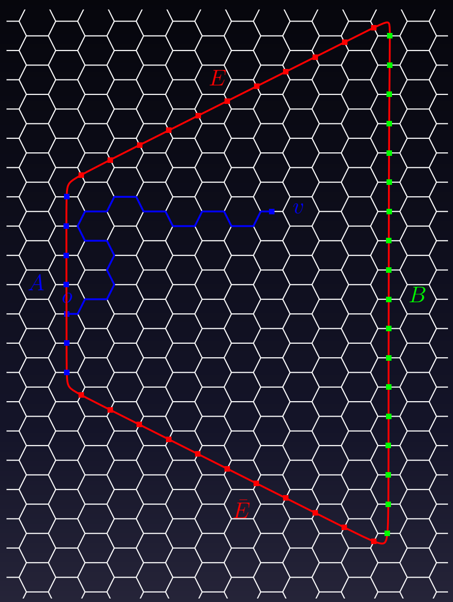

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the winding is constant on A, B, E and $\bar{E}$!

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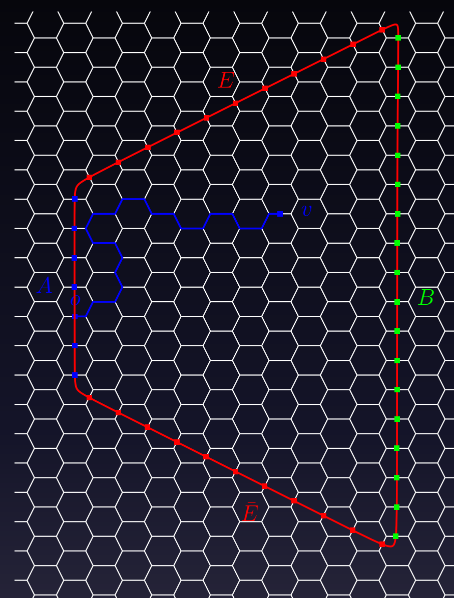

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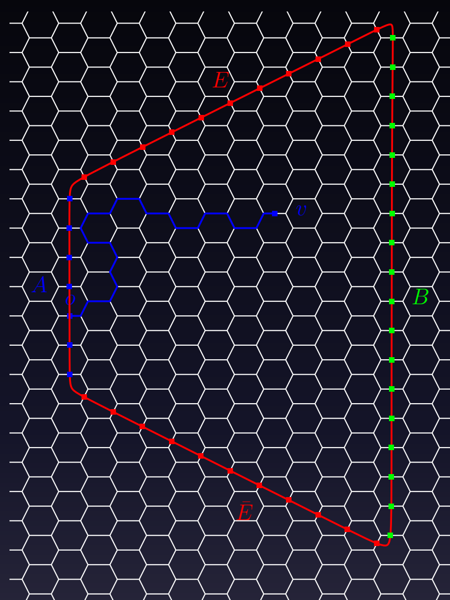

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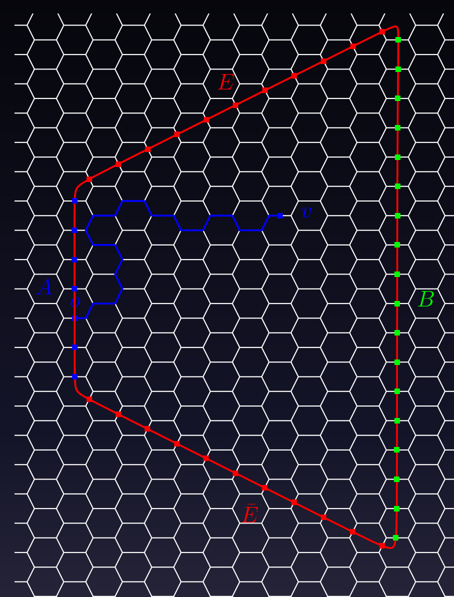

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taking $H \to \infty$ (height)

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