Spectral Analysis and Deformation Quantization

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Dedicated to Noriko Sakurai

Plan

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1. Introduction

1.1 Classical Mechanics

Hamiltonian system

- Phase space : $\mathbb{R}^{2n} = \{(x, p)\}(= \{(x_1, \cdot, x_n, \cdot, p_1, \cdot, p_n)\})$
- Symplectic 2-form : $\omega = dx \wedge dp(= \sum_k dx_k \wedge dp_k)$
- Hamiltonian function (Observable) : $H(x, p) \in C^\infty(\mathbb{R}^{2n})$
- Hamiltonian vector field : X_f

$$\omega(X_f, \cdot) = -df$$

The classical dynamics is described as the integral curve of the Hamiltonian vector field X_f

Evolution equation:

$$\frac{d\mathbf{x}}{dt} = X_f(\mathbf{x}(t))$$

1.2 Poisson algebra

Consider

- $C^\infty(\mathbb{R}^{2n})$
- Poisson bracket on $C^\infty(\mathbb{R}^{2n})$:

$$\{f,g\} = X_f g = \partial_x f \partial_p g - \partial_p f \partial_x g$$

The Poisson bracket satisfies

- $\{\cdot,\cdot\}$ is bilinear
- Skew symmetric :$\{f,g\} = -\{g,f\}$
- Jacobi identity : $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$
- Derivation rule : $\{f,gh\} = \{f,g\}h + g\{f,h\}$

- We call $(C^\infty(\mathbb{R}^{2n}),\cdot,\{\cdot\,,\,\cdot\})$ the **Poisson algebra**

1.3 Quantum Mechanics

Quantum Mechanics = Quantization Q of classical mechanics

- Space of state space (=Hilbert space) $\mathcal{H}$
- (Self-adjoint) operators $Q(f) : \mathcal{H} \to \mathcal{H}$ for $f \in C^\infty(\mathbb{R}^{2n})$
- Commutation Relations $[Q(f), Q(g)] = I\hbar Q(\{f, g\})$, where $\{f, g\}$ is Poisson bracket.

Q gives a homomorphism from the Poisson algebra $C^\infty(\mathbb{R}^{2n})$ to $End(\mathcal{H})$

In particular, we set $Q(x) = x\times$, and $Q(p) = \frac{\hbar}{i}\partial_x$.
Then, we have

$$[Q(x), Q(p)] = i\hbar (= i\hbar Q(\{x, p\}))$$

which gives most typical quantization.

2. Weyl Algebra and Deformation Quantization

2.1 Weyl algebra (of two generators)

We set up this quantization procedure algebraically as the Weyl algebra, where we consider the most simple situation.

- The algebra of two generators (u_1, u_2):
- The fundamental commutation relation

$$[u_1, u_2] = u_1 * u_2 - u_2 * u_1 = -i\hbar$$

- We denote this Weyl algebra by $(W_2, *)$

2.2 Deformation quantization

We realize the Weyl algebra on the space $\mathbb{C}[u_1, u_2]$ as follows:

Let J be the 2×2 matrix defined by $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

We fix J throughout this talk.

We define the star product as follows:

- Let K be an arbitrary 2×2-(complex) symmetric matrix
- Set
$$\Lambda = K + J$$
.
- Define a product $*_\Lambda$ on the space of polynomials $\mathbb{C}[u_1, u_2]$:

$$\begin{aligned} f *_K g &= f e^{\frac{i\hbar}{2}(\sum \overleftarrow{\partial_{u_i}} \Lambda^{ij} \overrightarrow{\partial_{u_j}})} g \\ &= \sum_k \frac{(i\hbar)^k}{k! 2^k} \Lambda^{i_1 j_1} \cdots \Lambda^{i_k j_k} \partial_{u_{i_1}} \cdots \partial_{u_{i_k}} f \ \partial_{u_{j_1}} \cdots \partial_{u_{j_k}} g. \end{aligned}$$

Remark

- The product $*_K$ is associative for any K
-
$$[f, g]_{*_K} = i\hbar\{f, g\} + \text{higher order term}$$
where $\{f, g\}$ is the Poisson bracket.

2.3 Remarks on the expression parameter

- This product depends on K, which is called the expression parameter.
- $(\mathbb{C}[u_1, u_2], *_{_K})$ are mutually isomorphic associative algebras for every K
  - The isomorphism class is the Weyl algebra $(W_2; *)$, for every K
- $*_K$-product formula gives a way of univalent expression for elements of $(W_2; *)$
  - Every element $A \in (W_2; *)$ is expressed in the form of ordinary polynomial, which we denote by $:A:_{_K} \in \mathbb{C}[u_1, u_2]$
- By using the product formulas, we can make topological completions of the algebra, and consider transcendental elements

2.4 Examples of expressions

Our expression of product is related to the "ordering problem".

Denote the symmetric matrix K by $K = \begin{pmatrix} \delta & c \\ c & \delta' \end{pmatrix}$.

The following expressions are common in physics:

- **Weyl expression =Weyl ordering**: $c = \delta = \delta' = 0$
- **Normal expression = Normal ordering**: $\delta = \delta' = 0$, $c = 1$

We can also consider various expressions by taking K.
For example:

- **Unit expression** : $K{=}I$,
- **More specific expression** : $(1{+}c)^2{-}\delta\delta'{=}0$

2.5 Intertwiners

Let $\mathfrak{S}(2)$ be the set of 2×2 symmetric (complex) metrices.

Proposition

- For every $K, K' \in \mathfrak{S}(2)$, we set

$$I_K^{K'}(f) = \exp\left(\frac{i\hbar}{4}\sum_{i,j}(K'^{ij} - K^{ij})\partial_{u_i}\partial_{u_j}\right) f \ (= I_0^{K'}(I_0^K)^{-1}(f)),$$

- This gives an isomorphism $I_K^{K'} : (\mathbb{C}[u_1, u_2]; *_K) \to (\mathbb{C}[u_1, u_2]; *_{K'})$
-
$$I_K^{K'}(f *_K g) = I_K^{K'}(f) *_K' I_K^{K'}(g),$$

  for any $f, g \in \mathbb{C}[u_1, u_2]$.
- We call $I_K^{K'}$ the intertwiner.

Remark Intertwiners do not change the algebraic structure $*$, but these change the expression of elements.

2.6 Infinitesimal intertwiners

- The "infinitesimal intertwiner" is given by the following evolution equation

$$dI_{\kappa}(K') = \frac{d}{dt}\Big|_{t=0} I_{\kappa}^{\kappa+t\kappa'} = \frac{i\hbar}{4} K'_{ij}\partial_{u_i}\partial_{u_j}$$

which is viewed as a "flat connection" on the trivial bundle $\coprod_{K \in \mathfrak{S}(2)} Hol(\mathbb{C}^2)$.

- The equation of parallel translation along a curve $K(t)$ is given by

$$\frac{d}{dt} f_t = dI_{\kappa}(\dot{K}(t)) f_t, \quad \dot{K}(t) = \frac{d}{dt} K(t),$$

3. Star exponential functions

3.1 General idea for star exponential function

- Let $H_* \in (W_2; *)$
- We consider the evolution equation

$$\frac{d}{dt} f_t = H_* * f_t, \quad f_0 = 1.$$

- If the solution of the above evolution equation exists then the solution is denoted by $e_*^{tH_*}$.
- we call $e_*^{tH_*}$ the $*$-exponential function of H_*

3.2 Star-exponential functions of quadratic functions

- In what follows we replace (u_1, u_2) by (u, v).
- We consider a special quadratic function $2u{\circ}v = u*v + v*u$
- Consider the $*$-exponential function of $2u{\circ}v$ under a general expression parameter $K{=}\begin{pmatrix} \delta & c \\ c & \delta' \end{pmatrix}$.
- Note that $:\frac{1}{i\hbar}u{\circ}v:_K = \frac{1}{i\hbar}uv{+}\frac{1}{2}c$

Proposition

$$:e_*^{t\frac{1}{i\hbar}2u\circ v}:_K = \frac{2}{\sqrt{\Delta^2-(e^t-e^{-t})^2\delta\delta'}}\ e^{\frac{1}{i\hbar}\frac{e^t-e^{-t}}{\Delta^2-(e^t-e^{-t})^2\delta\delta'}\Big((e^t-e^{-t})(\delta' u^2+\delta v^2)+2\Delta uv\Big)}.$$

where $\Delta{=}e^t{+}e^{-t}{-}c(e^t{-}e^{-t})$

3.3 Generic properties of Star exponential functions

We consider a special quadratic form $u{\circ}v$ and its star exponential function $e_*^{z\frac{2}{i\hbar}u{\circ}v}$, where we consider the complex parameter z.
We first note the following remarkable generic properties :

Proposition There is an open dense subset $\mathfrak{R}$ of $\mathfrak{S}(2)$ such that

- $e_*^{z\frac{2}{i\hbar}u{\circ}v}$ has no singular point on the real axis and the pure imaginary axis on $\mathfrak{R}$
- $e_*^{z\frac{2}{i\hbar}u{\circ}v}$ is a $Hol(\mathbb{C}^2)$-valued $2\pi i$-periodic function
  - More precisely, it is πi-periodic or alternating πi-periodic
- $e_*^{z\frac{2}{i\hbar}u{\circ}v}$ is rapidly decreasing along any line parallel to the real line.
- $e_*^{z\frac{2}{i\hbar}u{\circ}v}$ has periodic double branched singular points. Singular point set Σ_K is distributed πi-periodically along two lines parallel to the imaginary axis.
- The exponential law holds:

$$e_*^{z\frac{2}{i\hbar}u{\circ}v}*e_*^{z'\frac{2}{i\hbar}u{\circ}v}=e_*^{(z+z')\frac{2}{i\hbar}u{\circ}v}$$

- Set $:e_*^{0\frac{2}{i\hbar}u{\circ}v}:_K=1$. Then, the value $:e_*^{[0\sim z]\frac{2}{i\hbar}u{\circ}v}:_K$ is determined uniquely, where $[0{\sim}z]$ is a path from 0 to z avoiding Σ_K and evaluating at z.

We call $K\in\mathfrak{R}$ a **generic ordered expression**.

3.4 Generic set

Let $\mathfrak{R}$ be the open dense subset of general expressions K. There are three disjoint open subsets $\mathfrak{K}_{\pm}$ and $\mathfrak{K}_0$ of the space of expression parameters such that $\mathfrak{K}_+ \cup \mathfrak{K}_- \cup \mathfrak{K}_0$ is dense.

- If $K{\in}\mathfrak{K}_+$ (resp. $\mathfrak{K}_-$)
  - The singular set of $:e_*^{\frac{z}{i\hbar}2u\circ v}:_K$ appears πi-periodically only in the open right (resp. left) half plane, along two lines parallel to the imaginary axis
  - The $*$-exponential functions form a complex semi-group over the left (resp. right) half plane without sign ambiguity by requesting 1 at $t{=}0$
  - Moreover, $:e_*^{\pm z\frac{1}{i\hbar}2u\circ v}:_K$, is alternating πi-periodic on the imaginary axis, and $:e_*^{\pm\frac{z}{i\hbar}2u\circ v}:_K$ is rapidly decreasing of $e^{-|z|}$ order along any line parallel to the real line.
- If $K \in \mathfrak{K}_0$
  - The singular set appears in both left and right half-planes, but not on the imaginary axis. Both of these lines are parallel to the imaginary axis.
  - Moreover, $:e_*^{\pm\frac{z}{i\hbar}2u\circ v}:_K$, is πi-periodic on the imaginary axis.

4. Vacuum

4.1 Vacuum and bar-vacuum

Note that in a generic ordered expression $e_*^{\pm\frac{t}{i\hbar}2u\circ v}$ is rapidly decreasing with the growth order $e^{-|t|}$ along lines parallel to the real axis. Noting $v{*}u{=}u{\circ}v{+}\frac{1}{2}i\hbar$, we see the following:

Proposition

- In generic ordered expressions such that there is no singular point on the real axis
-
$$\lim_{t\to\infty} e_*^{t\frac{1}{i\hbar}2u*v}{=}0, \quad \lim_{t\to-\infty} e_*^{t\frac{1}{i\hbar}2v*u}{=}0,$$
- The following limit exists
$$\lim_{t\to-\infty} e_*^{t\frac{1}{i\hbar}2u*v}{=}\varpi_{00}, \quad \lim_{t\to\infty} e_*^{t\frac{1}{i\hbar}2v*u}{=}\overline{\varpi}_{00}.$$

We call ϖ_{00} and $\overline{\varpi}_{00}$ **vacuum** and **bar-vacuum** respectively. These are contained in the space $\mathbb{C}e^{Q(u,v)}$ of exponential functions of quadratic forms.

4.2 Explicit form

More precisely, in a fixed generic expression parameter $K=\begin{pmatrix} \delta & c \\ c & \delta' \end{pmatrix}$,

$:e_*^{t\frac{1}{i\hbar}2u\circ v}:_K$ is smooth rapidly decreasing in $\pm$ directions, and satisfies

$$:\varpi_{00}:_K = \lim_{t\to-\infty} :e_*^{t\frac{1}{i\hbar}2u*v}:_K = \frac{2}{\sqrt{(1+c)^2-\delta\delta'}} e^{-\frac{1}{i\hbar}\frac{1}{(1+c)^2-\delta\delta'}(\delta u^2-(1+c)2uv+\delta' v^2)},$$

$$:\overline{\varpi}_{00}:_K = \lim_{t\to\infty} :e_*^{t\frac{1}{i\hbar}2v*u}:_K = \frac{2}{\sqrt{(1-c)^2-\delta\delta'}} e^{\frac{1}{i\hbar}\frac{1}{(1-c)^2-\delta\delta'}(\delta u^2+(1-c)2uv+\delta' v^2)},$$

$$\lim_{t\to\infty} :e_*^{t\frac{1}{i\hbar}2u*v}:_K = 0, \quad \lim_{t\to-\infty} :e_*^{t\frac{1}{i\hbar}2v*u}:_K = 0.$$

without sign ambiguity.

4.3 Properties for vacuum

As ϖ_{00} is defined by the limit, we see $u{*}v{*}\varpi_{00} = 0 = \varpi_{00}{*}u{*}v$.
We also have the following:

Lemma

In generic expressions, we have $v{*}\varpi_{00}{=}0{=}\varpi_{00}{*}u$, $u{*}\overline{\varpi}_{00}{=}0{=}\overline{\varpi}_{00}{*}v$

We note the following

- If $p \neq 0$, then $\varpi_{00}{*}(u^p{*}\varpi_{00}){=}0, \textit{and}\ \ (\varpi_{00}{*}v^p){*}\varpi_{00}{=}0.$
- Moreover, for every polynomial $f(u, v)$,

$$\varpi_{00} * (f(u, v) * \varpi_{00}) = f(0, 0)\varpi_{00} = (\varpi_{00} * f(u, v)) * \varpi_{00}$$

5. Spectral analysis

5.1 Elementary matrix for vacuum

Set

$$E_{p,q} = \frac{1}{\sqrt{p!q!(i\hbar)^{p+q}}} u^p {*} \varpi_{00} {*} v^q$$

Proposition

In generic ordered expressions K, we have

- $E_{p,q}$ is the (p,q)-elementary matrix.

  that is,

  $$E_{p,q} {*} E_{r,s} = \delta_{q,r} E_{p,s}$$

  .
- We denote the K-expression $:E_{p,q}:_K$ of $E_{p,q}$ by $E_{p,q}(K)$.
- $E_{0,0}(K) = :\varpi_{00}:_K$.

5.2 Elementary matrix for bar-vacuum

Smilarly, set

$$\overline{E}_{p,q} = \frac{\sqrt{-1}^{p+q}}{\sqrt{p!q!(i\hbar)^{p+q}}} v^p {*} \overline{\varpi}_{00} {*} u^q$$

Proposition

In generic expressions K, we have

- $\overline{E}_{p,q}$ is the (p,q)-elementary matrix in generic ordered expressions
- The K-expression of $\overline{E}_{p,q}$ will be denoted by $\overline{E}_{p,q}(K)$
- $\overline{E}_{0,0}(K) = {:}\overline{\varpi}_{00}{:}_K$

Proposition

In generic expressions K, we have

$$E_{p,q} {*} \overline{E}_{r,s} = 0 = \overline{E}_{r,s} {*} E_{p,q}.$$

5.3 Spectral decomposition (1)

We first consider the case $K \in \mathfrak{K}_+$.

Proposition

In the K-ordered expression for $K \in \mathfrak{K}_+$, we have

- $$1 = \sum_{n=0}^{\infty} E_{n,n}(K), \quad :e_*^{t(\frac{1}{i\hbar}(u \circ v + \lambda))}:_K = \sum_{n=0}^{\infty} e^{t(n+\frac{1}{2})+\lambda t} E_{n,n}(K), \quad (\mathrm{Re}\ t \leq 0)$$
- If $\mathrm{Re}\, t \leq 0$, the r.h.s. converges in the space $Hol(\mathbb{C}^2)$, and the latter converges uniformly on every compact subset w.r.t. t.

Thus,
$\frac{1}{i\hbar} u \circ v$ looks as if it were an indeterminate moving in $\mathbb{N} + \frac{1}{2}$.

5.3 Spectral decomposition (2)

Similary. we consider the case $K \in \mathfrak{R}_-$.

Proposition

In the K-ordered expression for $K \in \mathfrak{R}_-$, we have

- $$I = \sum_{n=o}^{\infty} \overline{E}_{n,n}(K), \quad :e_*^{t(\frac{1}{i\hbar}(u\circ v+\lambda))}:_K = \sum_{n=0}^{\infty} e^{t(-n-\frac{1}{2})+\lambda t}\overline{E}_{n,n}(K), \quad (\mathrm{Re}\ t \geq 0).$$
- If $\mathrm{Re}\, t \geq 0$, the r.h.s. converges in the space $Hol(\mathbb{C}^2)$, and the latter converges uniformly on every compact subset w.r.t. t.

Thus,
$\frac{1}{i\hbar}u\circ v$ looks as if it were an indeterminate moving in $-(\mathbb{N}+\frac{1}{2})$.

5.4 Spectral decomposition (3)

The case $K \in \mathfrak{K}_0$ is little complicated to mention. A typical properties as follows: We have the following:

Proposition

Assume that $K \in \mathfrak{K}_0$ with some conditions. There is an open interval (a, b) (depending on K) with $a < 0 < b$ such that the following holds.

- $$1 = \sum_{n=-\infty}^{\infty} D_{n,n}(K)$$

  where

  $$D_{n,n} = \frac{1}{2\pi}\int_0^{2\pi} :e_*^{z(\frac{1}{i\hbar}u\circ v)}:_K e^{-zn}dt$$

- For z with $a < \mathrm{Rez} < b$,

  $$:e_*^{z\frac{1}{i\hbar}(u\circ v)}:_K = \sum_{n=-\infty}^{\infty} e^{zn}D_{n,n}(K), \quad \lambda \in \mathbb{C},$$

  where the r.h.s. converges in $Hol(\mathbb{C}^2)$.

  Thus,
  $\frac{1}{i\hbar}u\circ v$ looks as if it were an indeterminate moving in $\mathbb{Z}$.

6. Matrix representation

6.1 Representation of 1

We note the following:

Theorem

In generic ordered expressions K, we have $\sum_{n=0}^{\infty} E_{n,n}(K)$, $\sum_{n=0}^{\infty} \overline{E}_{n,n}(K)$ and $\sum_{n=-\infty}^{\infty} D_{n,n}(K)$ represent 1, which will be denoted by $:1:_{Kmat}$, where $:1:_{Kmat}$ is the multiplicative identity 1.

Hence every element of the Weyl algebra is expressed various ways depending on the expression parameter.

6.2 Matrix representations for fractional power (1)

Let m be a non-negative integer. we have

$$
{:}(z+\frac{1}{i\hbar}u{\circ}v)^m_*{:}_K = \begin{cases} \sum_{n=0}^{\infty}(z+n+\frac{1}{2})^m E_{n,n}(K), & K \in \mathfrak{K}_+ \\ \sum_{n=-\infty}^{\infty}(z+n)^m D_{n,n}(K), & K \in \mathfrak{K}_0 \\ \sum_{n=0}^{\infty}(z-n-\frac{1}{2})^m \overline{E}_{n,n}(K), & K \in \mathfrak{K}_- \end{cases}
$$

which depends on expression parameters.

Note that each of them converges in $Hol(\mathbb{C}^2)$.

6.3 Matrix representations for fractional power (2)

On the other hand, we may set

$$:(z+\frac{1}{i\hbar}u{\circ}v)^m_*:_{Kmat}=\begin{cases}\sum_{n=0}^{\infty}(z+n+\frac{1}{2})^m E_{n,n}(K), & K\in\mathfrak{K}_+\\ \sum_{n=-\infty}^{\infty}(z+n)^m D_{n,n}(K), & K\in\mathfrak{K}_0\\ \sum_{n=0}^{\infty}(z-n-\frac{1}{2})^m \overline{E}_{n,n}(K), & K\in\mathfrak{K}_-\end{cases}$$

for every $m\in\mathbb{Z}$

It gives another definition of $(z+\frac{1}{i\hbar}u{\circ}v)^m_*$

We will omit the suffix K if the expression paramater is not strictly specified and denote simply by by $(z+\frac{1}{i\hbar}u{\circ}v)^m_{mat}$.

6.4 Matrix representations for fractional power (3)

Theorem

- If $K\in\mathfrak{K}_+$, then $\sum_{n=0}^{\infty}(z+n+\frac{1}{2})^{-1}E_{n,n}(K)$ is a holomorphic mapping of $\mathbb{C}\backslash\{-(\mathbb{N}+\frac{1}{2})\}$ into $Hol(\mathbb{C}^2)$,
- If $K\in\mathfrak{K}_0$, then $\sum_{n=-\infty}^{\infty}(z+n)^{-1}D_{n,n}(K)$ is a holomorphic mapping of $\mathbb{C}\backslash\mathbb{Z}$ into $Hol(\mathbb{C}^2)$,
- If $K\in\mathfrak{K}_-$, then $\sum_{n=0}^{\infty}(z-n-\frac{1}{2})^{-1}\overline{E}_{n,n}(K)$ is a holomorphic mapping of $\mathbb{C}\backslash\{\mathbb{N}+\frac{1}{2}\}$ into $Hol(\mathbb{C}^2)$.

7. Matrix representations for several star functions

7.1 Matrix Representation of inverses (1)

Note that we have two inverses for $(z+\frac{1}{i\hbar}u\circ v)$:

$$(z+\frac{1}{i\hbar}u\circ v)^{-1}_{*+} = \int_{-\infty}^{0} e_*^{t(\frac{1}{i\hbar}u\circ v+\frac{1}{2})}dt$$

and

$$(z+\frac{1}{i\hbar}u\circ v)^{-1}_{*-} = \frac{1}{i}\int_{0}^{\infty} e_*^{t(\frac{1}{i\hbar}u\circ v+\frac{1}{2})}dt$$

We will study some properties for these inverses. To study the residue of a singular point z_0, we compute $\frac{1}{2\pi i}\int_{C_{z_0}}(z+\frac{1}{i\hbar}u\circ v)^{-1}_{*\pm}dz$, where C_{z_0} is a small circle with the center at z_0.

Theorem

- $\mathrm{Res}((z+\frac{1}{i\hbar}u\circ v)^{-1}_{*+}, -(n+\frac{1}{2}))$ is $\frac{1}{(i\hbar)^n n!}u^n{*}\varpi_{00}{*}v^n$ in generic ordered expressions. This is $E_{n,n}$.
- $\mathrm{Res}((z-\frac{1}{i\hbar}u\circ v)^{-1}_{*-}, -(n+\frac{1}{2}))$ is $-\frac{1}{(i\hbar)^n n!}v^n{*}\overline{\varpi}_{00}{*}u^n$ in generic ordered expressions. This is $-\overline{E}_{n,n}$.

7.2 Matrix representation of the inverses (2)

$$:\!\big(z+\frac{1}{i\hbar}u{\circ}v\big)^{-1}_{*+}\!:_{\kappa} = \sum_{k=0}^{\infty}(z+k+\frac{1}{2})^{-1}E_{k,k}(K),$$

on the domain $\{z; \mathrm{Re}\, z > -\frac{1}{2}\}$

Theorem
Assume that $K{\in}\mathfrak{K}_0$.
Then those three series

$$\sum_{n=0}^{\infty}(z+n+\frac{1}{2})^{-1}E_{n,n}(K), \quad \sum_{n=-\infty}^{\infty}(z+n)^{-1}D_{n,n}(K), \quad \sum_{n=0}^{\infty}(z-n-\frac{1}{2})^{-1}\overline{E}_{n,n}(K)$$

are holomorphic mappings respectively of $\mathbb{C}\backslash\{-(\mathbb{N}+\frac{1}{2})\}$, $\mathbb{C}\backslash\{\mathbb{Z}\}$, $\mathbb{C}\backslash\{\mathbb{N}+\frac{1}{2}\}$ into $Hol(\mathbb{C}^2)$.

7.3 Star-delta function

Note that the exponential function $:e_*^{t\frac{1}{i\hbar}u\circ v}:_K$ are rapidly decreasing of $e^{-\frac{1}{2}|t|}$ order in generic ordered expressions.
Thus, we define $*$-delta function by the $*$-Fourier transform of 1:

$$\delta_*(z+\frac{1}{\hbar}u\circ v)=\int_{\mathbb{R}}e_*^{-it(z+\frac{1}{\hbar}u\circ v)}dt,\quad |\mathrm{Im}z|<\frac{1}{2}.$$

It is easy to see that

$$\delta_*(z+\frac{1}{\hbar}u\circ v)=\frac{1}{2\pi i}((-iz+\frac{1}{i\hbar}u\circ v)_{*+}^{-1}-(-iz+\frac{1}{i\hbar}u\circ v)_{*-}^{-1}).$$

Hence this is holomorphic on the domain $|\mathrm{Im}z|<\frac{1}{2}$.

Theorem

- $\delta_*(z+\frac{1}{\hbar}u\circ v)$ is analytically continued on the space $\mathbb{C}\backslash i(\mathbb{Z}+\frac{1}{2})$ as an $Hol(\mathbb{C}^2)$-valued holomorphic functions of z with simple poles.
- The residue of $\delta_*(z+\frac{1}{\hbar}u\circ v)$ at $-i(n+\frac{1}{2})$, $i(n+\frac{1}{2})$ are $E_{n,n}$, $\overline{E}_{n,n}$ respectively.

8. Conclusions

- **Purpose**
  - We propose an idea for spectral analysis from deformation quantization point of view.
- **Principle**
  - We start from the "Quantum operator" and study common properties on spectral values for a star-product within its equivalence class.

8.2 Remarks

- **IOP=Independent of Ordering Principle**
  - Sudy properties which is independent of the expression parameter K
  - For general quadratic functions
  - For general dimensions
- **Study for special orderings**
  - Weyl ordering(not a generic ordering) : Moreno-Silva study this case for 2-dimensional Weyl algebra
  - Normal ordering(depend on the quadric functions): In progress

8.3 References

Joint work with H.Omori, N.Miyazaki, A.Yoshioka(in progress)

References

"Deformation expression for elements of Algebras"

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Thank you