

המחלקה למתמטיקה, בן-גוריון

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## אשנב למתמטיקה

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ביום שלישי, 3 בינואר, 2017

בשעה 18:30 – 20:00

באולם 101-

ההרצאה

### פונקציות הרמוניות וספירת מסלולים פשוטים

תינתן על-ידי

אריאל ידין

**תקציר:** המושג של "פונקציה הרמונית" חוזר לעבודות של לפלס ופוריה (במאה ה-18!) ויש לו חשיבות עצומה בפיסיקה, הנדסה, ומדעים בכללי. בעזרת "ההרמוניות" השונות ניתן לתאר את כל הגלים האפשריים. זה, למשל, מהווה את הבסיס לקידוד mp3 במאות ה-19 וה-20 הכלילי את המושג גם לאוביקטים מתמטיים מודרניים יותר, והחשיבות הגיאומטרית שלו הובנה יותר. אנחנו נשוחח על הגדרה גיאומטרית של פונקציה הרמונית, שהיא כללית למדי. נסביר איך ניתן להשתמש בפונקציות כאלה כדי לספור מסלולים פשוטים - שזו בעיה קשה בפני עצמה. אני אשתדל להסביר את כל המושגים המופיעים בתקציר במהלך ההרצאה.

# Harmonic Functions

Ariel Yadin

Ben Gurion University

Eshnav, Jan 2017



classical notions

random walks

gambler's ruin

counting simple paths

# heat equation



Jean-Baptiste  
Joseph Fourier  
(1768–1830)

$u(x, t)$  = heat at time  $t$ , point  $x \in [0, L]$

$$\frac{\partial}{\partial t} u = \alpha \cdot \frac{\partial^2}{\partial x^2} u$$

# heat equation



Jean-Baptiste  
Joseph Fourier  
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$u(x, t)$  = heat at time  $t$ , point  $x \in [0, L]$   
Fourier's law + conservation of energy:

$$\frac{\partial}{\partial t}u = \alpha \cdot \frac{\partial^2}{\partial x^2}u$$

# heat equation

$$\frac{\partial}{\partial t}u = \Delta u \quad \Delta = \frac{\partial^2}{\partial x^2}$$

$$f(x + \varepsilon) = f(x) + f'(x)\varepsilon + \frac{1}{2}f''(x)\varepsilon^2 + O(\varepsilon^3)$$

$$f(x - \varepsilon) = f(x) - f'(x)\varepsilon + \frac{1}{2}f''(x)\varepsilon^2 + O(\varepsilon^3)$$

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$$f(x + \varepsilon) + f(x - \varepsilon) = 2f(x) + f''(x)\varepsilon^2 + O(\varepsilon^3).$$

$$u(x, t + \delta) - u(x, t) = \delta \cdot \frac{1}{\varepsilon^2} \cdot \left( \frac{u(x+\varepsilon, t) + u(x-\varepsilon, t)}{2} - u(x, t) \right)$$

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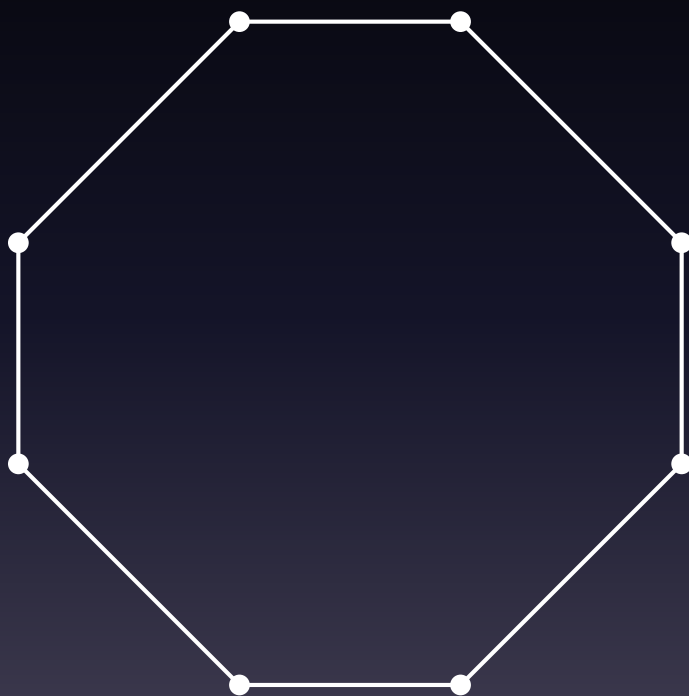
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# discrete geometry - graphs

A graph is a collection of vertices (nodes) and edges (connections).

graphs: cycle



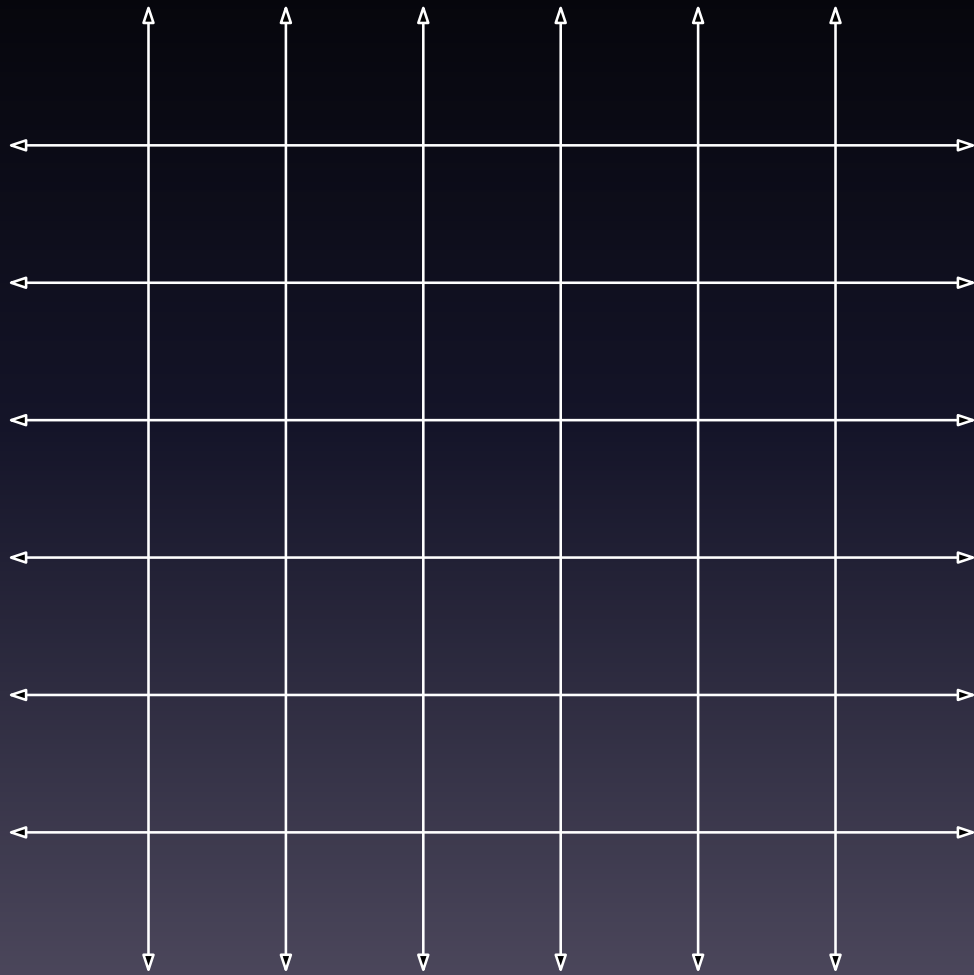
graphs: finite line



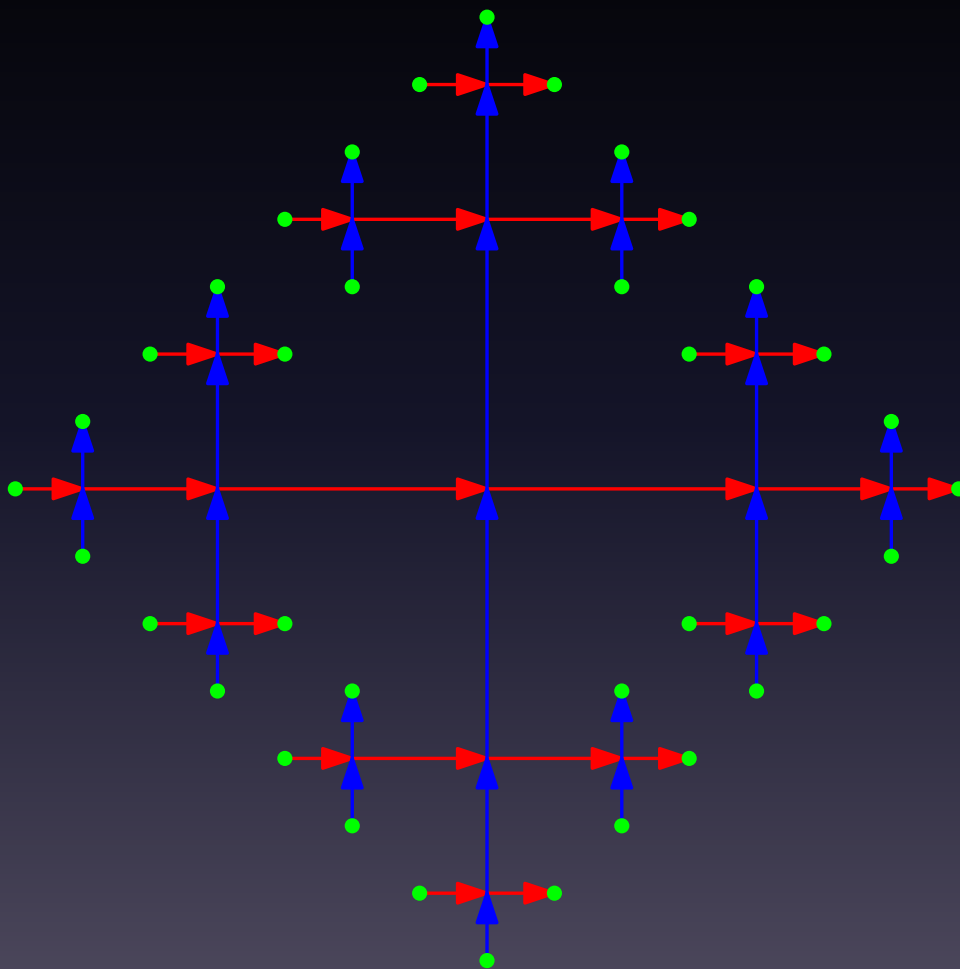
graphs:  $\mathbb{Z}$



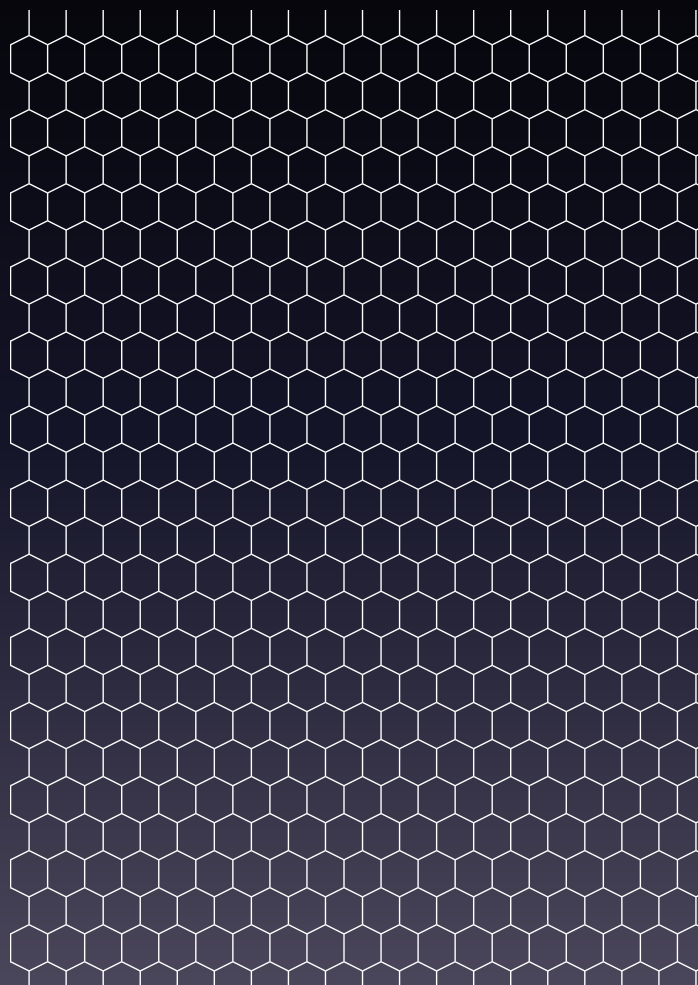
graphs:  $\mathbb{Z}^2$



graphs: regular tree



graphs: hexagonal lattice



# graphs: notation

- $x \sim y \iff x, y$  are neighbors
- $\deg(x)$  = the number of neighbors of  $x$
- paths give a metric (geometry)  
boundaries

$$\partial D = \{x \notin D : \exists y \sim x, y \in D\}$$

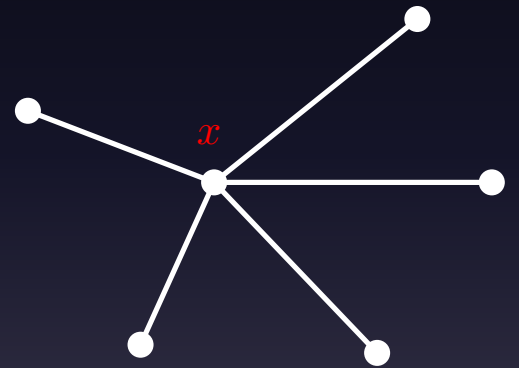




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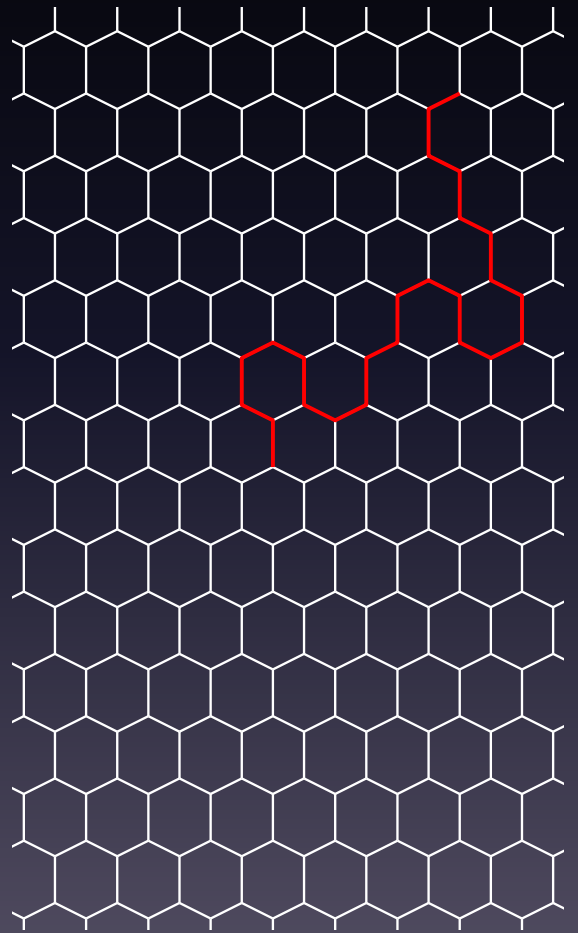
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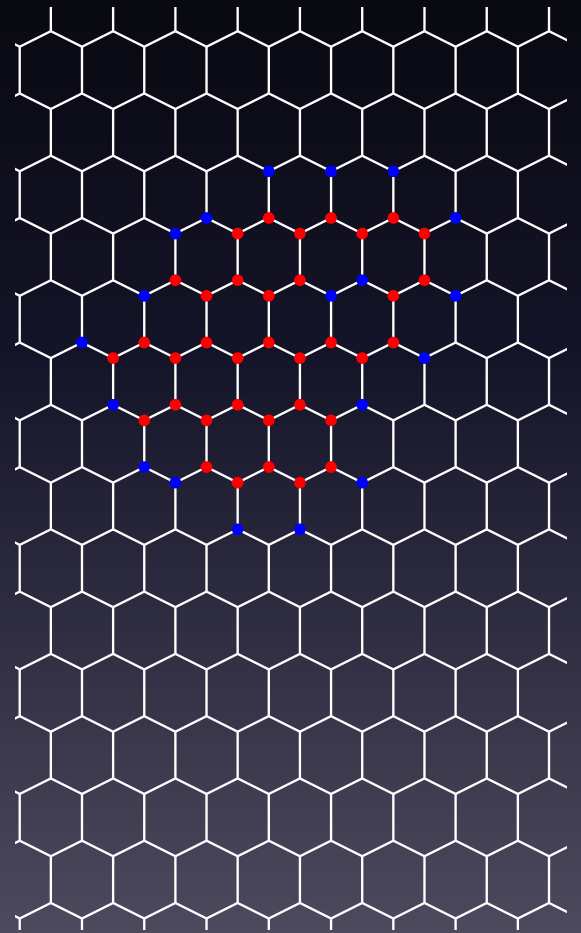
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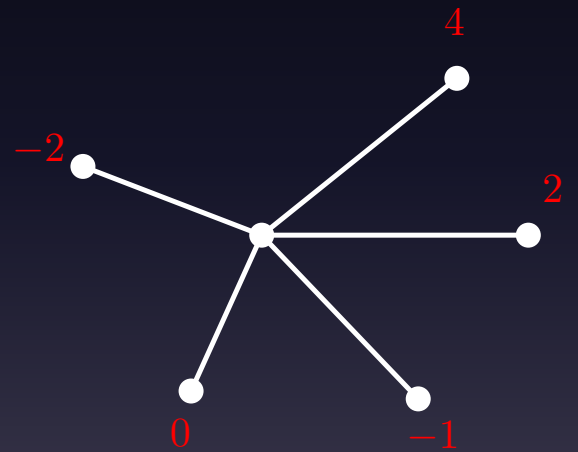


# harmonic function

Definition

A function is harmonic at  $x$  if

$$f(x) = \frac{1}{\deg(x)} \sum_{y \sim x} f(y)$$

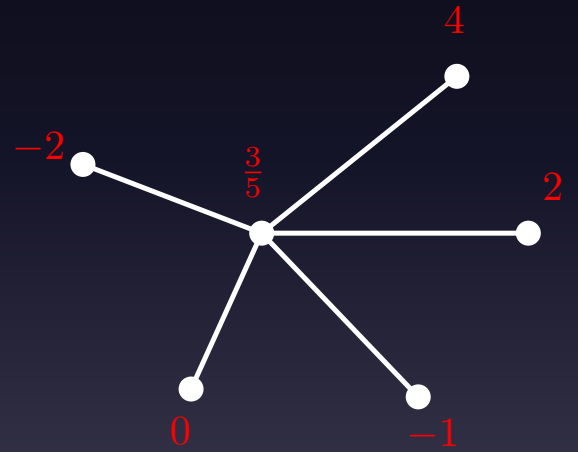


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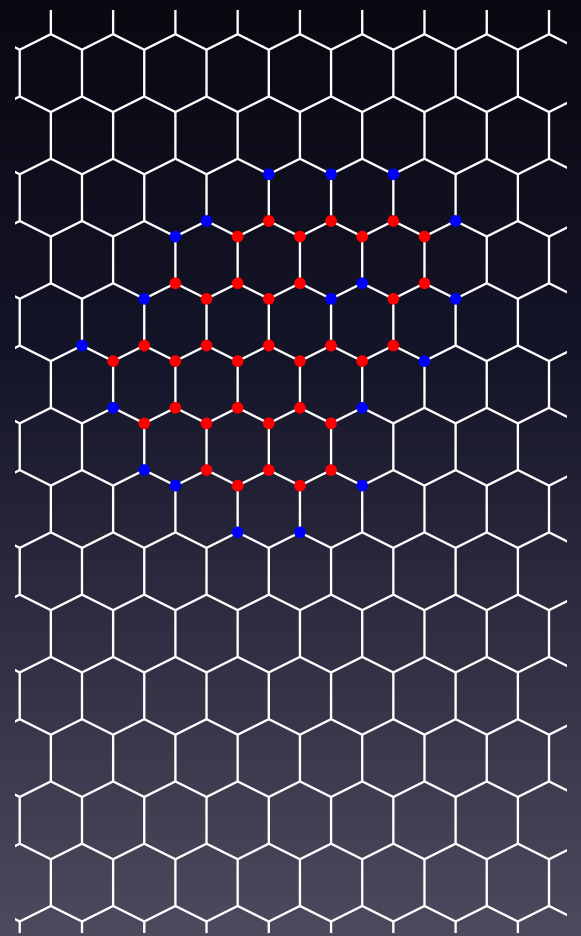
Definition

A function is harmonic at  $x$  if

$$\Delta f(x) := \frac{1}{\deg(x)} \sum_{y \sim x} (f(y) - f(x)) = 0$$

# harmonic functions

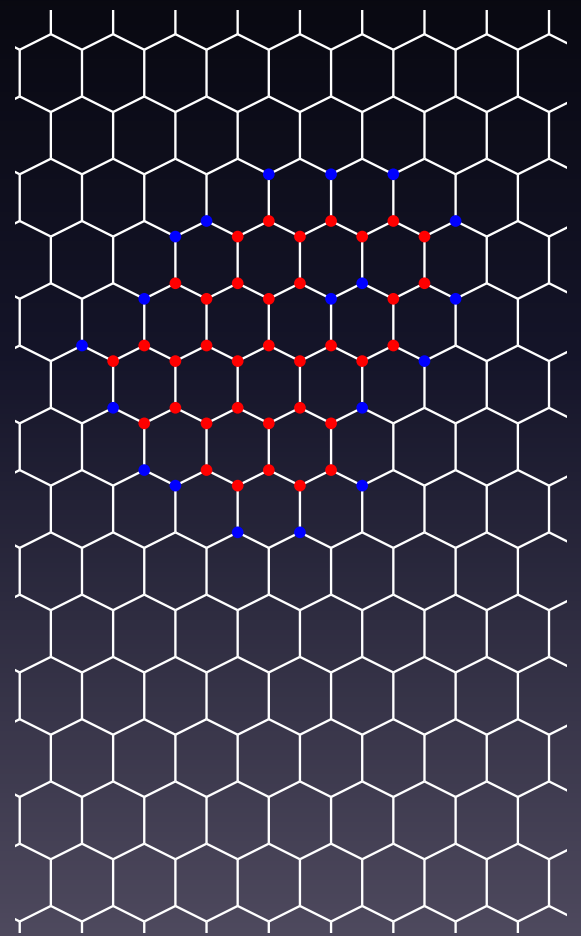
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- boundary conditions uniquely define harmonic functions
- Liouville's Theorem: A bounded function harmonic on all of  $\mathbb{Z}^d$  is constant.





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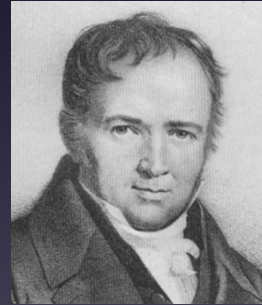


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Peter Gustav Lejeune  
Dirichlet (1805–1859)



Siméon Denis Poisson  
(1781–1840)

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Joseph Liouville  
(1809–1882)

# heat evolution

$$u(x, t + \delta) - u(x, t) = \delta \cdot \frac{2}{\varepsilon^2} \cdot \left( \frac{u(x+\varepsilon, t) + u(x-\varepsilon, t)}{2} - u(x, t) \right)$$

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harmonic functions are stable under heat evolution

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harmonic functions are stable under heat evolution

classical notions

random walks

gambler's ruin

counting simple paths



# random walk

Given a graph  $G$ , walk randomly on a graph: at every time step, independently of the past, choose a uniform neighbor and move to it

# Dirichlet problem solution

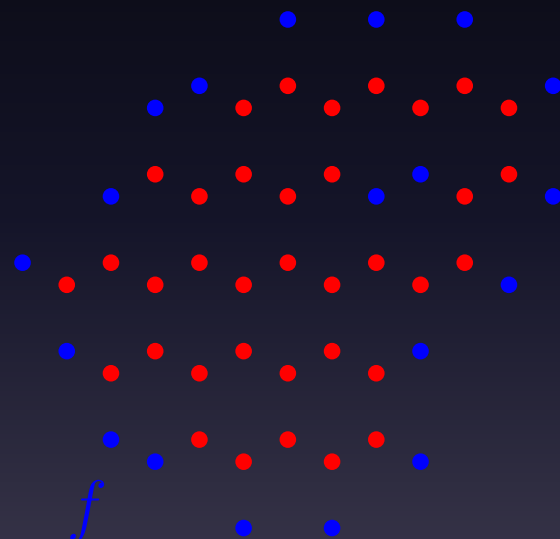
- in a graph  $G$  let  $D$  be some finite domain with a boundary

$$\partial D = \{x \notin D : \exists y \sim x, y \in D\}$$

- $f : \partial D \rightarrow \mathbb{R}$  boundary conditions
- the function

$$u(x) = \mathbb{E}[f(X_T) \mid X_0 = x]$$

is harmonic in  $D$  and coincides



# Dirichlet problem solution

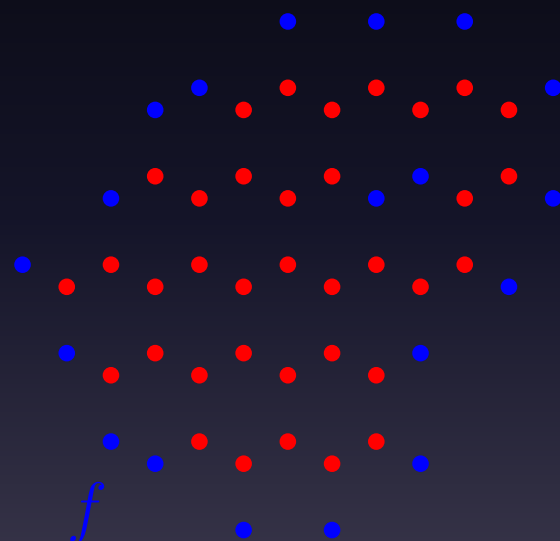
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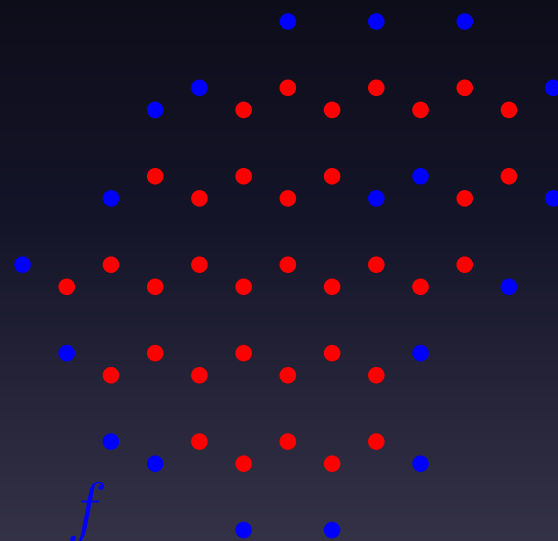
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is harmonic in  $D$  and coincides with the boundary conditions  $f$ .



# heat equation solution

- In a graph  $G$  let  $D$  be some finite domain with a boundary

$$\partial D = \{x \notin D : \exists y \sim x, y \in D\}.$$

- Let  $f : \partial D \cup D \rightarrow \mathbb{R}$  be some initial / boundary conditions.
- The function

$$u(x, t) = \mathbb{E}[f(X_{T \wedge t}) \mid X_0 = x]$$

is harmonic in  $D$  and coincides with the initial conditions  $f$ , and solves the heat equation

$$u(x, t + 1) - u(x, t) = \Delta u(x, t).$$

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# fair game

- play the following game:  
at each step, a coin is tossed  
you gain one coin, or lose one coin, each  
with probability  $\frac{1}{2}$
- let  $X_0, X_1, \dots$ , be the number of coins  
at step  $t$

question: starting with  $x$  coins, what is  
the probability to win  $N$  coins without  
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# gambler's ruin

- $f(x)$  = probability to reach  $N$  before 0 started at  $x$
- $f(0) = 0, f(N) = 1$
- $0 < x < N \Rightarrow f(x) = \frac{1}{2}f(x+1) + \frac{1}{2}f(x-1)$
- $f$  is harmonic in  $(0, N)$
- solution:  $f(x) = \frac{x}{N}$   
(unique by maximum principle)

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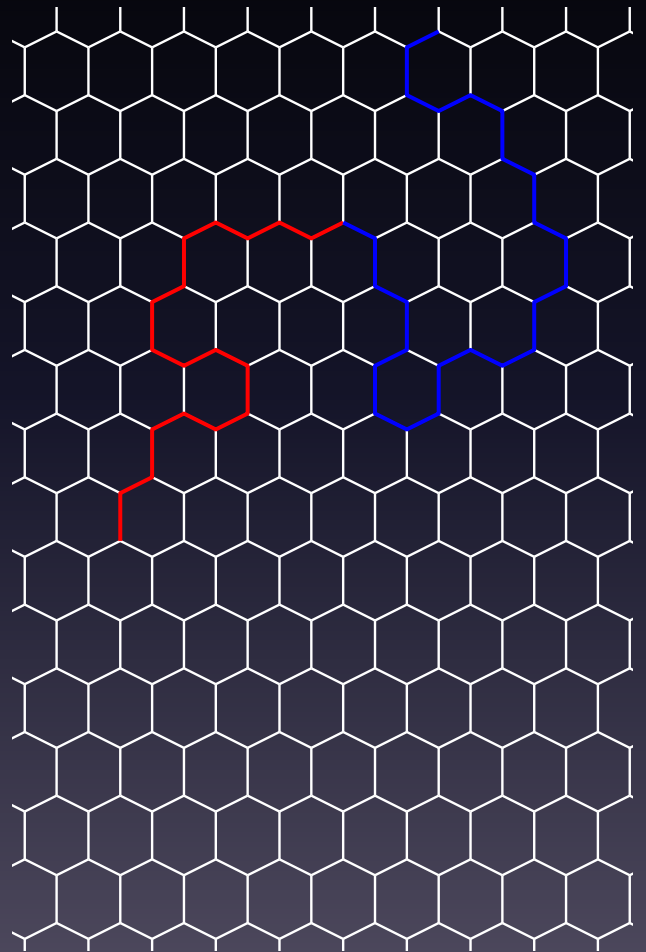


# SAW

- easy using Fekete's Lemma: the limit exists:

$$\mu = \mu(G) := \lim_{n \rightarrow \infty} |\text{SAW}_n|^{1/n}$$

- $\mu$  is called the **connective constant**



# connective constant

Example:  $\mu(\mathbb{Z}) =$



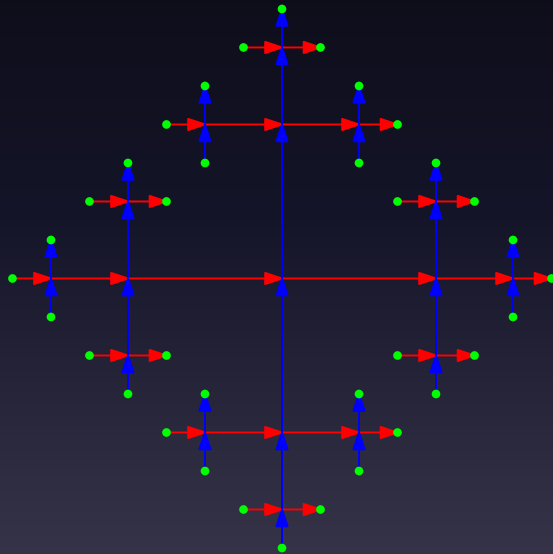
# connective constant

Example:  $\mu(\mathbb{Z}) = 1$



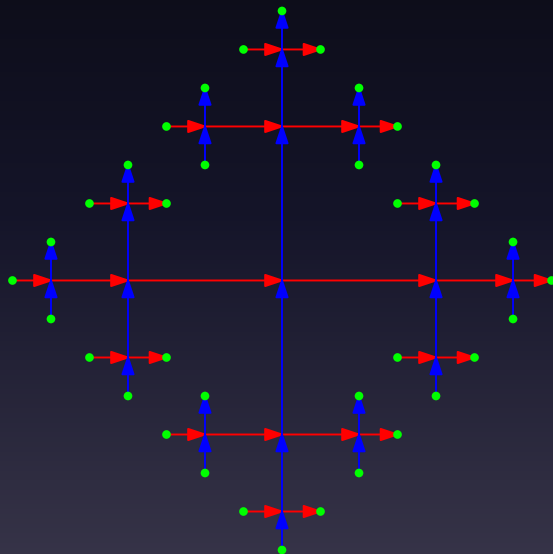
# connective constant

Example: regular tree,  $\mu(\mathbb{T}_d) =$



# connective constant

Example: regular tree,  $\mu(\mathbb{T}_d) = d - 1$



# connective constant

Example: the ladder  $\mu(L) =$



# connective constant

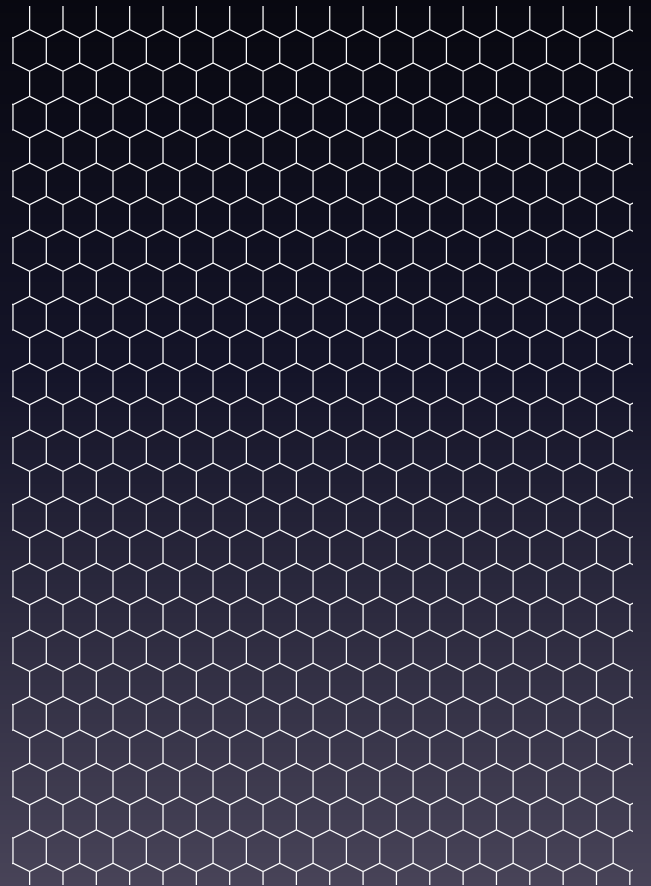
Example: the ladder  $\mu(L) = \frac{1+\sqrt{5}}{2}$



# connective constant

Theorem (Duminil-Copin &  
Smirnov 2010)

$$\mu(\mathbb{H}) =$$





# connective constant

Theorem (Duminil-Copin & Smirnov 2010)

$$\mu(\mathbb{H}) = \sqrt{2 + \sqrt{2}}$$



Hugo Duminil-Copin



Stas Smirnov

# calculating $\mu(\mathbb{H})$

- $\frac{1}{\mu}$  is the radius of convergence for the generating function

$$P(z) = \sum_{n=0}^{\infty} |\text{SAW}_n| z^n = \sum_{v \in \mathbb{H}} \sum_{\omega: o \rightarrow v} z^{|\omega|}$$

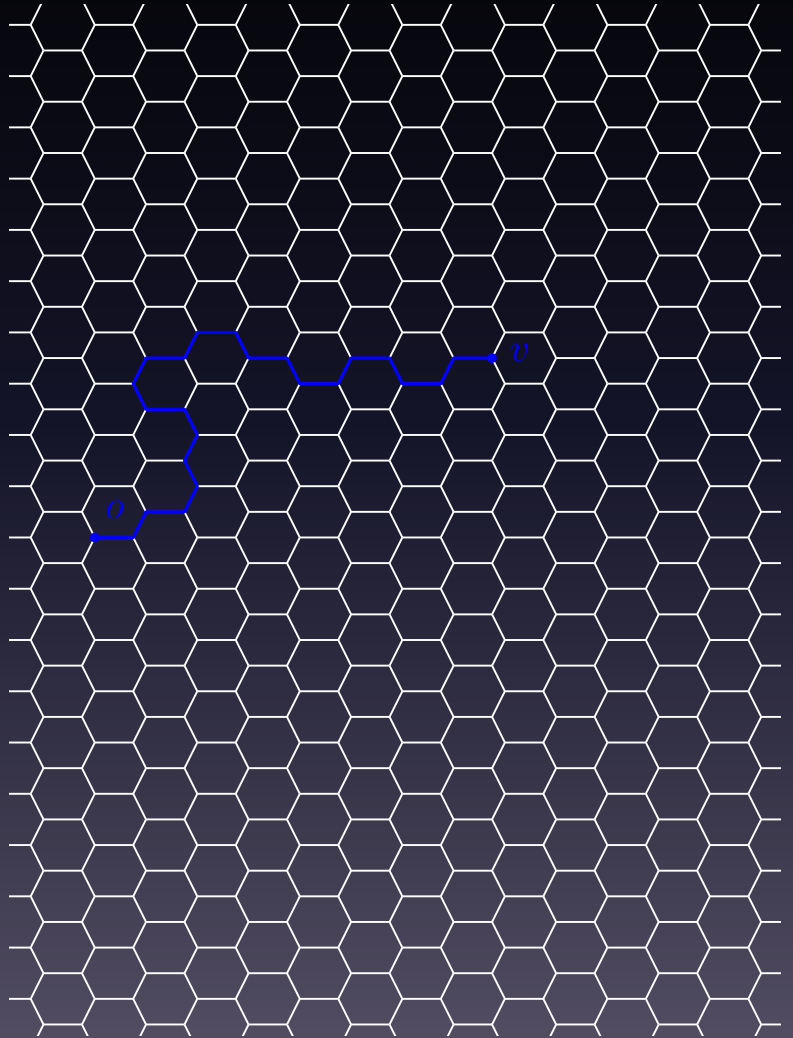
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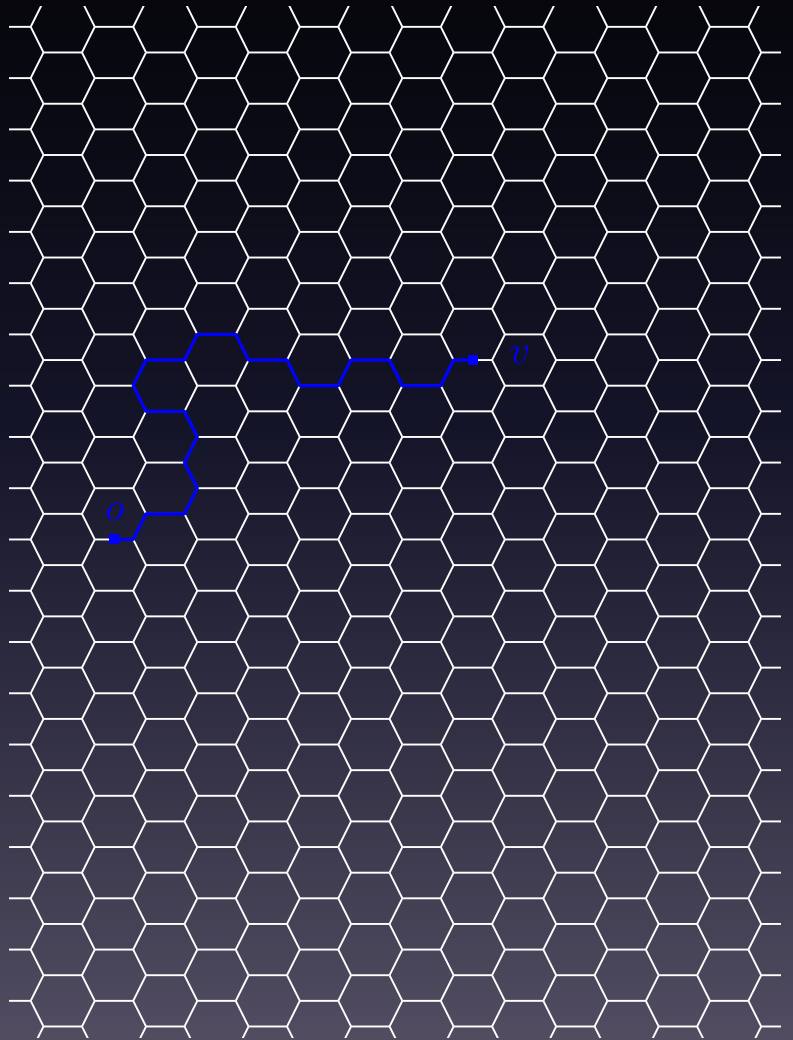
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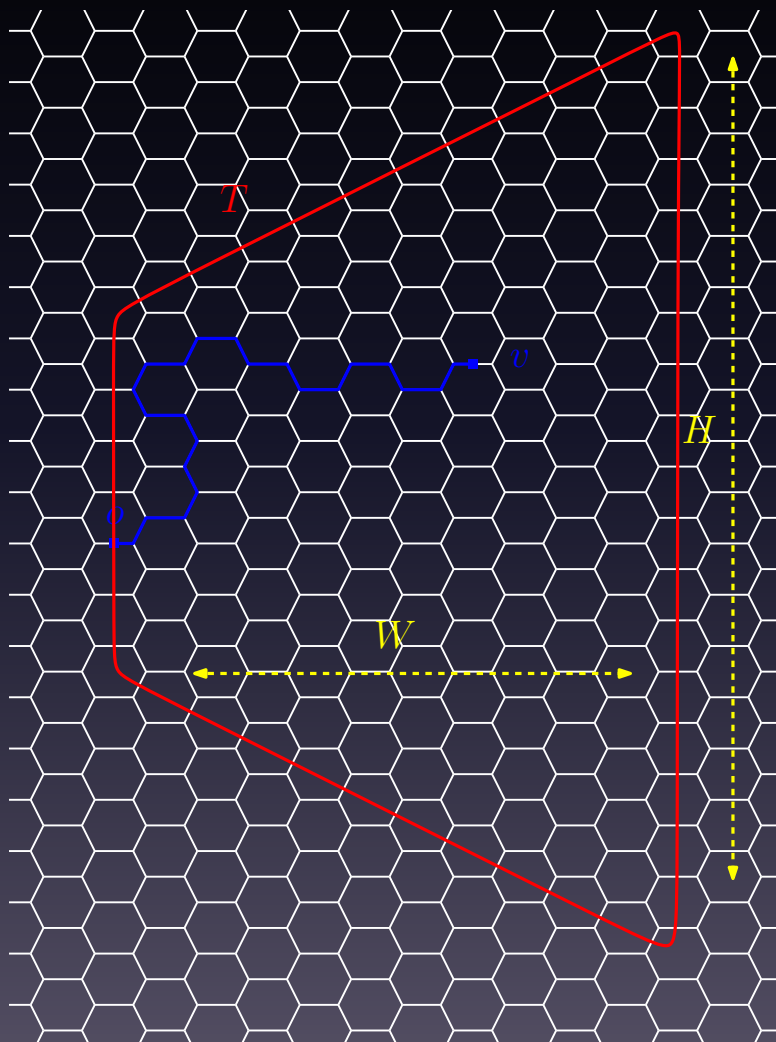
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# reductions

$$P(z) = \sum_{v \in T} \sum_{\omega: o \rightarrow v} z^{|\omega|}$$



- define a function on mid-edges in  $T$ :

$$F(p) = \sum_{\omega: o \rightarrow p} z^{|\omega|} e^{i\alpha\theta(\omega)}$$

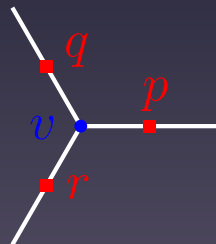
where  $\theta(\omega)$  is the winding of  $\omega$

- extend  $F$  to the vertices by averaging  $F$  on the mid edges around each vertex:

$$F(v) = (p - v)F(p) + (q - v)F(q) + (r - v)F(r)$$

where  $p, q, r$  are the mid-edges adjacent to  $v$

Question: can we find  $z, \alpha$  so that  $F$  is “holomorphic” ?  
that is, so that  $F$  is 0 on each vertex?



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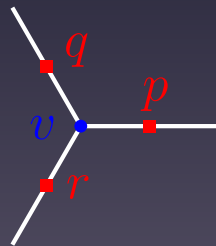
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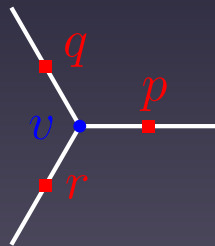
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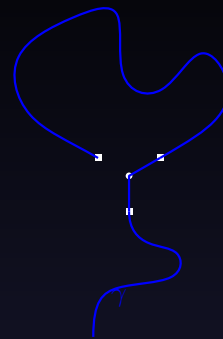
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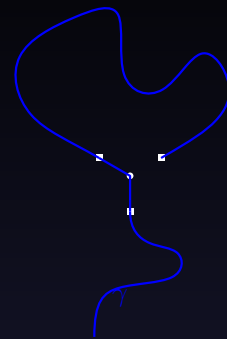


$$0 = e^{i\frac{4}{3}\pi} \cdot z^k e^{i\alpha\frac{4}{3}\pi} + e^{-i\frac{4}{3}\pi} \cdot z^k e^{-i\alpha\frac{4}{3}\pi} \\ + e^{-i\frac{4}{3}\pi} \cdot z e^{-i\alpha\frac{1}{3}\pi} + e^{i\frac{4}{3}\pi} \cdot z e^{i\alpha\frac{1}{3}\pi} + 1 \cdot 1$$



$$z^k \exp(i\alpha\frac{4}{3}\pi)$$

$$\text{end-mid} = \frac{4}{3}\pi$$



$$z^k \exp(-i\alpha\frac{4}{3}\pi)$$

$$\text{end-mid} = -\frac{4}{3}\pi$$



$$z \exp(-i\alpha\frac{1}{3}\pi)$$

$$\text{end-mid} = -\frac{1}{3}\pi$$



$$z \exp(i\alpha\frac{1}{3}\pi)$$

$$\text{end-mid} = \frac{1}{3}\pi$$



$$1$$

$$\text{end-mid} = 0$$

Equations:

$$0 = \cos\left((\alpha + 1)\frac{4}{3}\pi\right)$$
$$-1 = 2z \cos\left((\alpha + 4)\frac{1}{3}\pi\right)$$

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Solution:

$$(\alpha + 1)\frac{4}{3}\pi = \left(k + \frac{1}{2}\right)\pi$$

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$$\begin{aligned}\alpha &= \frac{6k - 5}{8} \\ z^{-1} &= 2 \cos\left(\frac{2k+1}{8}\pi\right)\end{aligned}$$

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Solution:

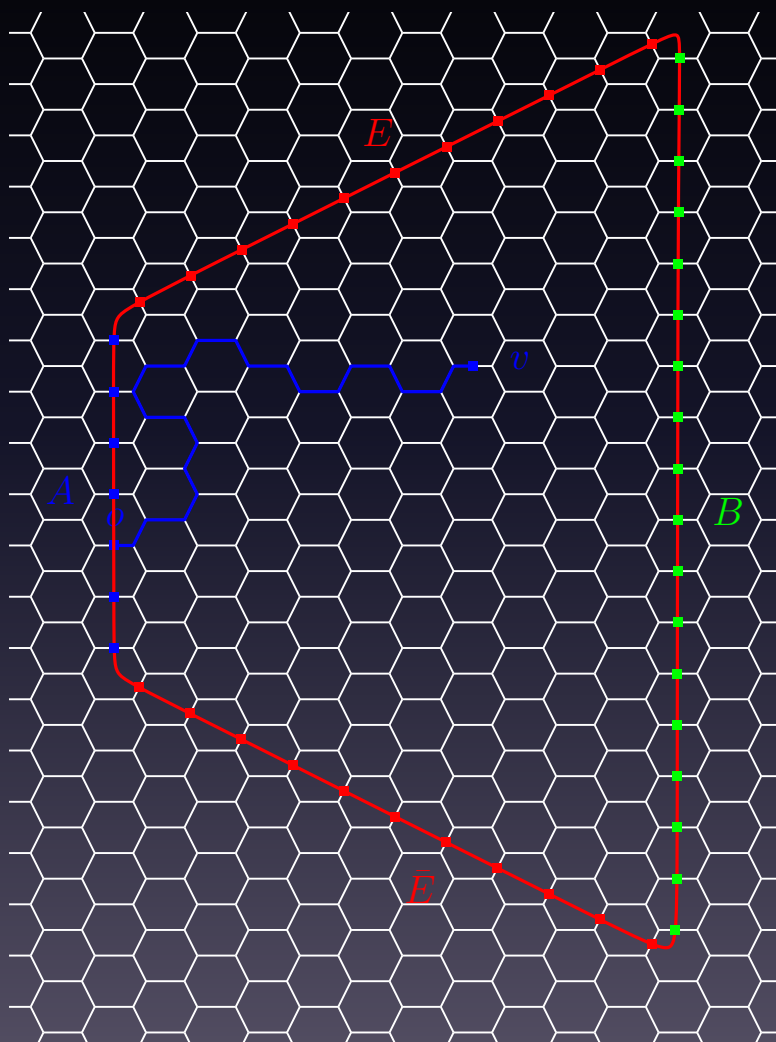
$$\begin{aligned}\alpha &= \frac{6k - 5}{8} \\ z^{-1} &= 2 \cos\left(\frac{2k+1}{8}\pi\right)\end{aligned}$$

*e.g.*  $k = 0$  and  $\alpha = -\frac{5}{8}$  and

$$z^{-1} = 2 \cos \frac{\pi}{8} = \sqrt{2 + \sqrt{2}}$$

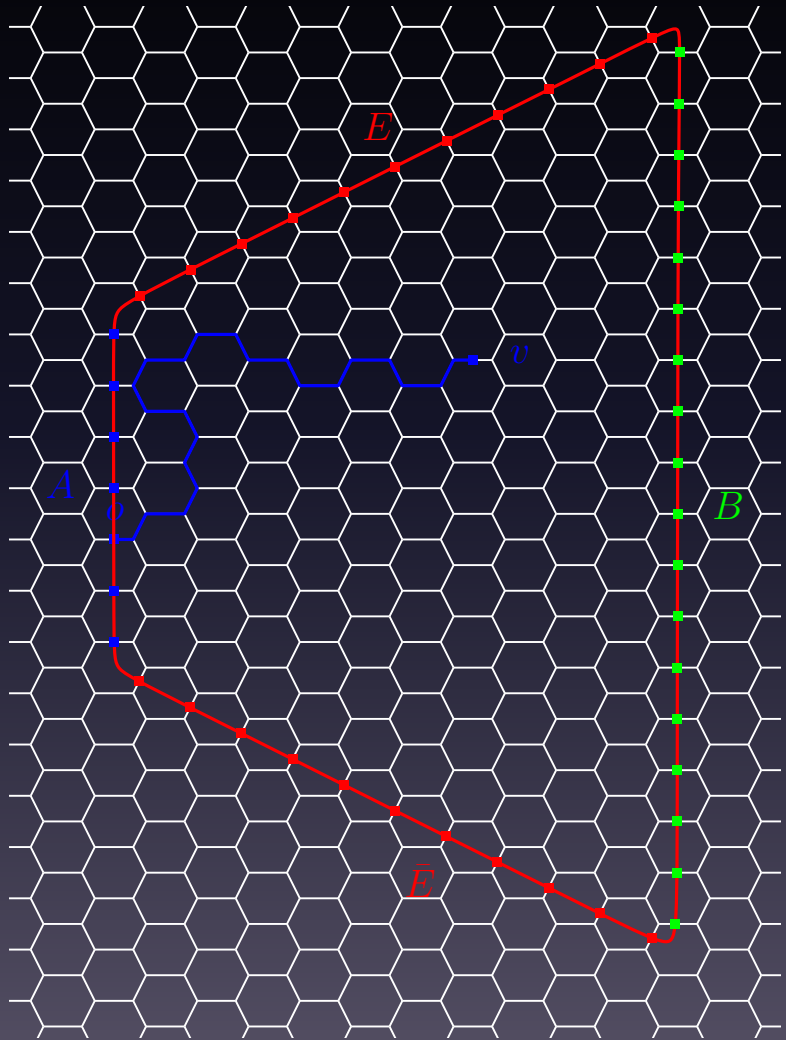
summary so far

$$P(z) = \sum_{v \in T} \sum_{\omega: o \rightarrow v} z^{|\omega|}$$



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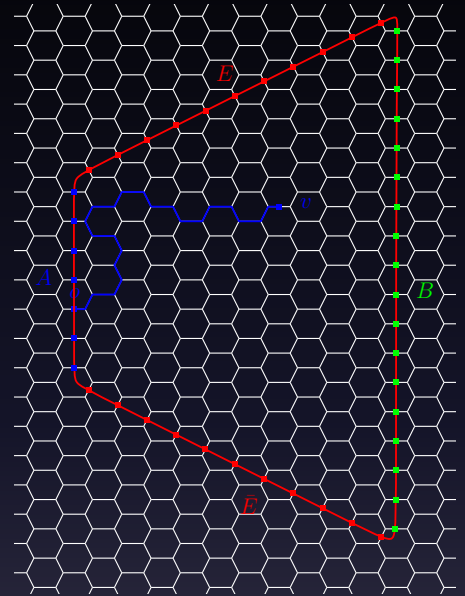




summary so far

$$P(x) = \sum_{\omega: o \rightarrow x} z^{|\omega|}$$

$$F(x) = \sum_{\omega: o \rightarrow x} z^{|\omega|} e^{i\alpha\theta(\omega)}$$



$$0 = \sum_{v \in T} F(v) = -F(A) + F(B) + \lambda F(E) + \bar{\lambda} F(\bar{E})$$

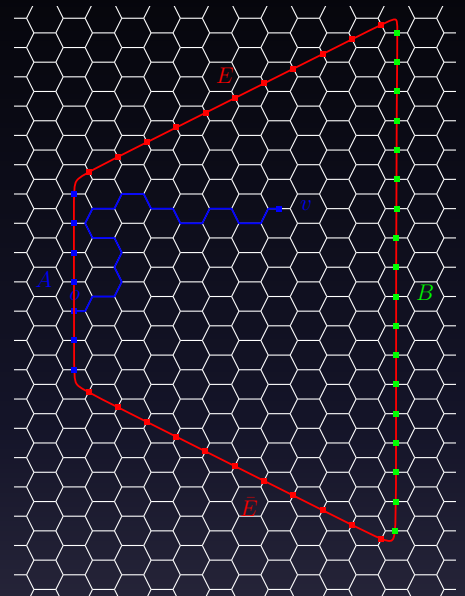
the winding is constant on  $A, B, E$  and  $\bar{E}$ !

$$0 = -1 - e^{i\alpha\pi} P(A^+) - e^{-i\alpha\pi} P(A^-) + P(B) + \lambda^2 P(E) + \lambda^{-2} P(\bar{E})$$

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(where  $\lambda = e^{i\frac{2}{3}\pi}$ )

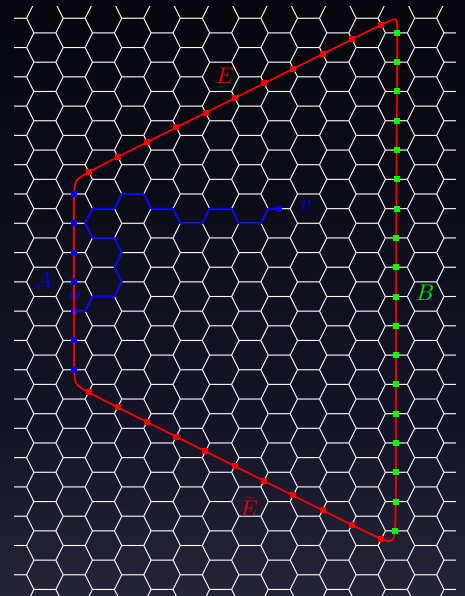
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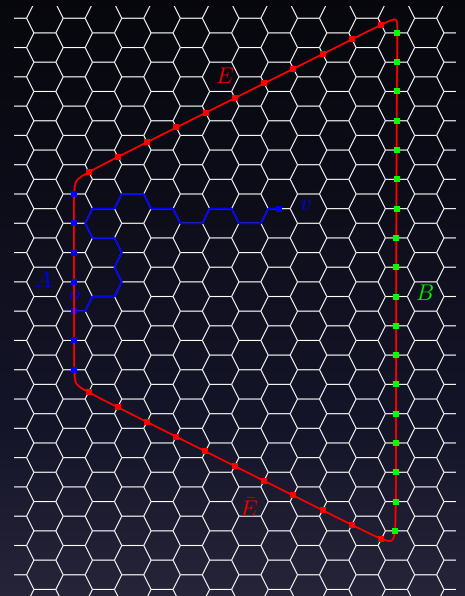
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taking  $H \rightarrow \infty$  (height)

$$1 = -\cos(\alpha\pi) \sum_{\omega:o \rightarrow A} z^{|\omega|} + \sum_{\omega:o \rightarrow B} z^{|\omega|}$$

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