Department of Mathematics, BGU

Arithmetic applications of o-minimality

On *Tuesday, May ,11 2021*

At *10:10 – 12:00*

In *online*

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will talk about

Local definable Oka

Abstract: We will discuss the proof of 2.13 and continue reading section 2

$E: X \to Y$

X, Y \ open subset complex alg varieties

E complex analytic, local iso.

S ~ collection alg. subvarieties of Y, "special". If

Z is special then

$E^{-1}(Z)$ is an alg. subvariety

$W \subseteq X$ is special iff

$E(W)$ is special.

$W \subseteq X \Rightarrow W^{alg}$ = union of alg. subvarieties of positive dim. in W.

W^{sp} = sub union cosisting of special subvarieties

E - exponential, special in

$Y = \mathbb{G}_m^n$ translate of subgroup

In $X = \mathbb{C}^n$ translate of $\mathbb{Q}$ - linear subspace.

<u>Want:</u> If $Z \subseteq Y$ algebraic then $E^{-1}(Z)^{alg} = E^{-1}(Z)^{sp}$

Ax-Lindemann-Weierstrass 1:

If $W \subseteq X$ alg subvariety not contained in a proper special subvariety then $E(W)$ is Zariski dense.

ALW1 implies the statement:

Take $a \in E^{-1}(Z)^{alg}$, $a \in W$

W positive dim alg subvariety, so

$E(W) \subseteq Z$ hence by ALW1
$E(W)$ is contained in special

subvariety. Take Y' to be the smallest one, if $Y' \not\supseteq Z$, replace Y by Y'.

Replace W by $V \in W \times Z$. So

V is an alg. subvariety of

$X \times Y$, and $V \cap \Gamma_E \supseteq U$

is an analytic subvariety.

Then $\dim(V) \geq \dim(U) + \dim(Y)$

$\dim(V) = \dim(W) + \dim(Z)$

As 1: $V \subseteq X \times Y$ alg sub, U conn. component of $V \cap \Gamma_E$, Assume $\pi_Y(V)$ is not contained in a proper special sub var. Then $\dim(V) \geq \dim(U) + \dim(Y)$

$\underline{Z} \supseteq \bigcup \{ a+H \mid a+H \subseteq X \}$

ALW2: If $W \subseteq X$ an alg. curve not contained in a proper special subvar. then $E(W)$ is dense in Y.

AS2: Let $V \subseteq X \times Y$ with infinite projection to X, and $V \cap \Gamma_E$ is infinite and $\pi_Y(\underline{V \cap \Gamma_E})$ not contained in a special subvariety then $\dim(V) \geq 1 + \dim(Y)$

ALW3: If $\gamma: D \to X$ non-constant alg map from a disc D to X is not $\subseteq$ in a special subvar. then $E \circ \gamma$ has dense image

ASP: If $\gamma : D \to \Gamma_E$ is such that $\pi_Y \circ \gamma$ is not contained in a special subvar. then $\dim(\gamma(D)^{\text{Zar}}) \geq 1 + \dim(Y)$

E is a solution to some alg. diff. equation.

This differential equation induces a diff. eq. on the possible maps γ.

$E : \mathbb{C}^n \to G_m^n$ usual exponential

$\gamma : D \to \Gamma_E$

$\gamma = (a_1, \dots, a_n, b_1, \dots, b_n)$

$b_i = E \circ a_i : D \to \mathbb{C}$

$b_i' = E' \circ a_i \cdot a_i' = \underline{E \circ a_i} \cdot a_i' = b_i \cdot a_i'$

$a_i' = \dfrac{b_i'}{b_i}$

$K, '$ differential field:

K field of char 0, $' : K \to K$

additive and $(xy)' = x'y + xy'$

$C_K = \{a \in K \mid a' = 0\}$

Theorem: (Ax) K is a differential field, $C = C_K$, $a_i, b_i \in K$ s.t.

$a_i' = \frac{b_i'}{b_i}$. If a_i are linearly independent over $\mathbb{Q}$ in K/C then

$td_C(\bar{a}, \bar{b}) \geq 1 + n$

Pf: Enough to prove: some non trivial product $B = b_1^{m_1} \cdots b_n^{m_n}$ is constant. If that is true, then

$0 = B' = m_1 b_1^{m_1 - 1} \cdot b_1' \cdots b_n^{m_n} + \cdots + m_n b_1^{m_1} \cdots b_n^{m_n - 1} \cdot b_n'$

$\Rightarrow m_1 \hat{a}_1 + \cdots + m_n \hat{a}_n = 0$

Let $M = \underline{\Omega_C K}$. Enough to show

that $dB = 0$, i.e. that $\underline{\frac{db_i}{b_i}}$ are

$\mathbb{Z}$-linearly dependent.

$K \xrightarrow{d} \Omega_C K$, $d(fg) = f\,dg + g\,df$

$f \in K$ $\quad df = d(f)$

There is a derivation

$D : M \to M$ determined by $D(d(f)) = d(f')$

$D(g\,d(f)) = g'\,d(f) + g\,d(f')$, $g, f \in K$

$D\left(\frac{df}{f}\right) = d\left(\frac{f'}{f}\right)$ for all $f \in K \Rightarrow$

$w_i = \underline{da_i} - \frac{db_i}{\underline{b_i}} \Rightarrow D(w_i) = 0$

$td \leq n \Rightarrow \dim \Omega_{\cancel{K}} \leq n \Rightarrow$

linear dependence

$\underbrace{g\,da}_{} + \sum_{i=1}^{n} f_i \underbrace{w_i}_{} = 0 \qquad a = a_1,\ a' = 1$

$D(da) = D(w_i) = 0$

Applying D, ~~get~~ $g'da + \sum f_i' w_i = 0$

$\Rightarrow g' = f_i' = 0$

Get constants c_i s.t.

$\underbrace{\sum c_i \frac{db_i}{b_i}} = \sum c_i\, da_i + g\,da = dv$

$\underline{v = g\,a + \sum c_i a_i \in K}$

$\underline{p(\bar{a}, \bar{b})}$ | $\dim_{\mathbb{Q}} \Omega_{L/K} = \text{tr.deg}(L)$

$dp =$

Prop: Suppose $b_i, v \in K$, $C \subseteq K$ (alg. closed) $\sum c_i \frac{db_i}{b_i} = dv$ in $M = \Omega_C K$. Then the $\frac{db_i}{b_i}$ are $\mathbb{Z}$-linearly dependent.

pf: Show that if c_i are indep. over $\mathbb{Q}$ then $db_i = 0$ for all i.

May assume $K = C(v, b_1, \dots, b_n)$

By induction, may assume that K is tr. deg. 1 over C, $K = C(X)$ on a smooth projective curve X. for every $x \in X$, there is a map $res_x : \Omega K \to C$ s.t. $res_x(dg) = 0$ for all g and $res_x\left(\frac{dg}{g}\right) = v_x(g)$.

$dv = \sum c_i \frac{db_i}{b_i}$ $\quad 0 = \sum_i c_i v_x(b_i) \Rightarrow v_x(b_i) = 0$

$\Rightarrow b_i \in C \Rightarrow db_i = 0.$

$E : X \to Y$

$E(Z)$ dense in Y

$Z \times Y \supseteq \Gamma_E$ dense

$\mathbb{H}$ - upper half plane

$G = PSL_2(\mathbb{R}) \circlearrowright \mathbb{H}$

$\cup |$

Γ

$\mathbb{H}/\Gamma$ - curve of genus 0

$j_\Gamma : H \to H/\Gamma$

$\Gamma = PSL_2(\mathbb{Z})$

$\Gamma_1 \sim \Gamma_2$, Γ_1, Γ_2 commensurable if

$\Gamma_1 \cap \Gamma_2$ is finite index in each

$\mathrm{comm}(\Gamma) = \{ g \in G \mid g^{-1} \Gamma g \sim \Gamma \}$

$\Gamma = PSL_2(\mathbb{Z}) \Rightarrow \mathrm{comm}(\Gamma) = \underset{g}{\overset{\in}{PSL_2}}(\mathbb{Q})$

A subset of $\mathbb{H}^2$ is special

if it is the graph of $g \in \mathrm{comm}(\Gamma)$

or fibre of a projection.

$\{ (\tau, g\tau) \mid \tau \in H \}$

τ $\quad g\tau$ $\quad$ geodesically dep.

<u>Thm</u> $W \subseteq \mathbb{H}^n$ alg. subvariety not contained in a special one, then the graph of

$$j_\Gamma^{\,n} \times j_\Gamma'^{\,n} \times j_\Gamma''^{\,n} \text{ on } W \text{ is}$$

dense in $W \times \mathbb{A}^{3n}$

<u>Outline</u>

There are differential algebraic polynomials χ_Γ and S (ind. of Γ) s.t.

1. For any mero. function a, $\chi_\Gamma(j_\Gamma \circ a) = S(a)$ for any vector field ∂ on the domain

2. $\mathcal{X}_\Gamma(y) = b \qquad \mathcal{X}_\Gamma(z) = c$

$b, c \in K$ differential field

if b, c are distinct and

$\dot{j}_b$, $\dot{j}_c$ solutions in an extension field not alg.

over K, $\underline{\dot{j}_b, \dot{j}_b', \dot{j}_b''}, \underline{\dot{j}_c, \dot{j}_c', \dot{j}_c''}$

are alg. independent over K.

3. If $\dot{j}_1, \dots, \dot{j}_n$ are solution of $\mathcal{X}_\Gamma(y) = b$. denote

$\bar{\dot{j}}_i = (\dot{j}_1, \dot{j}_1', \dot{j}_1'')$. Suppose

$\bar{\dot{j}}_1, \dots, \bar{\dot{j}}_n$ are dependent

Then for some i, k, $P(\dot{j}_i, \dot{j}_k) = 0$

for a Γ-special polynomial.

<u>Def:</u> A differential field K is differentially closed if any diff. alg. variety X over K with $X(L) \neq \emptyset$ for $L \supseteq K$, $X(K) \neq \emptyset$.

for every diff. field K

<u>Fact:</u> There is a first order theory DCF_K whose models are the differentially closed fields containing K.

DCF_K admits quantifier elimination, any diff. field extension L of K can be embedded over K in a model.

Def: A def. set X is <u>strongly minimal</u> if it is infinite but every def. subset is finite or co-finite (def with additional parameters).

- The definable set C given by $x' = 0$ is strongly minimal.

- Manin kernels: A simple abelian variety, $A^{\#}$ ~ smallest Zariski definable subgroup is strongly minimal set.

<u>Zil'ber trichotomy:</u> If X is strongly minimal and orthogonal to one of the above, then

it is geometrically trivial:

If $a_1, \ldots, a_n \in X(\mathcal{U})$ are

diff. alg. dependent then
some pair a_i, a_u are diff.
alg. dependent.

Ex: G alg. group (df), g_1, g_2

alg. ind. $g_1 \cdot g_2$ ind. of

g_1 and g_2

X, Y are non orthogonal if

there is an infinite $Z \subseteq X \times Y$
with fibres under the projections.
(Z might be defined over more
parameters).

Theorem 1. The equation

$\chi_f(y)=a$ is strongly minimal and geometrically trivial.

Theorem 2: If $a \neq b$.

then $\chi(y)=a$, $\chi(y)=b$

are orthogonal.

Schwarzian derivative

$$S(y)=\left(\frac{y''}{y'}\right)'-\frac{1}{2}\left(\frac{y''}{y'}\right)^2$$

$g \in PSL_2(\mathbb{R})$ G/H

$S(g)=0$, $S(g\circ f)=g\circ S(f)$

D fundamental domain for

the action of Γ, polygon

with finite number of vertices

$b_1, \dots, b_r$, with angles $\alpha_i \pi$

$$R_{j_\Gamma}(y) = \frac{1}{2}\sum_{i=1}^{r} \frac{1-\alpha_i^2}{(y-a_i)^2} + \sum_{i=1}^{r} \frac{A_i}{y-a_i}$$

$a_i = j(b_i)$

$$x_\Gamma(y) = S(y) + (y')^2 R_{j_\Gamma}(y)$$

$$X: S(y) + (y')^2 R(y) = 0$$

$R(y) \in \mathbb{C}(y)$

$$\frac{d^2u}{dy^2} + u^2 + \frac{1}{2}R(y) = 0$$

Thm If the Riccati equation does not have alg. solutions then X is strongly minimal.

$$\frac{d^2 z}{dy^2} = R(y) z \qquad E$$

v.s. of dim 2 over $\mathbb{C}$ V

$u = \frac{z'}{z}$

H Galois group

Riccati equation has alg. solution

$\Updownarrow$

Galois of linear equation is a proper subgp of $SL_2(V)$

~~$\forall$~~ X ~ ~~St~~rongly minimal order > 1, defined over the constants.

Claim: ~~A~~ny such X is geom. trivial.

Thm. 2.13 $X \subseteq \mathbb{C}^n$ def. open, then $\mathcal{O}_X$ is coherent

pf: $f_1, \dots, f_n \in \mathcal{O}_X(X)$, show

$I(\bar{f}) = \{\bar{g} \mid \bar{g} \cdot \bar{f} = 0\}$ loc. fin. gen

i.e. may pass to a cover by finitely many def. open subsets $U \subseteq X$, and find finitely many tuples $\bar{g}_1, \dots, \bar{g}_m \in I(\bar{f})(U)$ which gen. the relations on each def open subset V of U

In $I(f)(X)$ some standard elements: $\underbrace{(-f_i, 0..0, \underset{\uparrow i}{f_1}, ..0)}$ R_i

<u>Case 1:</u> Let U be the set where f_1 is invertible.

On U, the relations R_i generate: they are equivalent to $(-\frac{f_i}{f_1}, 0..0, \underset{g_i}{1}, 0..0)$

$(g_1, \dots, g_n) = \sum \frac{g_i}{f_1} \cdot R_i$, U is one of the elements in the cover.

$Y = X \setminus U$

Case 2: $f_1 = t^k + a_{k-1}(u)t^{k-1} \cdots + a_0(u)$

Let $U_0 \subseteq \mathbb{C}^{n-1}$ open definable.

Then by 2.11, may write each

$f_i = u_i \cdot f_1 + r_i$ where

r_i a poly. of deg $< k$ in

a nbhd of Y. If

$0 = \sum g_i f_i = \sum \left(\sum g_i u_i \right) f_1 + \sum_{i>1} g_i r_i$

So may assume all f_i are poly.

Use the same to write

$g_i = V_i f_1 + s_i \quad \deg(s_i) < k$

$0 = \sum g_i f_i = \quad f_1 + \dots + V_i f_1 f_i + s_i f_i =$

$-V_i f_i f_1$

$s_0 f_1 + s_2 f_2 \dots + s_n f_n$

$s_0 f_1 = -s_2 f_2 - \dots - s_n f_n$

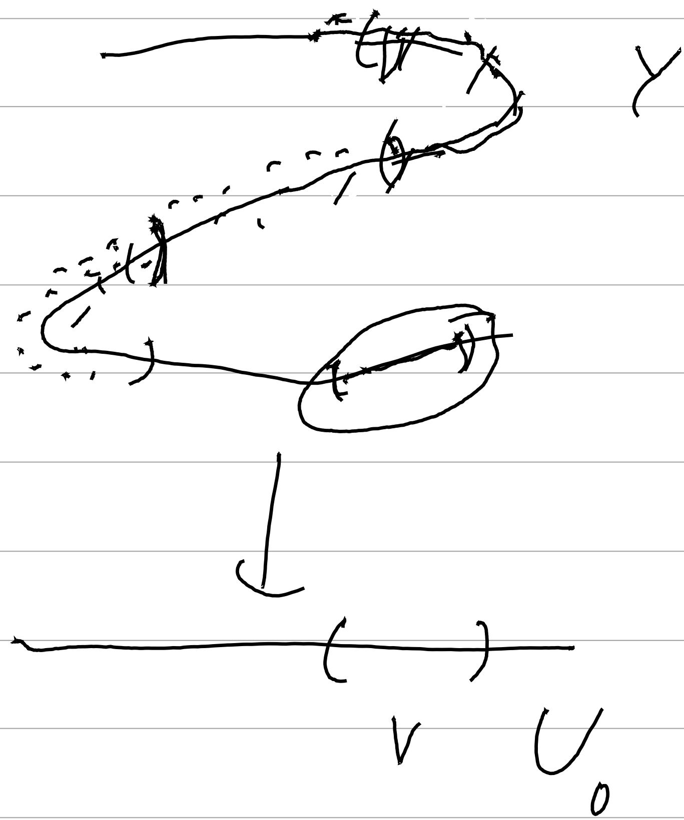
Think of the relation as elements of $\left(\mathcal{O}/_{f}M\right)^n = M$

If $\pi : Y \to U_0$ the projection $\pi_*(M)$ is finite generated over U_0. By induction there is a cover of U_0 for which the module of relation is f.g. over U_0. Take the inverse image of this cover on X.

<u>General case:</u> Can find an open cover and on open set in it a projection to $\mathbb{C}^{n-1}$ which is proper on Y, in particular with finite fibres. (lower col)

We assume that the ~~sheaf~~ sheaf of relations consists of poly. of deg $\leq k$. As a module over $\mathcal{O}_{U_0}$ it has finitely many generators, we take them $\underline{\bar{g}_1}, \dots, \underline{\bar{g}_\ell}$

I.e., if $\underline{h} \cdot \underline{\bar{f}} = 0$ on an open subset of the form $\pi^{-1}(V)$, $V \subseteq U_0$ then $h = \sum b_i \bar{g}_i|_{\pi^{-1}(V)}$



Components of sets of the form

$\pi^{-1}(V)$

<u>Lemma 2.14:</u> $f: X \to Y$ cont. X, Y dte spaces. finite deg, X separated (Hausdorff)

Given a cover $\underline{X}$ of X

Then can find a refinement $\underline{W}$ of $\underline{X}$ and a cover $\underline{Y}$ of Y s.t.

1. for each $Y_0 \in \underline{Y}$, $f^{-1}(Y_0)$ is a disjoint union of elements of $\underline{W}$

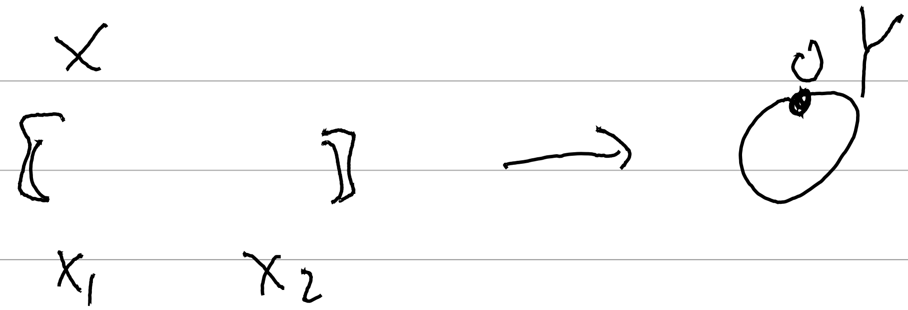
2. Every element of $\underline{W}$ is a component of $f^{-1}(Y_0)$ for some $Y_0 \in \underline{Y}$.

<u>pf:</u> Take a cell decomposition $\underline{D}$ of X, $\underline{C}$ of Y s.t. if $D \in \underline{D}$ and $X_0 \in \underline{X}$ then

1. either $D \subseteq X_0$ or ~~is~~ $D \cap X_0 = \phi$

and 2. $f^{-1}(C)$ is a disjoint union of elements of $\underline{D}$ for each $C \in \underline{C}$

3. $f|_D$ is a bijection onto a cell.



$\{\{0\}, \{1\}, (0,1)\}$ $\quad\quad$ $\{0\}, S^1 \setminus \{0\}$

For every $D \in \underline{D}$, $\overline{D}$ set of cells whose closure contains D.

and $X(D) = \bigcup \tilde{D}$ (similarly ~~for~~ for Y).

$\underline{W} = \{ X(D) \mid D \in \underline{D} \}$

$\underline{Y} = \{ Y(C) \mid C \in \underline{C} \}$

<u>Claim 1:</u> <u>$\underline{W}, \underline{Y}$ covers, $\underline{W}$ refines $\underline{X}$]</u>

<u>pf:</u> Clearly every element of $\underline{W}, \underline{Y}$ is def., $\bigcup \underline{W} = X$, $\bigcup \underline{Y} = Y$. Since each $D \in \underline{D}$ is either contained or disjoint from all elements of $\underline{X}$, all $X(D) \subseteq X_0$

for which $D \subseteq X_0$.

$X(D)$ open: take $x \in X(D)$

take U open nbhd of x

that intersects a mini # of

cells. Then $U \subseteq X(D)$ since

otherwise take $y \in U \setminus X(D)$

then y belongs to some cell

E, the closure $\overset{Z}{\text{of}}$ E is

disjoint ~~from~~ D, then

$U \setminus Z$ is an

open nbhd of x that
inters

(assume f open)

(a normality)

Claim 2: If $D, D' \in \underline{D}$ map to the same element $C \in \underline{\mathcal{D}}$ then $X(D)$, $X(D')$ disjoint.

Pf: Take $U \supseteq D$, $U' \supseteq D'$ open and disjoint. So $f(U)$, $f(U')$ nbhds of C. If $E \in \widetilde{D}, \widetilde{D'}$, $E \cap U \neq \emptyset$, $E \cap U' \neq \emptyset$, $f|_E$ is not a bijection.

Claim 3: $f^{-1}(Y(C))$ is a disjoint union of $X(D)$ for D s.t. $f(D) = C$.

Pf: need to show: if $C' \in \tilde{C}$, D' mapping to it ($f(D') = C'$), then there is D in its closure mapping to C.

LEMMA 2.14

MOSHE KAMENSKY

cover means definable open cover. If $\mathcal{C}$ is a cell decomposition of a space X, we denote the unique cell containing a point $x \in X$ by C_x. For a subset A of X, we let $\mathcal{C}(A)$ be the set of cells in $\mathcal{C}$ whose closure intersects A.

Claim 0.0.1. *For any $A \subseteq X$, $X(A) = \bigcup \mathcal{C}(A)$ is open*

Proof. Let $x \in X(A)$ and let U be an open neighbourhood of x intersecting a minimum number of cells. We claim that $U \subseteq X(A)$. Otherwise, let $y \in U \setminus X(A)$. Then the closure Z of D_y is disjoint from A. If $x \in Z$ then $D_x \subseteq Z$, contradicting that $D_x \in \mathcal{C}(A)$. Hence $U \setminus Z$ is also an open neighbourhood of x, intersecting less cells (since it does not intersect D_y), a contradiction. □

Lemma 0.0.2 (=Lemma 2.14)**.** *Assume $f : X \to Y$ is a finite map of definable spaces with X separable, $\mathcal{X}$ a cover of X. Then there is a refinement $\mathcal{W}$ of $\mathcal{X}$ and a cover $\mathcal{Y}$ of Y, such that*

(1) *For each $Y \in \mathcal{Y}$, $f^{-1}(Y)$ is a (finite) disjoint union of members of $\mathcal{W}$*
(2) *Each member of $\mathcal{W}$ is a component of $f^{-1}(Y)$ for some $Y \in \mathcal{Y}$*

Proof. We claim that there are cell decompositions $\mathcal{D}$ of X and $\mathcal{C}$ of Y such that:

(1) Each $D \in \mathcal{D}$ is either contained or is disjoint from each $X' \in \mathcal{X}$, and similarly for $\mathcal{C}$ and $f(X')$
(2) For each $D \in \mathcal{D}$, $f \restriction_D$ is a bijection with some $C \in \mathcal{C}$
(3) $f^{-1}(C)$ is a disjoint union of some elements of $\mathcal{D}$, for each $C \in \mathcal{C}$

This is achieved by applying Lemmas 2.7 and 2.8 of the paper, to the boolean algebra of subsets generated by $\mathcal{X}$ and its image, respectively. We say that $D \in \mathcal{D}$ is over $C \in \mathcal{C}$ if $f(D) = C$.

We next construct refined cell decompositions $\mathcal{D}_1$ and $\mathcal{C}_1$, as follows: if $D \neq D'$ are over the same cell in $\mathcal{C}$, then each is disjoint from the closure of the other (for example, by dimension). By separation[1], we may choose disjoint open sets $U \supseteq D$ and $U' \supseteq D'$. We do this, and add all such open sets to the cover $\mathcal{X}$. $\mathcal{D}_1$ and $\mathcal{C}_1$ are now cell decompositions satisfying the above conditions with respect to this new cover.

[1] It is not clear to me what separation axiom we need exactly. According to Wikipedia, what we are using is called "completely normal". It follows from regularity under the second-countable assumption, which we have. OTOH, we actually need a definable version, so I don't know if the implication holds...

Now let $X(D) = \bigcup \mathcal{D}_1(D)$, $Y(C) = \bigcup \mathcal{C}_1(C)$ and $\mathcal{W} = \{X(D)|D \in \mathcal{D}\}$ and $\mathcal{Y} = \{Y(C)|C \in \mathcal{C}\}$. The following claims show they satisfy the conditions.

Claim 0.0.3. *$\mathcal{W}$ and $\mathcal{Y}$ are covers, and $\mathcal{W}$ refines $\mathcal{X}$*

Proof. By claim 0.0.1, each of $\mathcal{W}$ and $\mathcal{Y}$ consists of open subsets. For each $D \in \mathcal{D}$ we have $D \subseteq X(D)$, since each D is a union of cells in $\mathcal{D}_1$. Hence $\bigcup \mathcal{W} = X$, so it is a cover (and similarly for $\mathcal{Y}$).

If $U \in \mathcal{X}$ and $D \in \mathcal{D}$ is a subset of U, then each element of $\mathcal{D}_1(D)$ is also contained in U, since otherwise it would be disjoint from it, hence its closure could not intersect D. Hence $X(D) \subseteq U$, so $\mathcal{W}$ refines $\mathcal{X}$. □

The following claims show that both conditions hold

Claim 0.0.4. *If $D, D' \in \mathcal{D}$ are distinct and map to the same cell in $\mathcal{C}$, then $X(D)$ and $X(D')$ are disjoint.*

Proof. This follows directly from the construction of $\mathcal{D}_1$: if $X(D)$ and $X(D')$ are not disjoint, neither are $\mathcal{D}_1(D)$ and $\mathcal{D}_1(D')$. Thus, there is $E \in \mathcal{D}_1$ whose closure intersects both D and D', contradicting the choice of U and U' above. □

Claim 0.0.5. *For every $C \in \mathcal{C}$, $f^{-1}(Y(C))$ is the disjoint union of $X(D)$ for $D \in \mathcal{D}$ mapping to C*

Proof. The union is disjoint by the previous claim, and is definitely contained in the pre-image. Hence it suffices to show that if $C' \in \mathcal{C}_1(C)$ for some $C \in \mathcal{C}$ and $D' \in \mathcal{D}_1$ is over C', then $D' \in \mathcal{D}_1(D)$ for some $D \in \mathcal{D}$ over C (so that $D' \subseteq X(D)$).

Since $C' \in \mathcal{C}_1(C)$, there is a curve γ inside C' whose limit is in C. Then γ lifts to a curve $\tilde{\gamma}$ in D', and by properness this curve has a limit x in X. We take $D = D_x$. □

□

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