Piercing edges with subsets in geometric hypergraphs

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Let P be a finite (nonempty) subset of $\mathbf{R}^n$ and $\epsilon > 0$ a fixed small parameter. Also let $\mathcal{R} \subset \mathcal{P}(\mathbf{R}^n)$ be a set of 'simple' geometric regions (for example, all halfspaces). Every region $H \in \mathcal{R}$ defines a hyperedge $H \cap P$ in the geometric hypergraph induced by $\mathcal{R}$ on vertex set P, but we consider only those 'heavy' H for which $|H \cap P| \geq \epsilon |P|$.

A subset $N \subset P$ pierces each such 'heavy' hyperedge if

$$\forall H \in \mathcal{R}, \qquad |H \cap P| \geq \epsilon |P| \implies P \cap N \neq \emptyset.$$

This N can have as few as

$$O\left(\frac{1}{\epsilon} \log \frac{1}{\epsilon}\right) \tag{$\star$}$$

points in total, by the seminal epsilon-net theorem of Haussler and Welzl ('87).

But what if we require that each H include not just a singleton, but a much larger subset? That is, take a collection $\mathcal{M} \subset \mathcal{P}(P)$ such that:

- $\min_{M \in \mathcal{M}} |M| \geq \lambda \epsilon |P|$ for some absolute constant $\lambda > 0$,
- every $H \in \mathcal{R}$ such that $|H \cap P| \geq \epsilon |P|$ includes some $M \in \mathcal{M}$.

Under some reasonable conditions on $\mathcal{R}$ we can upper-bound the cardinality of the smallest such $\mathcal{M}$, mirroring ($\star$). The proof involves packings in the Hamming cube and polynomial partitioning.

Roadmap

- Geometric hypergraphs: VC dimension and complexity
- Packing lemmas for hyperedges
- Polynomial partitioning
- Putting it all together: an Mnet theorem

Joint work with Kunal Dutta, Arijit Ghosh & Nabil H. Mustafa.