On bounded continuous solutions of the archetypal equation with rescaling

Gregory Derfel

We'll start from a brief general introduction in equations with rescaling, that does not require any prerequisites.

Then we turn to the problem indicated in the title. Namely, we study the "archetypal functional equation $y(x) = \int_{R^2} y(a(x-b))\,\mu(da, db)$ $(x \in R)$, equivalently, $y(x) = E\{y(\alpha(x-\beta))\}$, where E is expectation with respect to the distribution μ of random coefficients (α, β).

Particular cases include: (i) $y(x) = \sum_i p_i y(a_i(x-b_i))$ and (ii) $y'(x)+y(x) = \sum_i p_i y(a_i(x-b_i))$ (pantograph equation), both subject to the balance condition $\sum_i p_i = 1$ $(p_i > 0)$.

Existence of non-trivial (i.e. non-constant) bounded continuous solutions is governed by the value $K := \int_{R^2} \ln|a|\,\mu(da, db) = E\{\ln|\alpha|\}$; namely, under mild technical conditions no such solutions exist whenever $K < 0$, whereas if $K > 0$ (and $\alpha > 0$) then there is a non-trivial solution

In the critical case $K = 0$, we prove a Liouville theorem subject to the uniform continuity of $y(\cdot)$. The latter is guaranteed under a mild regularity assumption on the density of β conditioned on α, which is satisfied for a large class of examples including the pantograph equation (ii).

Further results are obtained in the supercritical case $K > 0$, including existence, uniqueness and a maximum principle. The case with $P(\alpha < 0) > 0$ is drastically different from that with $\alpha > 0$; in particular, we prove that a bounded solution $y(\cdot)$ possessing limits at $\pm\infty$ must be constant.

The proofs employ martingale techniques applied to the martingale $y(X_n)$, where (X_n) is an associated Markov chain with jumps of the form $x \to \alpha(x - \beta)$.