Joint and double coboundaries of commuting transformations – an application of operator theory to a problem in ergodic theory

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Abstract

Let T be a bounded linear operator on a Banach space X; the elements of $(I-T)X$ are called *coboundaries*. For two commuting operators T and S, elements of $(I-T)X \cap (I-S)X$ are called *joint coboundaries*, and those of $(I-T)(I-S)X$ are called *double coboundaries*. By commutativity, double coboundaries are joint ones. Are there any other?

Let θ and τ be commuting invertible measure preserving transformations of a standard probability space (Ω, Σ, m), with corresponding unitary operators U and V induced on $L_2(m)$. When θ and τ generate an aperiodic action of $\mathbb{Z}^2$ (i.e. $m\{s \in \Omega : \theta^j \tau^k s = s\} = 0$ when $(j,k) \neq (0,0)$), we prove the existence of a joint coboundary $g \in (I-U)L_2 \cap (I-V)L_2$ which is not in $(I-U)(I-V)L_2$.

For the proof, we use several elements from operator theory. Let E be the spectral measure on $\mathbb{T}^2$, obtained by Stone's general spectral theorem. Joint and double coboundaries are characterized using their scalar spectral measures. The identification of the support of E (due to Hastings) in terms of the joint spectrum, and properties of the maximal spectral type of E are used to show the existence of the required function g. The connection with ergodic theory is via a 2-dimensional Rokhlin lemma (due to Conze and to Katznelson-Weiss), which is used to identify the joint spectrum.