Some Recent Progress on the Diophantine Geometry of Curves

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Jerusalem-Beersheba, October, 2020

The main problem

X smooth projective curve over a number field F of genus $g \geq 2$.

Effective Mordell problem:

$$\boxed{\textbf{Find a terminating algorithm: } X \mapsto X(F)}$$

The **effective Mordell conjecture** (Szpiro, Vojta, ABC, ...) makes this precise using height inequalities:

$$h(x) \leq C(X, F)$$

for all $x \in X(F)$ and some (more or less) specific C.

The non-abelian method of Chabauty is concerned with non-Archimedean analogues using moduli of principal bundles and non-abelian Hodge theory.

Principal bundles in Diophantine geometry: a little history

Weil in 1929 constructed an embedding

$$j : X \hookrightarrow J_X,$$

where J_X is an abelian variety of dimension g.

That is, over $\mathbb{C}$,

$$J_X(\mathbb{C}) = \mathbb{C}^g/\Lambda = H^0(X(\mathbb{C}), \Omega^1_{X(\mathbb{C}))})^*/H_1(X, \mathbb{Z}).$$

The map j is defined over $\mathbb{C}$ by fixing a basepoint b and

$$j(x)(\alpha) = \int_b^x \alpha \mod H_1(X, \mathbb{Z}),$$

for $\alpha \in H^0(X(\mathbb{C}), \Omega^1_{X(\mathbb{C})})$.

Principal bundles in Diophantine geometry: a little history

But Weil's point was that J_X is also a projective algebraic variety defined over F, and if $b \in X(F)$, then the map j is also defined over F.

The reason is that J_X is a moduli space of line bundles of degree 0 on X and

$$j(x) = \mathcal{O}(x) \otimes \mathcal{O}(-b).$$

The main application is that

$$j : X(F) \hookrightarrow J(F).$$

Weil also proved that $J(F)$ is a finitely-generated abelian group, and hoped, without success, that this could be somehow used to control $X(F)$.

Principal bundles in Diophantine geometry: a little history

In the 1938 paper 'Généralisation des fonctions abéliennes', Weil studied

$$Bun_X(GL_n) = GL_n(K(X))\backslash GL_n(\mathbb{A}_{K(X)})/[\prod_x GL_n(\widehat{\mathcal{O}_x})]$$

as a 'non-abelian Jacobian'.

Proved a number of foundational theorems, including the fact that vector bundles of degree zero admit flat connections, beginning non-abelian Hodge theory.

Principal bundles in Diophantine geometry: a little history

This paper was very influential in geometry, leading to the paper of Narsimhan and Seshadri:

$$Bun_X(GL_n)_0^{st} \simeq H^1(X, U(n))^{irr}.$$

This was extended by Donaldson, influencing this work on smooth manifolds and gauge theory, and by Simpson to

$$Higgs(GL_n) \simeq H^1(X, GL_n).$$

Serre on Weil's paper:

> *'a text presented as analysis, whose significance is essentially algebraic, but whose motivation is arithmetic'*

Arithmetic principal bundles

Go back to Hodge theory of Jacobian:

$$X(\mathbb{C}) \longrightarrow J_X(\mathbb{C}) \simeq \mathrm{Ext}^1_{MHS,\mathbb{Z}}(\mathbb{Z}, H_1(X(\mathbb{C}), \mathbb{Z})).$$

$$X(F) \longrightarrow J_X(F) \otimes \mathbb{Z}_p \simeq \mathrm{Ext}^1_{\mathrm{Gal}(\bar{\mathbb{Q}}/F),f}(\mathbb{Z}_p, H_1^{et}(\bar{X}, \mathbb{Z}_p))$$
$$\simeq H^1_f(\mathrm{Gal}(\bar{\mathbb{Q}}/F), \pi_1^{p,ab}(\bar{X}, b)).$$

This suggests the possibility of extending the constructions to non-abelian homotopy and moduli space of non-abelian structures:

– over $\mathbb{C}$, Hain's 'higher Albanese varieties;'

– over $F_v/\mathbb{Q}_p$, p-adic period spaces;

– over global fields, Selmer schemes and variants.

Arithmetic principal bundles

Construction generally proceeds via a category $\mathcal{C}$ of sheaves on $\bar{X}$ such that points $b \in \bar{X}$ give fibre functors

$$F_b : \mathcal{C} \longrightarrow \mathcal{V}.$$

Then we get

$$\pi_{\mathcal{C}}(\bar{X}, b) := \mathrm{Aut}^*(F_b)$$

and

$$\pi_{\mathcal{C}}(\bar{X}; b, x) = \mathrm{Isom}^*(F_b, F_x),$$

which is a principal bundle for $\pi_{\mathcal{C}}(\bar{X}, b)$.

The basic case is when $\mathcal{C}$ is the category of finite étale covering spaces, and $\mathcal{V}$, the category of finite sets, which leads to profinite $\hat{\pi}(\bar{X}, b)$ and $\hat{\pi}(\bar{X}; b, x)$.

Arithmetic principal bundles

When we use the Tannakian category

$$\mathrm{Un}(\bar{X}, \mathbb{Q}_p)$$

of unipotent $\mathbb{Q}_p$-local systems, there are the fibre functors

$$F_b, F_x : \mathrm{Un}(\bar{X}, \mathbb{Q}_p) \longrightarrow \mathrm{Vect}_{\mathbb{Q}_p}$$

and we get the $\mathbb{Q}_p$ pro-unipotent completions

$$U(\bar{X}, b) := \mathrm{Aut}^{\otimes}(F_b),$$

$$P(\bar{X}; b, x) := \mathrm{Isom}^{\otimes}(F_b, F_x).$$

The role of the universal covering space is played by the universal unipotent $\mathbb{Q}_p$-local system $\mathcal{E}$ pointed at b, which is equipped with a comultiplication

$$\Delta : \mathcal{E} \longrightarrow \mathcal{E} \otimes \mathcal{E}.$$

Arithmetic principal bundles

$$U(\bar{X}, b) = \mathcal{E}_b^{gp} := \{a \in \mathcal{E}_b \mid \Delta(a) = a \otimes a\};$$

$$P(\bar{X}; b, x) = \mathcal{E}_x^{gp} := \{p \in \mathcal{E}_x \mid \Delta(p) = p \otimes p\}.$$

Arithmetic principal bundles

One can consider many other fundamental groups, for example,

$$\pi_{\mathcal{L}}(\bar{X}, b)$$

the completion with respect to a specific local system $\mathcal{L}$: Tannaka group of the Tannakian category generated by $\mathcal{L}$. (Lawrence and Venkatesh)

There is also the *relative completion*

$$\pi_{R\mathcal{L}}(\bar{X}, b),$$

the Tannaka group of the category generated by $\mathcal{L}$ allowing extensions. (Noam Kantor's Oxford thesis.)

One can also consider reductive completions, algebraic completions, or more complicated homotopy types, e.g., differential graded algebras and modules in suitable homotopy categories.

Arithmetic principal bundles

Key Arithmetic Fact:

When X, b and x are defined over F or F_v, these give rise to groups abd principal bundles with $G_F = \mathrm{Gal}(\bar{F}/F)$ or G_{F_v}-action.

Arithmetic principal bundles: the unipotent case

Focus on $F = \mathbb{Q}$ and $G = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. (Netan Dogra generalises to number fields.)

Localisation diagram

$$
\begin{tikzcd}
X(\mathbb{Q}) \arrow[r] \arrow[d, "j"'] & \prod_{v\in S} X(\mathbb{Q}_v) \arrow[d, "\prod_{v\in S} j_v"'] \\
H^1_f(G, U(\bar{X}, b)) \arrow[r, "loc"] & \prod_{v\in S} H^1(G_v, U(\bar{X}, b))
\end{tikzcd}
$$

The effect is that the moduli spaces become pro-algebraic varieties over $\mathbb{Q}_p$ and the lower row of this diagram is an algebraic map.

Arithmetic principal bundles: the unipotent case

That is, the key object of study is

$$H^1_f(G, U(\bar{X}, b))$$

the **Selmer scheme** of X, defined to be the subfunctor of $H^1(G, U(\bar{X}, b))$ satisfying local conditions at all v: unramified at $v \notin S$ and crystalline at p.
The local portion at p of the diagram

$$\begin{tikzcd} X(\mathbb{Q}_p) \arrow[d, "j_p"'] \arrow[dr, "j_{DR}"] & \\ H^1_f(G_p, U) \arrow[r, "\simeq"] & U^{DR}/F^0 \end{tikzcd}$$

is computable in terms of p-adic Hodge theory and *iterated integrals*, which, in particular, shows that the image is Zariski dense.

Arithmetic principal bundles: the unipotent case

$$\begin{tikzcd}
X(\mathbb{Q}) \arrow[r] \arrow[d, "j"'] & \prod_{v\in S} X(\mathbb{Q}_v) \arrow[d, "\prod_{v\in S} j_v"] \\
H^1_f(G, U(\bar{X}, b)) \arrow[r, "loc"] & \prod_{v\in S} H^1(G_v, U(\bar{X}, b))
\end{tikzcd}$$

Conjecture:

$$X(\mathbb{Q}) = pr_p[H^1_f(G, U) \times_{\prod_{v\in S} H^1_f(G_v, U(X,b))} [\prod_{v\in S} X(\mathbb{Q}_v)]],$$

where

$$pr_p : \prod_{v\in S} X(\mathbb{Q}_v) \longrightarrow X(\mathbb{Q}_p).$$

Arithmetic principal bundles: the unipotent case

$$\begin{tikzcd}
X(\mathbb{Q}) \arrow[rr] \arrow[d, "j"'] & & \prod_{v\in S} X(\mathbb{Q}_v) \arrow[d, "\prod_{v\in S} j_v"'] & \\
H^1_f(G, U(\bar{X}, b)) \arrow[rr, "loc"] & & \prod_{v\in S} H^1(G_v, U(\bar{X}, b)) \arrow[r, "\alpha"] & \mathbb{Q}_p
\end{tikzcd}$$

If α is an algebraic function vanishing on the image of *loc*, then

$$\alpha \circ \prod_v j_v$$

gives a defining equation for $X(\mathbb{Q})$ inside $\prod_{v\in S} X(\mathbb{Q}_v)$.

Arithmetic principal bundles: the unipotent case

To make this concretely computable, we take the projection

$$pr_p : \prod_{v \in S} X(\mathbb{Q}_v) \longrightarrow X(\mathbb{Q}_p)$$

and try to compute

$$\cap_\alpha pr_p(Z(\alpha \circ \prod_v j_v)) \subset X(\mathbb{Q}_p).$$

Conjecture (Non-Archimedean effective Mordell)

$$\cap_\alpha pr_p(Z(\alpha \circ \prod_v j_v)) = X(\mathbb{Q})$$

and this set is effectively computable.

Arithmetic principal bundles: the unipotent case

Some motivation comes from the fact that the previous diagram breaks into levels

$$\begin{tikzcd} X(\mathbb{Q}) \arrow[r] \arrow[d, "j"'] & \prod_v X(\mathbb{Q}_v) \arrow[d, "\prod_v j_v"'] & \\ H^1_f(G, U_n(X,b)) \arrow[r, "loc"] & \prod_v H^1(G_v, U_n(X,b)) \arrow[r, "\alpha_n"] & \mathbb{Q}_p \end{tikzcd}$$

So we could define

$$X(\mathbb{Q}_p)_n = \cap_{\alpha_n} pr_p(Z(\alpha_n))$$

and conjecture that

$$X(\mathbb{Q}) = \cap_n X(\mathbb{Q}_p)_n.$$

Arithmetic principal bundles: the unipotent case

Standard motivic conjectures (Bloch-Kato, Fontaine-Mazur,...) give bounds on the dimensions of

$$H^1_f(G, U_n(X, b))$$

and imply that for each n, there are α_n algebraically independent from the functions α_i for $i < n$.

In fact, many interesting examples give equality already at $n = 2$.

Diophantine geometry: remark on non-abelian reciprocity

There is a **non-abelian class field theory** with coefficients in a fairly general variety X over a number field F generalising CFT with coefficients in $\mathbb{G}_m$.

This consists (with some simplifications) of a filtration

$$X(\mathbb{A}_F) = X(\mathbb{A}_F)_1 \supset X(\mathbb{A}_F)_2 \supset X(\mathbb{A}_F)_3 \supset \cdots$$

and a sequence of maps

$$rec_n : X(\mathbb{A}_F)_n \longrightarrow \mathfrak{G}_n(X)$$

to a sequence of groups such that

$$X(\mathbb{A}_F)_{n+1} = rec_n^{-1}(0).$$

Diophantine geometry: remark on non-abelian reciprocity

Here,

$$\mathfrak{G}_n(X) = H^1(G_F, \mathrm{Hom}(Z^n(\hat{\pi}_1(\bar{X}, b)), \mu_\infty))^\vee,$$

where Z^n refers to the lower central series. The reciprocity maps measure the obstruction to a collection of local torsors being a global torsor while going up the levels.

Diophantine geometry: remark on non-abelian reciprocity

$$
\begin{array}{ccccccccc}
\cdots & rec_3^{-1}(0) & \subset & rec_2^{-1}(0) & \subset & rec_1^{-1}(0) & \subset & X(\mathbb{A}_F) \\
 & \| & & \| & & \| & & \| \\
\cdots & X(\mathbb{A}_F)_4 & \subset & X(\mathbb{A}_F)_3 & \subset & X(\mathbb{A}_F)_2 & \subset & X(\mathbb{A}_F)_1 \\
 & \downarrow{\scriptstyle rec_4} & & \downarrow{\scriptstyle rec_3} & & \downarrow{\scriptstyle rec_2} & & \downarrow{\scriptstyle rec_1} \\
\cdots & \mathfrak{G}_4(X) & & \mathfrak{G}_3(X) & & \mathfrak{G}_2(X) & & \mathfrak{G}_1(X)
\end{array}
$$

Diophantine geometry: remark on non-abelian reciprocity

Put

$$X(\mathbb{A}_F)_\infty = \cap_{n=1}^\infty X(\mathbb{A}_F)_n.$$

Theorem (Non-abelian reciprocity)

$$X(F) \subset X(\mathbb{A}_F)_\infty.$$

Conjecture

$$pr_p(X(\mathbb{A}_F)_\infty) = X(\mathbb{Q}) \subset X(\mathbb{Q}_p).$$

Computing rational points

[Dan-Cohen, Wewers]

For $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$,

$$X(\mathbb{Z}[1/2]) = \{2, -1, 1/2\} \subset \{D_2(z) = 0\} \cap \{D_4(z) = 0\},$$

where

$$D_2(z) = \ell_2(z) + (1/2)\log(z)\log(1-z),$$

$$D_4(z) = \zeta(3)\ell_4(z) + (8/7)[\log^3 2/24 + \ell_4(1/2)/\log 2]\log(z)\ell_3(z)$$
$$+[(4/21)(\log^3 2/24 + \ell_4(1/2)/\log 2) + \zeta(3)/24]\log^3(z)\log(1-z),$$

and

$$\ell_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}.$$

Numerically, the inclusion appears to be an equality.

Computing rational points

[Balakrishnan, Dan-Cohen, K., Wewers], [Bianchi arXiv:1904.04622v1]

$X = E \setminus O$, where E is an elliptic curve of rank 1 written as

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6.$$

$$\alpha = dx/(2y + a_1 + a_3), \qquad \beta = x\alpha.$$

Choose p an ordinary prime of good reduction. S, set of primes of bad reduction.

Let $h : E(\mathbb{Z}) \longrightarrow \mathbb{Q}_p$be the cyclotomic p-adic height, written in terms of local p-adic Neron functions:

$$h = \lambda_p + \sum_{v \neq p} \lambda_v.$$

Computing rational points

For each $v \in S$, have a finite set

$$W_v = \lambda_v(X(\mathbb{Z}_v)) \cup \{0\}$$

and

$$W = \prod_{v \in S} W_v.$$

For $w = (w_v) \in W$, let

$$\|w\| = \sum w_v.$$

Let $c = h(P)/\log_\alpha^2(P)$ for P a point of infinite order, and

$$C = \frac{a_1^2 + 4a_2}{12} + \mathbf{E}_2(E, \alpha),$$

where $\mathbf{E}_2$ is Katz's p-adic Eisenstein series of weight 2.

Computing rational points

Then

Theorem

$$X(\mathbb{Z}) \subset X(\mathbb{Z}_p)_2 = \cup_w \{ \int_b^z \beta\alpha + (c + C/2) \log_\alpha^2(z) = \|w\| \}$$

When E has CM, c can be expressed as a ratio of p-adic L-values.

Proposition (Bianchi)

$$X(\mathbb{Q}) \cap X(\mathbb{Z}_p)_2 = X(\mathbb{Z}).$$

In practice, this can be used to efficiently compute $X(\mathbb{Z})$ by using several p (Mordell-Weil sieve) [Balakrishnan, Besser, Mueller].

Computing rational points

Given a point $z \in X(\mathbb{Z}_p)_2$ need to figure out which ones are in $X(\mathbb{Q})$. Write P for a generator of free-part, so we are looking for N such that

$$z = NP + \text{torsion} \in X(\mathbb{Z}_p)_2 \Rightarrow z \in X(\mathbb{Z})$$

Need to figure out possible N.

If there were such an N, we would have

$$N = \log_\alpha z / \log_\alpha P.$$

We can restrict possibilities for N now using several primes.

Computing rational points

[Balakrishnan, Dogra, Mueller, Tuitmann, Vonk (arXiv 1711.05846, 'Explicit Chabauty-Kim theory for the split modular curve of level 13,' to be published in Annals of Math.)]

Let

$$X_s^+(N) = X(N)/C_s^+(N),$$

where $X(N)$ is the compactification of the moduli space of pairs

$$(E, \phi : E[N] \simeq (\mathbb{Z}/N)^2),$$

and $C_s^+(N) \subset GL_2(\mathbb{Z}/N)$ is the normaliser of a split Cartan subgroup.

Bilu-Parent-Rebolledo had shown that $X_s^+(p)(\mathbb{Q})$ consists entirely of cusps and CM points for all primes $p > 7$, $p \neq 13$. They called $p = 13$ the 'cursed level'.

Computing rational points

Theorem (BDMTV)

$$X_s^+(13)(\mathbb{Q}) = X_s^+(13)(\mathbb{Q}_{19})_2$$

has exactly 7 points, consisting of the cusp and 6 CM points.

This concludes an important chapter of a conjecture of Serre:

There is an absolute constant A such that

$$G \longrightarrow \mathrm{Aut}(E[p])$$

is surjective for all non-CM elliptic curves $E/\mathbb{Q}$ and primes $p > A$.

Computing rational points

[Burcu Baran]

$$y^4 + 5x^4 - 6x^2y^2 + 6x^3z + 26x^2yz + 10xy^2z - 10y^3z$$

$$-32x^2z^2 - 40xyz^2 + 24y^2z^2 + 32xz^3 - 16yz^3 = 0$$

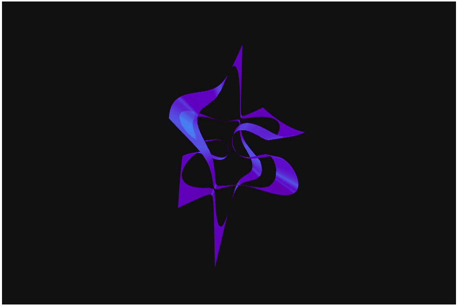


Figure: The cursed curve

{(1:1:1), (1:1:2), (0:0:1), (-3:3:2), (1:1:0), (0,2:1), (-1:1:0) }

Computing rational points

Explain by way of recent work of Dogra, Le Fourn, and Siksek.

We have an exact sequence

$$0 \longrightarrow \wedge^2 V/\mathbb{Q}_p(1) \longrightarrow U_2 \longrightarrow V \longrightarrow 0,$$

where $V = T_p \otimes \mathbb{Q}$ and the $\mathbb{Q}_p(1)$ comes from the Weil pairing.

Suppose one has a correspondence

$$Z \subset X \times X$$

such that

$$[Z] \in H^2(\bar{X} \times \bar{X})(1)$$

lives in $\wedge^2 H^1(\bar{X})(1) = H^2(\bar{J})(1)$ and the corresponding map

$$\wedge^2 V \longrightarrow \mathbb{Q}_p(1)$$

kills $\mathbb{Q}_p(1)$.

Computing rational points

Then we get a pushout extension

$$0 \longrightarrow \mathbb{Q}_p(1) \longrightarrow A_Z \longrightarrow V \longrightarrow 0,$$

and the diagram

$$\begin{tikzcd} X(\mathbb{Q}) \arrow[r] \arrow[d, "j"'] & \prod_v X(\mathbb{Q}_v) \arrow[d, "\prod_v j_v"] \\ H^1_f(G, A_Z) \arrow[r, "loc"] & \prod_v H^1_f(G_v, A_Z) \end{tikzcd}$$

Denote by

$$X(\mathbb{Q}_p)_Z \subset X(\mathbb{Q}_p)$$

the common zero set of functions obtained from this diagram.

Computing rational points

There is a unique line bundle $L \longrightarrow J$ such that

$$c_1(L) = [Z] \in H^2(\bar{J})(1)$$

and $L|X$ is trivial, so that the choice of a basepoint $\tilde{b} \in L_e^\times$ determines a lifting

$$\begin{tikzcd} & L^\times \arrow[d] \\ X \arrow[ur] \arrow[r, hook] & J \end{tikzcd}$$

We can define a p-adic height with respect to L

$$h_L = \sum_v \lambda_v : L^\times(\mathbb{A}_\mathbb{Q}) \longrightarrow \mathbb{Q}_p.$$

Computing rational points

Theorem (Dogra, Le Fourn, Siksek)

Suppose $X = X_0^+(N)$ or $X_{ns}^+(N)$. Then for any homologically non-trivial Z as above, $X(\mathbb{Q}_p)_Z$ is finite, and can be effectively computed.

In fact, if

$$Z = \sum_f a_f \mathbf{1}_f,$$

where f runs over cuspidal eigenforms of weight 2, then $X(\mathbb{Q}_p)_Z$ can be described by means of an equation

$$\lambda_p(x) = \sum_f [\frac{h(c_f, c_f)}{\log_f(c)^2} \log_f(x)(a_f \log_f(x) + \sum_g a_g \log_f(\Delta_g))]$$

where c is a Heegner point coming from the modular curve and Δ_g is the Chow-Heegner cycle associated to the modular form g.

Computing rational points

Note that

$$X(\mathbb{Q}) \subset X(\mathbb{Q}_p)_2 \subset X(\mathbb{Q})_Z.$$

Thus, if $X(\mathbb{Q}) = X(\mathbb{Q}_p)_Z$, get equality everywhere, and conjecture is verified.

In fact, need only check $X(\mathbb{Q}) = \cap_Z X(\mathbb{Q}_p)_Z$.

This was checked recently for $X_s^+(13)$, but also for $X_0^+(p)$ when

$$p = 67, 73, 97, 103, 107, 109$$

by Jennifer Balakrishnan, Steffen Mueller, Netan Dogra, and Kiran Kedlaya.

All these examples have $\mathrm{rank} J(\mathbb{Q}) = g$.

Here as well, can try to apply Mordell-Weil sieve to $L^\times(\mathbb{Q})$.

Some speculations on rational points and critical points

Would like to think of

$$H^1(G, U(X,b)) \longrightarrow \prod_v H^1(G_v, U(X,b))$$

as being like

$$\mathbb{S}(M,G) \subset \mathcal{A}(M,G),$$

where $\mathcal{A}$ is some space of connections and $\mathbb{S}$ solutions to Euler-Lagrange equations.

In particular, functions cutting out the image of localisation should be thought of as 'classical equations of motion' for gauge fields.

Some speculations on rational points and critical points

When X is smooth and projective, $X(\mathbb{Q}) = X(\mathbb{Z})$, and we are actually interested in

$$Im(H^1(G_S, U)) \cap \prod_{v\in S} H^1_f(G_v, U) \subset \prod_{v\in S} H^1(G_v, U),$$

where

$$H^1_f(G_v, U) \subset H^1(G_v, U)$$

is a subvariety defined by some integral or Hodge-theoretic conditions.

In order to apply symplectic techniques, replace U by

$$T^*(1)U := (LieU)^*(1) \rtimes U.$$

Some speculations on rational points and critical points

Then

$$\prod_{v\in S} H^1(G_v, T^*(1)U)$$

is a symplectic variety and

$$Im(H^1(G_S, T^*(1)U)), \qquad \prod_{v\in S} H^1_f(G_v, T^*(1)U)$$

are Lagrangian subvarieties.

Thus, the (derived) intersection

$$\mathcal{D}_S(X) := Im(H^1(G_S, T^*(1)U)) \cap \prod_{v\in S} H^1_f(G_v, T^*(1)U)$$

has a $[-1]$-shifted symplectic structure.

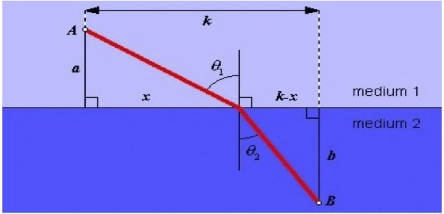
Zariski-locally the critical set of a function. [Ben-Basset, Brav, Bussi, Joyce]

Some speculations on rational points and critical points

$$
\begin{tikzcd}
X(\mathbb{Z}) \arrow[r] \arrow[d, "j^g"'] & j_S^{-1}(\mathcal{D}_S(X)) \arrow[r, hook] \arrow[d, "j_S"'] & \prod_{v \in S} X(\mathbb{Q}_v) \arrow[d, "j_S"'] \\
H^1_f(G_S, T^*(1)U) \arrow[r, "loc_S"] & \mathcal{D}_S(X) \arrow[r, hook] & \prod_{v \in S} H^1(G_v, T^*(1)U_n)
\end{tikzcd}
$$

From this view, the global points can be obtained by pulling back 'Euler-Lagrange equations' via a period map.

Some speculations on rational points and critical points



For integers $n > 2$ the equation

$$a^n + b^n = c^n$$

cannot be solved with positive integers a, b, c.

Figure: Pierre de Fermat (1607-1665)