LEMMA 2.14

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cover means definable open cover. If $\mathcal{C}$ is a cell decomposition of a space X, we denote the unique cell containing a point $x \in X$ by C_x. For a subset A of X, we let $\mathcal{C}(A)$ be the set of cells in $\mathcal{C}$ whose closure intersects A.

Claim 0.0.1. *For any $A \subseteq X$, $X(A) = \bigcup \mathcal{C}(A)$ is open*

Proof. Let $x \in X(A)$ and let U be an open neighbourhood of x intersecting a minimum number of cells. We claim that $U \subseteq X(A)$. Otherwise, let $y \in U \setminus X(A)$. Then the closure Z of D_y is disjoint from A. If $x \in Z$ then $D_x \subseteq Z$, contradicting that $D_x \in \mathcal{C}(A)$. Hence $U \setminus Z$ is also an open neighbourhood of x, intersecting less cells (since it does not intersect D_y), a contradiction. □

Lemma 0.0.2 (=Lemma 2.14)**.** *Assume $f : X \to Y$ is a finite map of definable spaces with X separable, $\mathcal{X}$ a cover of X. Then there is a refinement $\mathcal{W}$ of $\mathcal{X}$ and a cover $\mathcal{Y}$ of Y, such that*

1. *For each $Y \in \mathcal{Y}$, $f^{-1}(Y)$ is a (finite) disjoint union of members of $\mathcal{W}$*
2. *Each member of $\mathcal{W}$ is a component of $f^{-1}(Y)$ for some $Y \in \mathcal{Y}$*

Proof. We claim that there are cell decompositions $\mathcal{D}$ of X and $\mathcal{C}$ of Y such that:

1. Each $D \in \mathcal{D}$ is either contained or is disjoint from each $X' \in \mathcal{X}$, and similarly for $\mathcal{C}$ and $f(X')$
2. For each $D \in \mathcal{D}$, $f \restriction_D$ is a bijection with some $C \in \mathcal{C}$
3. $f^{-1}(C)$ is a disjoint union of some elements of $\mathcal{D}$, for each $C \in \mathcal{C}$

This is achieved by applying Lemmas 2.7 and 2.8 of the paper, to the boolean algebra of subsets generated by $\mathcal{X}$ and its image, respectively. We say that $D \in \mathcal{D}$ is over $C \in \mathcal{C}$ if $f(D) = C$.

We next construct refined cell decompositions $\mathcal{D}_1$ and $\mathcal{C}_1$, as follows: if $D \neq D'$ are over the same cell in $\mathcal{C}$, then each is disjoint from the closure of the other (for example, by dimension). By separation[1], we may choose disjoint open sets $U \supseteq D$ and $U' \supseteq D'$. We do this, and add all such open sets to the cover $\mathcal{X}$. $\mathcal{D}_1$ and $\mathcal{C}_1$ are now cell decompositions satisfying the above conditions with respect to this new cover.

[1]It is not clear to me what separation axiom we need exactly. According to Wikipedia, what we are using is called "completely normal". It follows from regularity under the second-countable assumption, which we have. OTOH, we actually need a definable version, so I don't know if the implication holds...

Now let $X(D) = \bigcup \mathcal{D}_1(D)$, $Y(C) = \bigcup \mathcal{C}_1(C)$ and $\mathcal{W} = \{X(D) | D \in \mathcal{D}\}$ and $\mathcal{Y} = \{Y(C) | C \in \mathcal{C}\}$. The following claims show they satisfy the conditions.

Claim 0.0.3. *$\mathcal{W}$ and $\mathcal{Y}$ are covers, and $\mathcal{W}$ refines $\mathcal{X}$*

Proof. By claim 0.0.1, each of $\mathcal{W}$ and $\mathcal{Y}$ consists of open subsets. For each $D \in \mathcal{D}$ we have $D \subseteq X(D)$, since each D is a union of cells in $\mathcal{D}_1$. Hence $\bigcup \mathcal{W} = X$, so it is a cover (and similarly for $\mathcal{Y}$).

If $U \in \mathcal{X}$ and $D \in \mathcal{D}$ is a subset of U, then each element of $\mathcal{D}_1(D)$ is also contained in U, since otherwise it would be disjoint from it, hence its closure could not intersect D. Hence $X(D) \subseteq U$, so $\mathcal{W}$ refines $\mathcal{X}$. □

The following claims show that both conditions hold

Claim 0.0.4. *If $D, D' \in \mathcal{D}$ are distinct and map to the same cell in $\mathcal{C}$, then $X(D)$ and $X(D')$ are disjoint.*

Proof. This follows directly from the construction of $\mathcal{D}_1$: if $X(D)$ and $X(D')$ are not disjoint, neither are $\mathcal{D}_1(D)$ and $\mathcal{D}_1(D')$. Thus, there is $E \in \mathcal{D}_1$ whose closure intersects both D and D', contradicting the choice of U and U' above. □

Claim 0.0.5. *For every $C \in \mathcal{C}$, $f^{-1}(Y(C))$ is the disjoint union of $X(D)$ for $D \in \mathcal{D}$ mapping to C*

Proof. The union is disjoint by the previous claim, and is definitely contained in the pre-image. Hence it suffices to show that if $C' \in \mathcal{C}_1(C)$ for some $C \in \mathcal{C}$ and $D' \in \mathcal{D}_1$ is over C', then $D' \in \mathcal{D}_1(D)$ for some $D \in \mathcal{D}$ over C (so that $D' \subseteq X(D)$).

Since $C' \in \mathcal{C}_1(C)$, there is a curve γ inside C' whose limit is in C. Then γ lifts to a curve $\tilde{\gamma}$ in D', and by properness this curve has a limit x in X. We take $D = D_x$. □

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