

$$E: X \rightarrow Y$$

open subset ~~X~~

X, Y complex alg varieties

E complex analytic, local iso.

S -collection alg. subvarieties

of Y , "special" if

z is special then

$E^{-1}(z)$ is an alg. subvariety

$W \subseteq X$ is special iff

$E(W)$ is special.

$W \subseteq X \Rightarrow W^{\text{alg}} = \text{closure of}$
alg. subvarieties of positive dim.
in W .

$W^{\text{sp}} = \text{sub variety consisting of special}$
subvarieties

$E \sim \text{exponential, special in}$

$Y = G_m^n$ translate of subgroup

in \mathbb{A}^n translate of

\mathbb{Q} -linear subspace.

Want: If $z \in Y$ algebraic

then $E^{-1}(z)^{\text{alg}} = E^{-1}(z)^{\text{sp}}$

Ax - Lindemann - Weierstrass:

If $W \subseteq X$ alg subvariety
not contained in a proper special
subvariety then $E(W)$ is Zariski
dense.

ALW1 implies the statement:

Take $a \in E^{-1}(z)^{alg}$, $a \in W$

W positive dim alg subvariety, so

$E(W) \subseteq z$ hence by ALW1

$E(W)$ is contained in special

subvariety, take Y' to be

the smallest one, if $Y' \not\subseteq z$,

replace Y by Y' .

Replace W by $V \times W \times Z$. So

V is an alg. subvariety of

$X \times Y$, and $V \cap \Gamma_E \cong U$

is an analytic subvariety.

Then $\dim(V) \geq \dim(U) + \dim(Y)$

$$\underline{\dim(V) = \dim(W) + \dim(Z)}$$

As 1: $V \subseteq X \times Y$ alg sub, U conn.

component of $V \cap \Gamma_E$. Assume $\pi_Y(V)$ is not contained in a proper special sub var. Then $\dim(V) \geq \dim(U) + \dim(Y)$

$$\underline{\tau} \supseteq \bigcup \{ a+H \mid a \in H \subseteq X \}$$

ALW2: If $W \subseteq X$ an alg. curve not contained in a proper special subvar. then $E(W)$ is dense in Y .

AS2: Let $V \subseteq X \times Y$ with infinite projection to X , and

$V \cap \Gamma_E$ is infinite and $\pi_Y(\frac{V \cap \Gamma_E}{0})$ not contained in a special subvariety then $\dim(V) \geq 1 + \dim(Y)$

ALW3: If $\gamma: D \rightarrow X$ non-constant alg map from a disc D to X is not \subseteq in a special subvar. then $E(\gamma)$ has dense image

Prop: If $\gamma: D \rightarrow E$ is such that
 $\pi_Y \circ \gamma$ is not contained in a special
 subvar. then $\dim(\gamma(D)) \geq 1 + \dim(Y)$

E is a solution to some

alg. diff. equation.

This differential equation induces
 a diff. eq. on the possible maps γ .

$E: \mathbb{C}^n \rightarrow G_n^h$ usual exponential

$$\gamma: D \rightarrow \Gamma_E$$

$$\gamma = (a_1, \dots, a_n, b_1, \dots, b_n)$$

$$b_i = E \circ a_i: D \rightarrow \mathbb{C}$$

$$b_i' = E' \circ a_i \cdot a_i' = E \circ a_i \cdot a_i' = b_i \cdot a_i'$$

$$a_i' = \frac{b_i'}{b_i}$$

$K, ' \quad$ differential field:

K field of char 0, $' : K \rightarrow K$

additive and $(xy)' = x'y + xy'$

$$C_K = \{a \in K \mid a' = 0\}$$

Theorem: (A, X) K is a differential field, $C = C_K$, $a_i, b_i \in K$ s.t.

$a_i' = \frac{b_i'}{b_i}$: If a_i are pairwise

independent over C in K/C then

$$\text{ed}_C(\bar{a}, \bar{b}) \geq 1+n$$

Pf: Enough to prove: some

non trivial product $B = b_1^{m_1} \cdots b_n^{m_n}$

is constant. If that is false; then

$$0 = B' = m_1 b_1^{m_1-1} b_1' \cdots b_n^{m_n} + \dots + m_n b_1^{m_1} \cdots b_n^{m_n-1} b_n'$$

$$\Rightarrow m_1 a_1 + \dots + m_n a_n = 0$$

Let $M = \underline{\Omega_c K}$. Enough to show
 that $d\beta = 0$, i.e. that $\frac{db_i}{b_i}$ are
 \mathbb{Z} -linearly dependent.

$$K \xrightarrow{d} \underline{\Omega_c K}, \quad d(fg) = f dg + g df$$

$$f \in K \quad df = d(f)$$

There is a derivation

$$D: M \rightarrow M \quad \text{determined by } D(d(f)) = d(f')$$

$$D(g d(f)) = g' d(f) + g d(f'), \quad g, f \in K$$

$$D\left(\frac{df}{f}\right) = d\left(\frac{f'}{f}\right) \quad \text{for all } f \in K \Rightarrow$$

$$\omega_i = \underline{\underline{da_i}} = \underline{\underline{\frac{db_i}{b_i}}} \Rightarrow D(\omega_i) = 0$$

$$td \leq n \Rightarrow \dim M_{\mathbb{R}} \leq n \Rightarrow$$

linear dependence

$$g da + \sum_{i=1}^n f_i \omega_i = 0 \quad a = a, \quad a' = 1$$

$$D(da) = D(\omega_i) = 0$$

Applying D , get $g' da + \sum f_i' \omega_i = 0$

$$\Rightarrow g' = f_i' = 0$$

Get constants c_i s.t.

$$\sum c_i \frac{db_i}{b_i} = \sum c_i da_i + g da = dV$$

$$V = g a + \sum c_i a_i \in K$$

\bullet $c_i \in \mathbb{P}(\overline{\mathbb{Q}})$ | $\dim_{\mathbb{Q}} \Omega_{\mathbb{Q}}^k = \text{tr. deg } (k)$
 $dP =$

Prop: Suppose $b_i, v \in K, C \subseteq K$

(als. closed) $\sum c_i \frac{db_i}{b_i} = dv$ in

$M = \Omega_C^1$. Then the $\frac{db_i}{b_i}$ are

\mathbb{Z} -linearly dependent.

pf: Show that if c_i are indep:

over \mathbb{Q} then $db_i = 0$ for all i .

May assume $K = C(v, b_1, \dots, b_n)$

By induction, may assume that K is tr. deg. 1 over $C, K = C(X)$ on a smooth projective curve X .

for every $x \in X$, there is a map $\text{res}_x: \Omega_K \rightarrow C$ s.t. $\text{res}_x(dg) = 0$

for all g and $\text{res}_x(\frac{dg}{g}) = v_x(g)$.

$dv = \sum c_i \frac{db_i}{b_i} \implies 0 = \sum c_i v_x(b_i) \implies v_x(b_i) = 0$
 $\implies b_i \in C \implies db_i = 0$

$$E: X \rightarrow Y$$

$E(z)$ dense in Y

$$z \in Y \ni \Gamma_E \text{ dense}$$

\mathbb{H} - upper half plane

$$G = \text{PSL}_2(\mathbb{R}) \curvearrowright \mathbb{H}$$
$$\cup$$
$$\Gamma$$

\mathbb{H}/Γ - curve of genus 0

$$j_\Gamma: \mathbb{H} \rightarrow \mathbb{H}/\Gamma$$

$$\Gamma = \text{PSL}_2(\mathbb{Z})$$

$\Gamma_1 \sim \Gamma_2, \Gamma_1, \Gamma_2$ commensurable if

$\Gamma_1 \cap \Gamma_2$ is finite index in each.

$$\text{Comm}(\Gamma) = \{g \in G \mid g^{-1}\Gamma g \sim \Gamma\}$$

$$\Gamma = \text{PSL}_2(\mathbb{Z}) \Rightarrow \text{Comm}(\Gamma) = \underset{g}{\text{PSL}_2(\mathbb{Q})}$$

A subset of \mathbb{H}^2 is special

if it is the graph of $g \in \text{Comm}(\Gamma)$

or fibre of a projection.

$$\{(\tau, g\tau) \mid \tau \in \mathbb{H}\}$$

τ $g\tau$ geodesically disp.

Thm $W \subseteq \mathbb{H}^n$ ^{alg.} subvariety

not contained in a special one,

then the graph of

$$\prod_{\Gamma}^n \times \prod_{\Gamma}^n \times \prod_{\Gamma}^n \quad \text{on } W \text{ is}$$

dense in $W \times \mathbb{A}^{3n}$

Outline

These are differential algebraic polynomials X_{Γ} and

S (ind. of Γ) s.t.

1. For any univ. function a ,

$X_{\Gamma}(\frac{d}{dt} a) \geq S(a)$ for any vector field $\frac{d}{dt}$ on the domain

$$2. \quad \chi_{\Gamma}(y) = b \quad \chi_{\Gamma}(z) = c$$

$b, c \in K$ differential field
if b, c are distinct and
in an extension field

j_b, j_c solutions \checkmark not alg.

over K , $\underbrace{j_b, j_b', j_b'', j_c, j_c', j_c''}_{\text{are alg. independent over } K}$

3. If j_1, \dots, j_n are solutions
of $\chi_{\Gamma}(y) = b$. Denote

$\vec{j}_i = (j_i, j_i', j_i'')$. Suppose

$\vec{j}_1, \dots, \vec{j}_n$ are dependent

Then for some i, k , $P(\vec{j}_i, \vec{j}_k) = 0$

for a Γ -special polynomial.

Def: A differential field

K is differentially closed

if any diff. alg. variety

X over K with $X(L) \neq \emptyset$

for $L \supseteq K$, $X(K) \neq \emptyset$.

for every diff. field K

Fact: There is a first order theory DCF_K whose models are the differentially closed fields containing K .

DCF_K admits quantifier elimination, any diff. field extension L of K can be embedded over K in a model.

Def, A def. set X is
strongly minimal if it is
infinite but every def-subset is
finite or co-finite (def with
additional parameters).

- The definable set C given
by $X=0$ is strongly minimal

- Main kernels: A (simple algebraic
variety, $A^\#$ smallest Zariski
definable subgroup is strongly
minimal set.)

Zil'ber trichotomy: If X
is strongly minimal and orthogonal
to one of the above, then

it is geometrically trivial:

If $a_1, \dots, a_n \in X(M)$ are

diff. alg. dependent then
some pair a_i, a_j are d.f.
alg. dependent.

Ex G alg. group, g_1, g_2
(d.f.)

alg. ind. $g_1 * g_2$ ind. of

g_1 and g_2

X, Y are non-orthogonal if

there is an infinite $Z \in X \times Y$
with fibres under the projections.
(Z might be defined over some
parameters).

Theorem 1: The equation

$X_p(y) = a$ is strongly minimal and geometrically trivial.

Theorem 2: If $a \neq b$.

then $X(y) = a$, $X(y) = b$
are orthogonal.

Schwarzian derivative

$$S(y) = \left(\frac{y''}{y'} \right)' - \frac{1}{2} \left(\frac{y''}{y'} \right)^2$$

$g \in \text{PSL}_2(\mathbb{R})$ \mathcal{G}/\mathcal{H}

$$S(g) = 0, \quad S(g \circ f) = g \circ S(f)$$

\mathbb{D} fundamental domain for
 the action of Γ , polygon
 with finite number of vertices
 b_1, \dots, b_r , with angles $\alpha_i \pi$

$$R_{j_r}(y) = \frac{1}{2} \sum_{i=1}^r \frac{1 - \alpha_i^2}{(y - a_i)^2} + \sum_{i=1}^r \frac{A_i}{y - a_i}$$

$$a_i = j(b_i)$$

$$X_\Gamma(y) = S(y) + (y')^2 R_{j_r}(y)$$

$$X: \underline{S(y) + (y')^2 R(y) = 0}$$

$$R(y) \in \mathbb{C}(y)$$

$$\frac{d^2 u}{dy^2} + u^2 + \frac{1}{2} P(y) = 0$$

Then if the Riccati

equation does not have

alg. solutions then X

is strongly minimal.

$$\frac{dz}{dy^2} = R(y)z \quad \mathbb{E} \quad \text{v.s. of dim 2 over } \mathbb{C} \quad V$$

$$u = \frac{z'}{z}$$

H Galois group

Riccati equation has alg. solution



Galois of linear equation is a proper subgroup of $SL_2(V)$

~~X~~ X ~ Strongly minimal
order > 1 , defined
over \mathbb{C} constants.

Claim: Any such X is

Geom. trivial.

Thm. 2.13 $X \subseteq \mathbb{C}^n$ def.

open, then \mathcal{O}_X is coherent

Pf: $f_1, \dots, f_n \in \mathcal{O}_X(X)$, show

$$\underline{\underline{I(\tilde{F})}} = \{ \tilde{g} \mid \tilde{g} \cdot \tilde{F} = 0 \} \quad \text{loc. fin. gen.}$$

i.e. may pass to a cover

by finitely many def. open

subsets $U \subseteq X$, and find
finitely many tuples

$\tilde{g}_1, \dots, \tilde{g}_m \in I(\tilde{F})(U)$ which gen.

the relations on each def. open
subset V of U

In $I(f)(X)$ some standard

$$p(\text{parents}) : \underbrace{(-f_i, 0, f_i, \dots, 0)}_{i} \in R_i$$

Case 1: Let U be the set

where f_i is invertible.

On U , the relations R_i generate:

$$\text{they are equivalent to } \left(-\frac{f_i}{f_i}, 0, 0, \dots, 0 \right)_{g_i}$$

$$(g_1, \dots, g_n) = \sum_{f_i} g_i \cdot R_i, \quad U \text{ is}$$

one of the elements in the cover.

$$Y = X \setminus U$$

Case 2: $f_1 = t^k + a_{k-1}(u)t^{k-1} + \dots + a_0(u)$

Let $V_0 \in \mathbb{C}^{n-1}$ open definable.

Then by 2.11, may write each

$f_i = u_i \cdot f_1 + v_i$ where

v_i a poly. of deg $< k$ in

a hld of Y . \square

$0 = \sum g_i f_i = \left(\sum a_i u_i \right) f_1 + \sum_{i>1} g_i v_i$

So may assume all f_i are poly.

Use the same to write

$$g_i = V_i f_i + \underbrace{S_i}_{\text{X}}$$

$\deg(s_i) < k$

$$0 = \sum g_i f_i = \underbrace{f_1 + \dots + V_i f_i}_{-V_i f_i} + S_i f_i =$$

$$\underbrace{S_0 f_1 + S_2 f_2 - \dots + S_n f_n}$$

$$S_0 f_1 = -S_2 f_2 - \dots - S_n f_n$$

Think of the relation as

$$\text{elements of } \underbrace{\left(\frac{0}{F^M}\right)^n}_{= M}$$

If $\pi: Y \rightarrow U_0$ the projection
 $\pi_*(M)$ is finite generated
over U_0 . By induction there
is a cover of U_0 for which
the module of relations is f.g.

over U_0 . Take the inverse
image of this cover on X

General case: Can find an

open cover and on open U_i
in it a projection to \mathbb{C}^{1-n}
which is proper on Y , in
particular with finite fibres

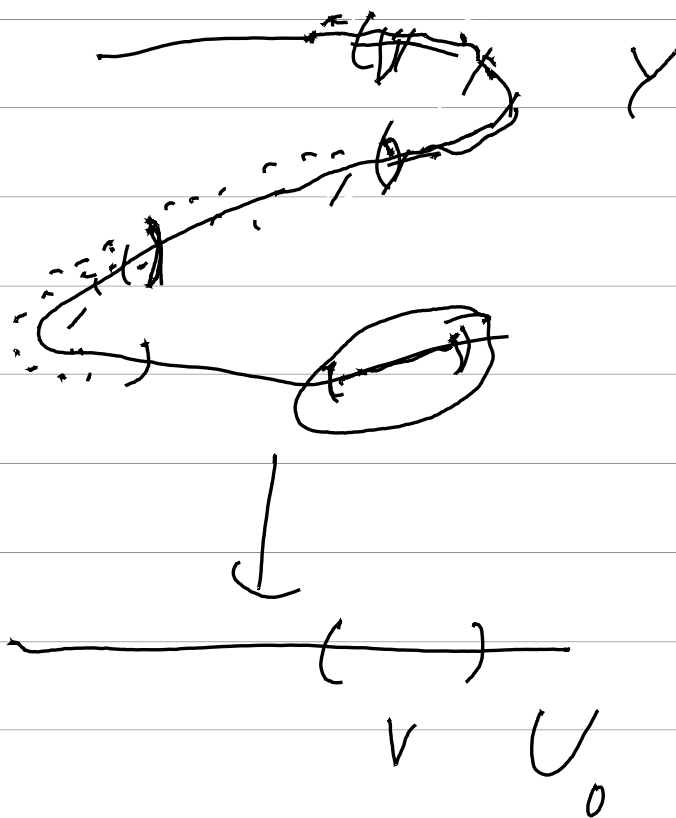
We assume that the set of relations consists of poly. of $\deg \leq k$. As a module over \mathcal{O}_U it has finitely many generators, we take them $\bar{g}_1, \dots, \bar{g}_r$

I.e., if $\bar{h} \cdot \bar{f} = 0$ on

an open subset of \mathbb{A}^n then

$\pi^{-1}(V)$, $V \subseteq U$ then

$$h = \sum b_i \bar{g}_i|_{\pi^{-1}(V)}$$



Components of sets of the form

$$\pi^{-1}(V)$$

Lemma 2.14: $f: X \rightarrow Y$ cont.

X, Y are spaces.

finite def, X separated (Hausdorff)

Given a cover \underline{X} of X

Then can find a refinement \mathcal{W} of \mathcal{X} and a cover \mathcal{Y} of Y s.t.

1. for each $Y_0 \in \mathcal{Y}$, $f^{-1}(Y_0)$ is a disjoint union of elements of \mathcal{W}

2. Every element of \mathcal{W} is a component of $f^{-1}(Y_0)$ for some $Y_0 \in \mathcal{Y}$.

Pf: Take a cell decomposition

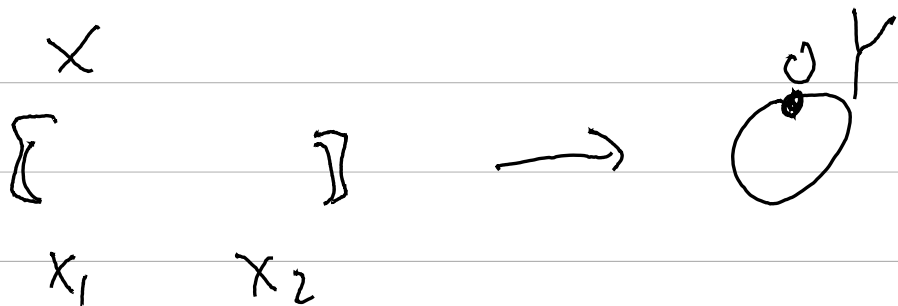
\mathcal{D} of X , \mathcal{C} of Y
s.t. if $D \in \mathcal{D}$ and $X_0 \in X$ then

1. either $D \subseteq X_0$ or $D \cap X_0 = \emptyset$

and 2. $f^{-1}(C)$ is a disjoint

union of elements of \mathcal{D} for
each $C \in \underline{C}$

3. $f|_D$ is a bijection onto
a cell.



$\{\{0\}, \{1\}, (0, 1)\}$ $\{0\}, \{1\}, \{0, 1\}$

For every $D \in \underline{D}$, \mathcal{D} set
of cells whose closure contains D .

and $X(D) = U\tilde{D}$ (similarly for Y).

$$\underline{W} = \{ X(D) \mid D \in \underline{D} \}$$

$$\underline{Y} = \{ Y(C) \mid C \in \underline{C} \}$$

Claim 1: $\underline{W}, \underline{Y}$ covers,

\underline{W} refines \underline{X}

pf: Clearly every element

of $\underline{W}, \underline{Y}$ is def., $U\underline{W} = X$

$U\underline{Y} = X$. Since each $D \in \underline{D}$ is either contained or disjoint from all elements of \underline{X} , $X(D) \subseteq X_0$

for which $D \subseteq X_0$.

$X(D)$ open, take $x \in X(D)$

take U open nbhd of x

that intersects D in min # of

cells. Then $U \cap X(D)$ is

otherwise take $y \in U \setminus X(D)$

then y belongs to some cell

E , the closure ^{\bar{E}} of E is

disjoint from D , then

$U \setminus \bar{E}$ is an

open nbhd of x that
inters

(assume f open) (normality)
claim 2: \forall If $D, D' \in \underline{\mathcal{D}}$

map to the same element $c \in \underline{C}$

then $X(D), X(D')$ disjoint.

PF: Take $U \supseteq D, U' \supseteq D'$

U, U' are disjoint. So

$f(U), f(U')$ subsets of C

If $E \in \tilde{\mathcal{D}}, \tilde{\mathcal{D}}'$, $E \cap U \neq \emptyset$

$E \cap U' \neq \emptyset$ $f|_E$ is not a bijection

claim 3: $f^{-1}(Y(c))$ is

a disjoint union of

\emptyset $X(D)$

for D s.t. $f(D) = c$.

pf: need to show! if

$c' \in \bar{C}$, D' mapping

to it $(f(b') = c')$, then

then is D' in its closure

mapping to C .