♣ VISUALIZING $\mathbb{R}^N$ AND SOME NEW DUALITIES

Alfred Inselberg
School of Mathematical Sciences
Tel Aviv University
Tel Aviv, Israel

With parallel coordinates the perceptual barrier imposed by our 3-dimensional habitation is breached enabling the visualization of multidimensional problems. The highlights, interlaced with interactive demonstrations, are intuitively developed showing how M-dimensional objects are recognized recursively from their $(M-1)$-dimensional subsets. It emerges that *a hyperplane in N-dimensions is represented by $(N-1)$ indexed points.* Points representing lines have two indices, those representing planes in $\mathbb{R}^3$ have three indices and so on. In turn, this yields powerful geometrical algorithms (e.g. for intersections, containment, proximities) and applications including classification.

A smooth surface in 3-D is the envelope of its tangent planes each represented by 2 planar points. As a result it is represented by two planar regions, and a hypersurface in N-dimensions by $(N-1)$ regions. This is equivalent to *representing a surface by its normal vectors.* Developable surfaces are represented by curves revealing the surface characteristics. *Convex surfaces in any dimension* are recognized by hyperbola-like regions. Non-orientable surfaces yield stunning patterns unlocking new geometrical insights. Non-convexities like folds, bumps, concavities are visible. The patterns persist in the presence of errors. Intuition gained from the $\mathbb{R}^3$ representations leads to generalizations for $\mathbb{R}^N$ with beautiful new dualities like **cusp in** $\mathbb{R}^N \leftrightarrow (N-1)$ **"swirls" in** $\mathbb{R}^2$**,** **"twist" in** $\mathbb{R}^N \leftrightarrow (N-1)$ **cusps in** $\mathbb{R}^2$**.** The methodology extends to spaces of dimension $\aleph_0$ and $\aleph_1$.

EYE-CANDY

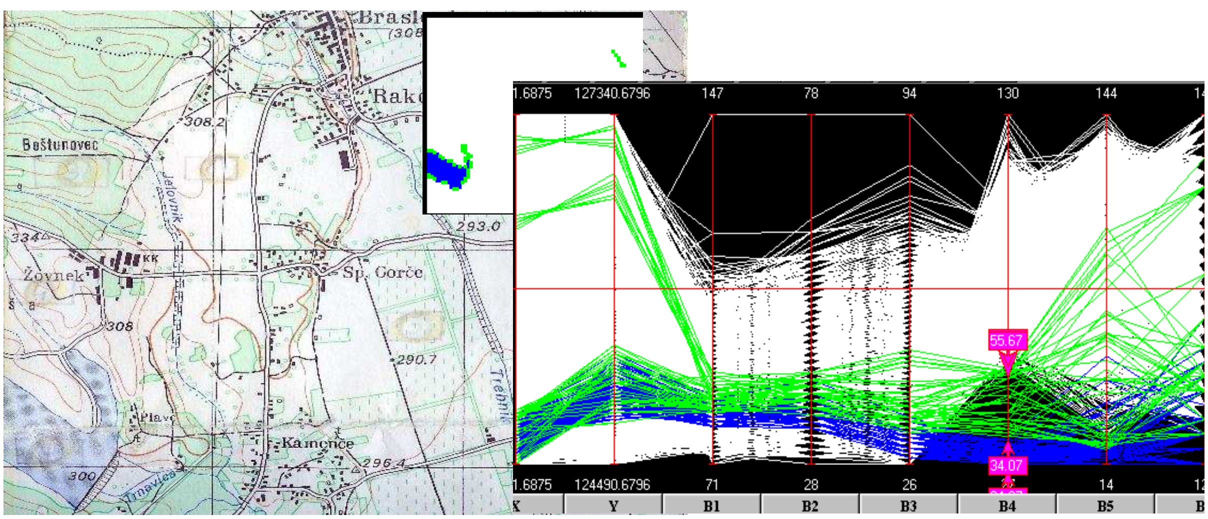

Figure 1: Exploratory Data Analysis, ground emissions measured by satellite over a region (left) are displayed on the right. In the middle, water (in blue) and the lake's edge (in green) are discovered by the indicated queries.

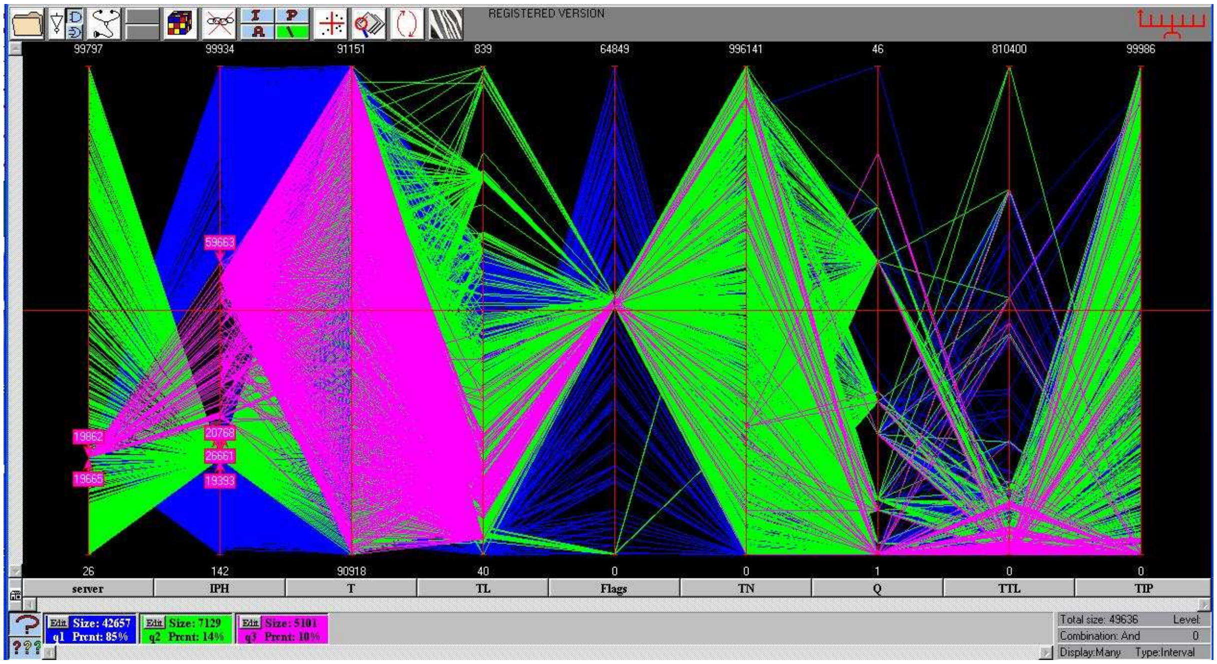

Figure 2: Detecting Network Intrusion from Internet Traffic Flow Data. Note the many-to-one relations.

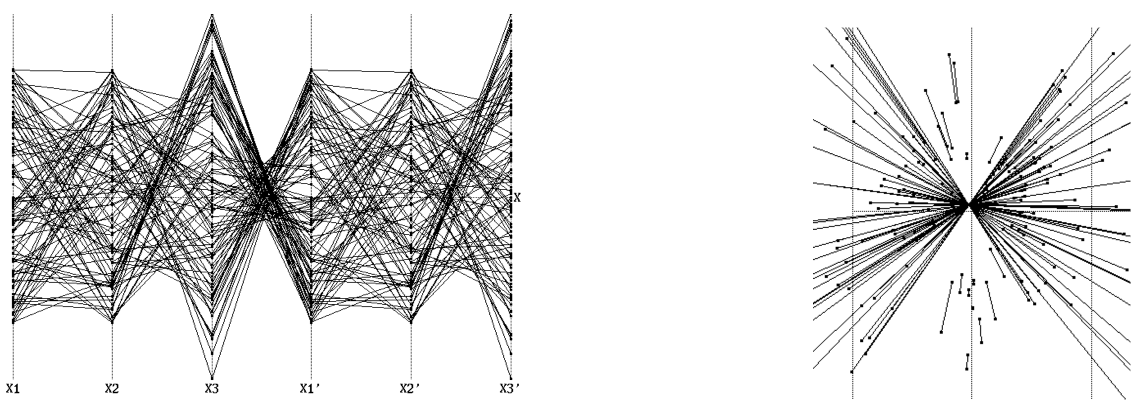

Figure 3: (left) Polygonal lines on the first 3 axes represent randomly chosen coplanar points. There is no discernible pattern. (right) Seeing coplanarity! Two points represent a line which is determined from the intersection (two points) of the corresponding two polygonal lines. **All** straight lines joining these pairs of points intersect. A plane is recognized from the representation of its *lines*. The *recursive* visualization generalizes to higher dimensions.

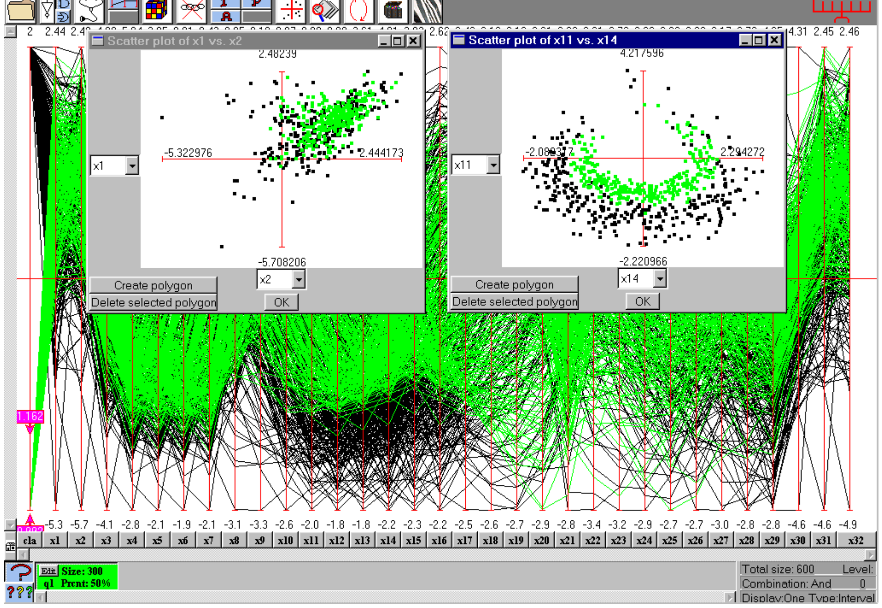

Figure 4: In the background is a dataset with 32 variables and 2 categories. On the left is the plot of the first two variables in the original order, on the right are the best two variables after classification. The algorithms discovers the best 9 variables (features) needed to describe the classification rule, with 4% error, and orders them according to their predictive power.

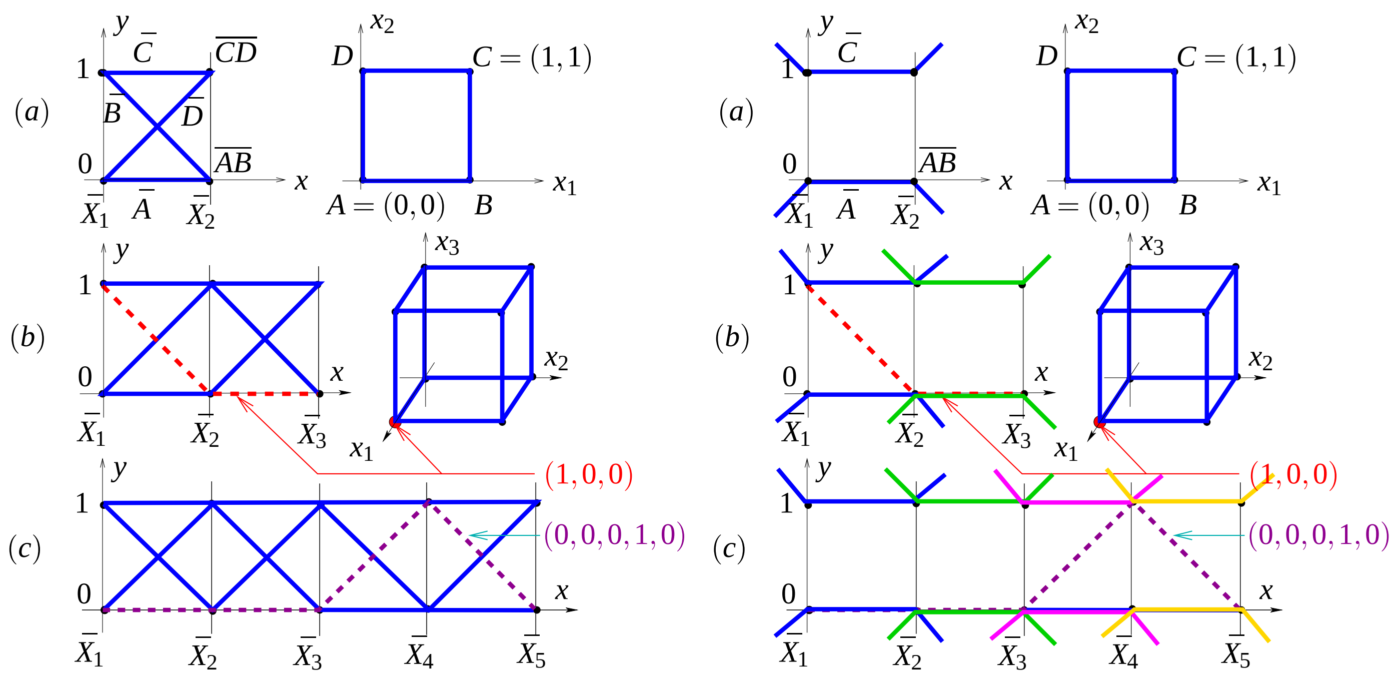

Figure 5: Square, cube and hypercube in 5-D on the left represented by their vertices and on the right by the tangent planes. Note the hyperbola-like (with 2 assymptotes) regions showing that the object is convex.

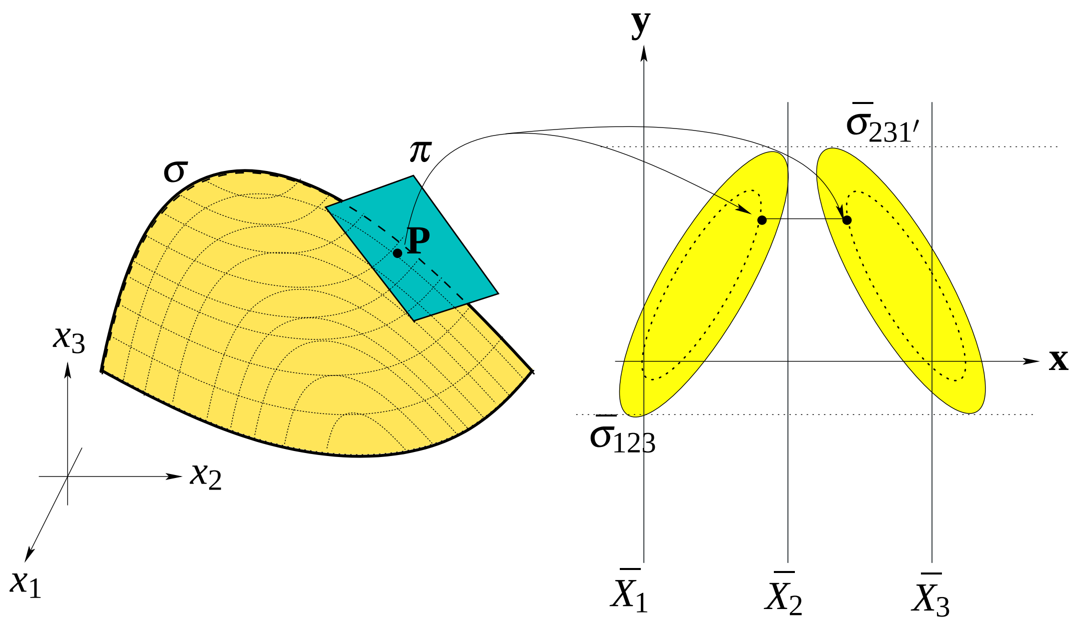

Figure 6: In 3-D a surface σ is represented by two linked planar regions $\bar{\sigma}_{123}$, $\bar{\sigma}_{231'}$. They consist of the pairs of points representing all its tangent planes. In N-dimensions a hypersurface is represented by $(N-1)$ regions as the hypercube above.

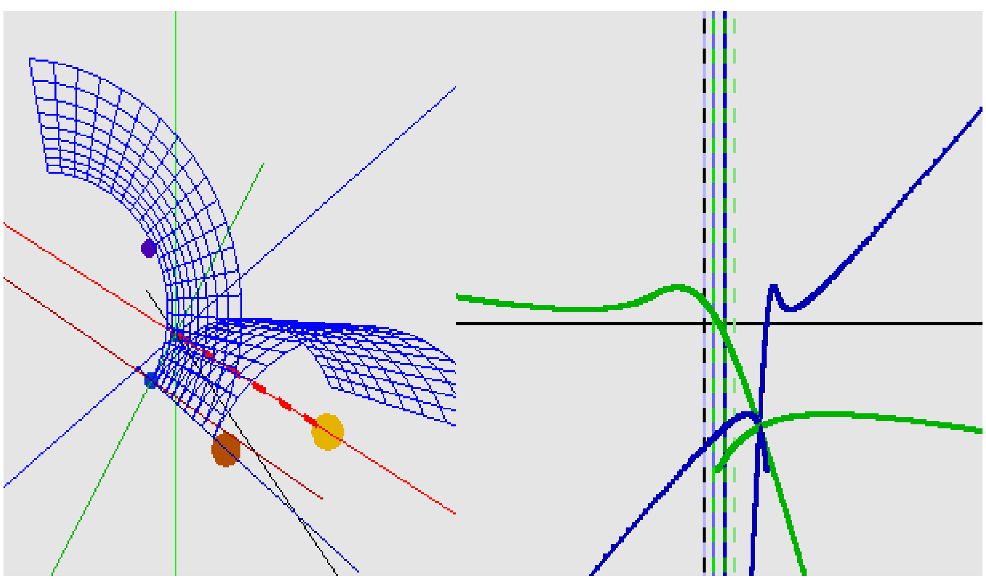

Figure 7: Developable surfaces are represented by curves. Note the two dualities *cusp* $\leftrightarrow$ *inflection point* and *bitangent (tangent at two points) plane* $\leftrightarrow$ *crossing point*. Three such curves represent the corresponding hypersurface in 4-D and so on.

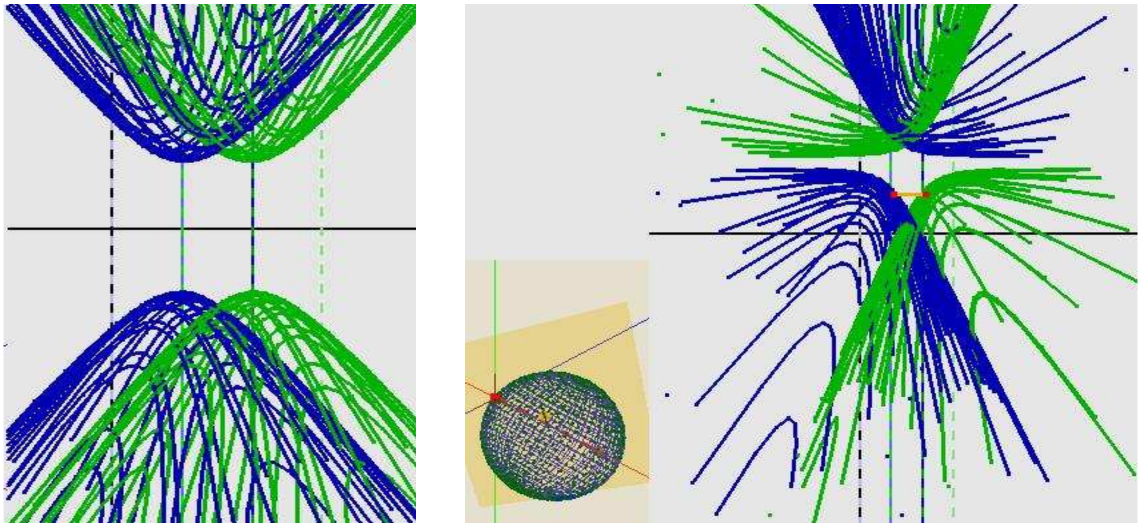

Figure 8: Representation of a sphere centered at the origin (left) and after a translation along the x_1 axis (right) causing the two hyperbolas to rotate in opposite directions illustrating the *rotation* $\leftrightarrow$ *translation* duality. In N-D a sphere is represented by $(N-1)$ such hyperbolic regions — pattern repeats as for the hypercube above.

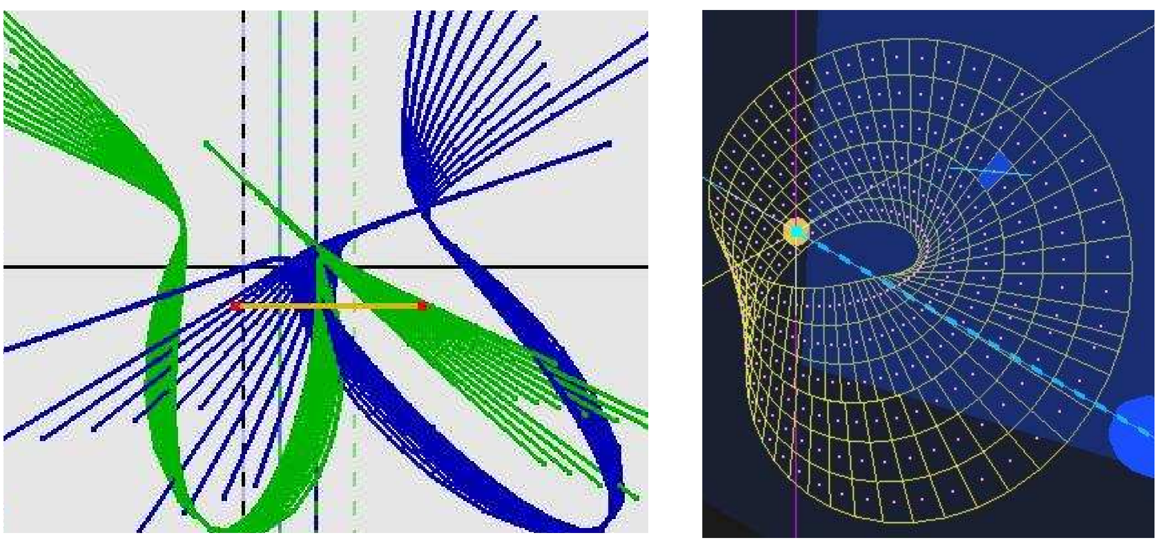

Figure 9: Möbius strip and its representation. Two cusps on the left represent the *twist* as an "inflection-point in 3-D" – see the duality in Fig 7. A tangent plane is represented by the indicated pair of points. The N-dimensional analogue of the of Möbius strip is represented by $(N-1)$ such regions with cusps.

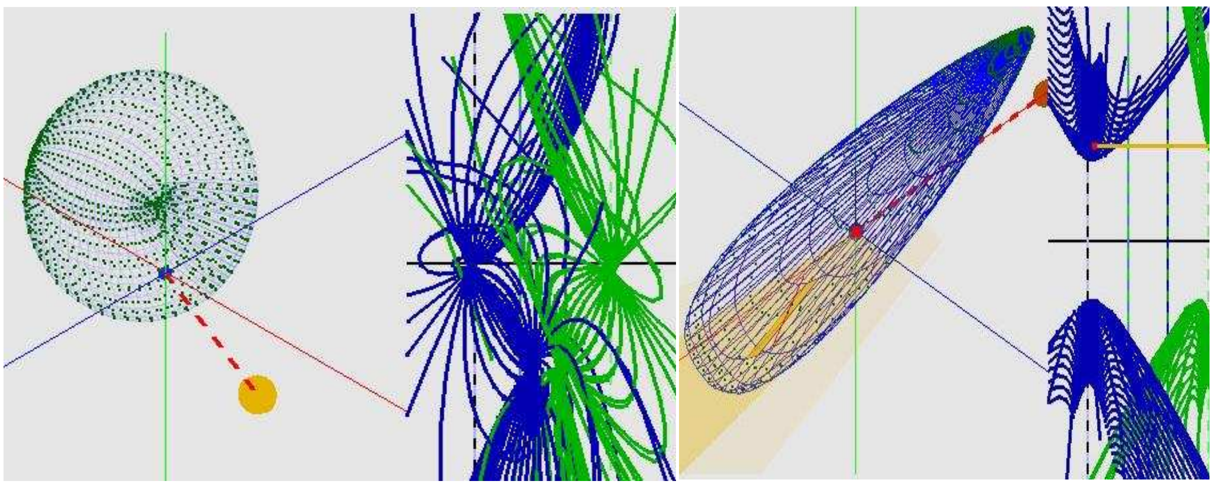

Figure 10: Representation of a surface with two *3-D cusps* – only one is visible in the perspective. Each cusp in 3-D is mapped into a pair of *''swirls''*. The two pairs of swirls in the representation show that the surface has two cusps. On the right is a convex surface and its representation by hyperbola-like regions. In general a convex hypersurface in $\mathbb{R}^N$ is represented by $(N-1)$ hyperbora-like (each having two assymptotes) regions.