Course Notes:

Algebraic Geometry – Schemes 2

Spring 2018-19

Amnon Yekutieli

Contents

1. Review of Prior Material

Lecture 1, 27 Feb 2019

We fix a nonzero commutative base ring $\mathbb{K}$. All rings will be commutative $\mathbb{K}$-rings by default.

Let X be a topological space. Recall that a *presheaf* of $\mathbb{K}$-modules on X is a functor

$$\mathcal{M} : \mathsf{Open}(X)^{\mathrm{op}} \to \mathsf{Mod}\,\mathbb{K},$$

where $\mathsf{Open}(X)$ is the category of open sets of X, and $\mathsf{Mod}\,\mathbb{K}$ is the category of $\mathbb{K}$-modules.

More concretely, the presheaf $\mathcal{M}$ is the data of a $\mathbb{K}$-module $\Gamma(U, \mathcal{M})$ for every open set $U \subseteq X$, and a module homomorphism

$$\mathrm{rest}_{V/U} : \Gamma(U, \mathcal{M}) \to \Gamma(V, \mathcal{M})$$

for every inclusion $V \subseteq U$. The conditions are that

$$\mathrm{rest}_{W/U} = \mathrm{rest}_{W/V} \circ \mathrm{rest}_{V/U}$$

for every double inclusion $W \subseteq V \subseteq U$, and that $\mathrm{rest}_{U/U} = \mathrm{id}$ for every U. We often use the abbreviation

$$m|_V := \mathrm{rest}_{V/U}(m) \in \Gamma(V, \mathcal{M})$$

for a section $m \in \Gamma(U, \mathcal{M})$.

The presehaves of $\mathbb{K}$-modules on X form a category. The morphisms are the obvious ones. We denote it by $\mathsf{Mod}^{\mathrm{pre}}\,\mathbb{K}_X$. Here $\mathbb{K}_X$ is the constant sheaf on X with values in $\mathbb{K}$ (but we didn't define sheaves yet...)

Given a presheaf $\mathcal{M}$ and a point $x \in X$, we have the *stalk* $\mathcal{M}_x$ of $\mathcal{M}$ at x. This is a $\mathbb{K}$-module. Recall the formula:

$$\mathcal{M}_x = \lim_{U \to} \Gamma(U, \mathcal{M}),$$

where the direct limit is on the open neighborhoods U of x. Taking the stalk at x is a functor

$$\mathsf{Mod}^{\mathrm{pre}}\,\mathbb{K}_X \to \mathsf{Mod}\,\mathbb{K}.$$

A presheaf $\mathcal{M}$ is a *sheaf* if it satisfies the two sheaf axioms. These can be encoded as follows: for every open set $U \subseteq X$ and every open covering $U = \bigcup_{i \in I} U_i$, the sequence of $\mathbb{K}$-modules

$$0 \to \Gamma(U, \mathcal{M}) \xrightarrow{\epsilon} \prod_{i_0 \in I} \Gamma(U_{i_0}, \mathcal{M}) \xrightarrow{\delta^0 - \delta^1} \prod_{i_0, i_1 \in I} \Gamma(U_{i_0} \cap U_{i_1}, \mathcal{M})$$

is exact. Here the homomorphisms ϵ, δ^i are induced by the restrictions. For more details see [Ye4, Sec 3 and Prop 7.10].

The sheaves of $\mathbb{K}$-modules on X, also called $\mathbb{K}_X$*-modules*, form a full subcategory of $\mathsf{Mod}^{\mathrm{pre}}\,\mathbb{K}_X$, that we denote by $\mathsf{Mod}\,\mathbb{K}_X$.

The *sheafification functor* assigns to each presheaf $\mathcal{M}$ a sheaf $\mathrm{Sh}(\mathcal{M})$, and a homomorphism of presheaves

$$\tau_{\mathcal{M}} : \mathcal{M} \to \mathrm{Sh}(\mathcal{M}),$$

which is universal for homomorphisms into sheaves. Namely: if $\mathcal{N}$ is a sheaf and $\phi : \mathcal{M} \to \mathcal{N}$ is a homomorphism of presehaves, then there is a unique homomorphism of sheaves $\phi' : \mathrm{Sh}(\mathcal{M}) \to \mathcal{N}$ such that $\phi = \phi' \circ \tau_{\mathcal{M}}$.

Exercise 1.1. Read the proof of the sheafication, including an understanding of the Godement sheaf $\mathrm{GSh}(\mathcal{M})$. This is [Ye4, Thm 6.1]. We will need this next week when we define the structure sheaf of an affine scheme.

Exercise 1.2. State the categorical property of the functor Sh, as an adjoint (from which side?) to the inclusion functor $\mathsf{Mod}\,\mathbb{K}_X \to \mathsf{Mod}^{\mathrm{pre}}\,\mathbb{K}_X$.

The sheafication does not change the stalks: for every $x \in X$ the homomorphism:

$$\tau_{\mathcal{M},x} : \mathcal{M}_x \xrightarrow{\simeq} \mathrm{Sh}(\mathcal{M})_x$$

is bijective.

A *ringed space* over $\mathbb{K}$ is a pair $(X, \mathcal{O}_X)$, consisting of a topological space X, and a sheaf of $\mathbb{K}$-rings $\mathcal{O}_X$ on X.

Let $(X, \mathcal{O}_X)$ be such a ringed space. A sheaf of $\mathcal{O}_X$-modules, also called an *$\mathcal{O}_X$-module*, is a sheaf of $\mathbb{K}$-modules on X, together with a structure of a $\Gamma(U, \mathcal{O}_X)$-module for every open set $U \subseteq X$, which respects restrictions to open subsets.

The category of $\mathcal{O}_X$-modules is denoted by $\mathsf{Mod}\,\mathcal{O}_X$. The morphisms are the obvious ones.

The sheafifcation functor respects the $\mathcal{O}_X$-module structure: if $\mathcal{M} \in \mathsf{Mod}^{\mathrm{pre}}\,\mathcal{O}_X$ then $\mathrm{Sh}(\mathcal{M}) \in \mathsf{Mod}\,\mathcal{O}_X$, and $\tau_{\mathcal{M}}$ is $\mathcal{O}_X$-linear.

Let $\phi : \mathcal{M} \to \mathcal{N}$ be a homomorphism in $\mathsf{Mod}\,\mathcal{O}_X$. Its *kernel* is the $\mathcal{O}_X$-module $\mathrm{Ker}(\phi)$ such that

$$\Gamma(U, \mathrm{Ker}(\phi)) = \mathrm{Ker}\Big(\Gamma(U, \phi) : \Gamma(U, \mathcal{M}) \to \Gamma(U, \mathcal{N})\Big).$$

The *image* of ϕ is the $\mathcal{O}_X$-module

$$\mathrm{Im}(\phi) := \mathrm{Sh}(\mathrm{Im}^{\mathrm{pre}}(\phi)),$$

where $\mathrm{Im}^{\mathrm{pre}}(\phi)$ is the presheaf defined by

$$\Gamma(U, \mathrm{Im}^{\mathrm{pre}}(\phi)) = \mathrm{Im}\Big(\Gamma(U, \phi) : \Gamma(U, \mathcal{M}) \to \Gamma(U, \mathcal{N})\Big).$$

Note that $\mathrm{Ker}(\phi)$ is a subsheaf of $\mathcal{M}$ and $\mathrm{Im}(\phi)$ is a subsheaf of $\mathcal{N}$.

A sequence of homomorphisms

$$\boldsymbol{S} = \Big(\cdots \mathcal{M}^i \xrightarrow{\phi^i} \mathcal{M}^{i+1} \xrightarrow{\phi^{i+1}} \mathcal{M}^{i+1} \cdots\Big)$$

in $\mathsf{Mod}\,\mathcal{O}_X$ is called *exact* if for every point $x \in X$ the sequence of homomorphisms

$$\cdots \mathcal{M}^i_x \xrightarrow{\phi^i_x} \mathcal{M}^{i+1}_x \xrightarrow{\phi^{i+1}_x} \mathcal{M}^{i+1}_x \cdots$$

in $\mathsf{Mod}\,\mathcal{O}_{X,x}$ is exact. We know that $\boldsymbol{S}$ is exact iff for every i there is equality

$$\mathrm{Im}(\phi^{i-1}) = \mathrm{Ker}(\phi^i)$$

of these subsheaves of $\mathcal{M}^i$.

Given an open set $U \subseteq X$ we write $\mathcal{O}_U := \mathcal{O}_X|_U$. Thus $(U, \mathcal{O}_U)$ is also a ringed space. There is a restriction functor

$$\mathsf{Mod}\,\mathcal{O}_X \to \mathsf{Mod}\,\mathcal{O}_U,\ \mathcal{M} \mapsto \mathcal{M}|_U.$$

Sheaves, and homomorphisms between sheaves, can be glued.

Notation: given an open covering $X = \bigcup_{i \in I} U_i$, and indices $i, j, \ldots \in I$, we often write

$$U_{i,j,\ldots} := U_i \cap U_j \cap \cdots . \tag{1.3}$$

Theorem 1.4 (Gluing Sheaf Homomorphisms)**.** *Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{O}_X$-modules, let $X = \bigcup_{i \in I} U_i$ be an open covering, and let*

$$\phi_i : \mathcal{M}|_{U_i} \to \mathcal{N}|_{U_i}$$

be homomorphisms of $\mathcal{O}_{U_i}$-modules satisfying the condition

$$\phi_i|_{U_{i,j}} = \phi_j|_{U_{i,j}} : \mathcal{M}|_{U_{i,j}} \to \mathcal{N}|_{U_{i,j}}.$$

(*This is the* 0*-cocycle condition.*)

Then there is a unique homomorphism of $\mathcal{O}_X$-modules

$$\phi : \mathcal{M} \to \mathcal{N}$$

such that

$$\phi|_{U_i} = \phi_i : \mathcal{M}|_{U_i} \to \mathcal{N}|_{U_i}$$

for all i.

Theorem 1.5 (Gluing Sheaves). *Suppose $X = \bigcup_{i \in I} U_i$ is an open covering. For every i let $\mathcal{M}_i$ be an $\mathcal{O}_{U_i}$-module, and for every i, j let*

$$\phi_{i,j} : \mathcal{M}_i|_{U_{i,j}} \xrightarrow{\simeq} \mathcal{M}_j|_{U_{i,j}}$$

be an isomorphism of $\mathcal{O}_{U_{i,j}}$-modules. The condition is that

$$\phi_{j,k}|_{U_{i,j,k}} \circ \phi_{i,j}|_{U_{i,j,k}} = \phi_{i,k}|_{U_{i,j,k}}$$

as isomorphisms

$$\mathcal{M}_i|_{U_{i,j,k}} \xrightarrow{\simeq} \mathcal{M}_k|_{U_{i,j,k}}$$

for all i, j, k. (This is the 1-cocycle condition.)

Then there is an $\mathcal{O}_X$-module $\mathcal{M}$, together with isomorphisms

$$\phi_i : \mathcal{M}|_{U_i} \xrightarrow{\simeq} \mathcal{M}_i$$

of $\mathcal{O}_X|_{U_i}$-modules, such that

$$\phi_{i,j} \circ \phi_i|_{U_{i,j}} = \phi_j|_{U_{i,j}} : \mathcal{M}|_{U_{i,j}} \xrightarrow{\simeq} \mathcal{M}_j|_{U_{i,j}}.$$

Moreover, the $\mathcal{O}_X$-module $\mathcal{M}$, with the collection of isomorphisms $\{\phi_i\}$, is unique up to a unique isomorphism.

Exercise 1.6. Read and make sure you understand these last two thms. Proofs can be found here: [Ye4, Thm 7.3] and [Ye4, Thm 7.4].

Let $\mathcal{M}, \mathcal{N} \in \mathsf{Mod}\, \mathcal{O}_X$. The $\mathcal{O}_X$-module $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ is defined by

$$\Gamma(U, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})) := \mathrm{Hom}_{\mathsf{Mod}\, \mathcal{O}_X|_U}(\mathcal{M}|_U, \mathcal{N}|_U).$$

The $\mathcal{O}_X$-module $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$ is the sheaf associated to the presheaf

$$U \mapsto \Gamma(U, \mathcal{M}) \otimes_{\Gamma(U, \mathcal{O}_X)} \Gamma(U, \mathcal{N}). \tag{1.7}$$

Exercise 1.8. Find an example of $\mathcal{M}$ and $\mathcal{N}$ such that the presheaf tensor product (1.7) is not a sheaf. (Hint: line bundles on $\mathbf{P}^1$)

Let $f : Y \to X$ be a map of topological spaces. For a $\mathbb{K}_Y$-module $\mathcal{N}$, its pushforward, or *direct image*, is the $\mathbb{K}_X$-module $f_*(\mathcal{N})$ defined by

$$\Gamma(U, f_*(\mathcal{N})) := \Gamma(f^{-1}(U), \mathcal{N})$$

for open sets $U \subseteq X$. We get a functor

$$f_* : \mathsf{Mod}\, \mathbb{K}_Y \to \mathsf{Mod}\, \mathbb{K}_X. \tag{1.9}$$

For a $\mathbb{K}_X$-module $\mathcal{M}$, the pullback, or *inverse image*, is the $\mathbb{K}_Y$-module $f^{-1}(\mathcal{M})$ defined by

$$\Gamma(V, f^{-1}(\mathcal{M})) := \lim_{U \to} \Gamma(U, \mathcal{M}),$$

where $V \subseteq Y$ is open, and U runs over the open sets in X that contain $f(V)$. We get a functor

$$f^{-1} : \mathsf{Mod}\, \mathbb{K}_X \to \mathsf{Mod}\, \mathbb{K}_Y. \tag{1.10}$$

There is adjunction: an isomorphism of $\mathbb{K}$-modules

$$\mathrm{Hom}_{\mathsf{Mod}\, \mathbb{K}_X}(\mathcal{M}, f_*(\mathcal{N})) \cong \mathrm{Hom}_{\mathsf{Mod}\, \mathbb{K}_Y}(f^{-1}(\mathcal{M}), \mathcal{N}) \tag{1.11}$$

which is functorial in $\mathcal{M}$ and $\mathcal{N}$.

Exercise 1.12. Prove the adjunction formula (1.11).

Exercise 1.13. Prove that the functor f^{-1} in (1.10) is exact.

Exercise 1.14. Prove that the functor f_* in (1.9) is left exact. Find and example showing that it is not exact. (Hint: line bundles on $\mathbf{P}^1$)

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be ringed spaces. A map of ringed spaces

$$(f, \psi) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X) \tag{1.15}$$

consists of a map of topological spaces $f : Y \to X$, together with a homomorphism of $\mathbb{K}_X$-rings

$$\psi : \mathcal{O}_X \to f_*(\mathcal{O}_Y). \tag{1.16}$$

We shall often use the notation (f, f^*) instead of (f, ψ). The notation (f, ψ) is common in most textbooks, but it is a bit redundant, so we will only use it when it is need to clarify matters. Note that in many cases (see examples below) the homomorphism f^* is literally pullback of functions; and this makes the notation (f, f^*) good heuristically. On the other hand, we sometimes omit all mention of the structure sheaves, and just talk about a map $f : Y \to X$ of locally ringed spaces.

References

[Gro] A. Grothendieck, Sur quelques points d'algèbre homologique, Tôhoku Math. J. **9** (1957), 119-221.
[Har] R. Hartshorne, "Algebraic Geometry", Springer-Verlag, New-York, 1977.
[HltSt] P.J. Hilton and U. Stammbach, "A Course in Homological Algebra", Springer, 1971.
[Lee] John M. Lee, "Introduction to Smooth Manifolds", LNM **218**, Springer, 2013.
[KaSc] M. Kashiwara and P. Schapira, "Sheaves on manifolds", Springer-Verlag, 1990.
[Mac1] S. Maclane, "Homology", Springer, 1994 (reprint).
[Mac2] S. Maclane, "Categories for the Working Mathematician", Springer, 1978.
[Rot] J. Rotman, "An Introduction to Homological Algebra", Academic Press, 1979.
[Row] L.R. Rowen, "Ring Theory" (Student Edition), Academic Press, 1991.
[We] C. Weibel, "An introduction to homological algebra", Cambridge Studies in Advanced Math. **38**, 1994.
[Ye1] A. Yekutieli, "Derived Categories", prepublication, eprint `https://arxiv.org/abs/1610.09640`.
[Ye2] A. Yekutieli, "Commutative Algebra", Course Notes, `http://www.math.bgu.ac.il/~amyekut/teaching/2017-18/comm-alg/course_page.html`.
[Ye3] A. Yekutieli, "Homological Algebra", Course Notes, `http://www.math.bgu.ac.il/~amyekut/teaching/2017-18/hom-alg/course_page.html`.
[Ye4] A. Yekutieli, "Algebraic Geoemtry – Schemes 1", Course Notes, `https://www.math.bgu.ac.il/~amyekut/teaching/2018-19/schemes-1/course_page.html` .

Department of Mathematics, Ben Gurion University, Be'er Sheva 84105, Israel.
Email: `amyekut@math.bgu.ac.il`, *Web*: `http://www.math.bgu.ac.il/~amyekut`.