Course Notes:

Algebraic Geometry – Schemes 2

Spring 2018-19

Amnon Yekutieli

Contents

1. Review of Prior Material

Lecture 1, 27 Feb 2019

We fix a nonzero commutative base ring $\mathbb{K}$. All rings will be commutative $\mathbb{K}$-rings by default.

Let X be a topological space. Recall that a *presheaf* of $\mathbb{K}$-modules on X is a functor

$$\mathcal{M} : \mathsf{Open}(X)^{\mathrm{op}} \to \mathsf{Mod}\,\mathbb{K},$$

where $\mathsf{Open}(X)$ is the category of open sets of X, and $\mathsf{Mod}\,\mathbb{K}$ is the category of $\mathbb{K}$-modules.

More concretely, the presheaf $\mathcal{M}$ is the data of a $\mathbb{K}$-module $\Gamma(U, \mathcal{M})$ for every open set $U \subseteq X$, and a module homomorphism

$$\mathrm{rest}_{V/U} : \Gamma(U, \mathcal{M}) \to \Gamma(V, \mathcal{M})$$

for every inclusion $V \subseteq U$. The conditions are that

$$\mathrm{rest}_{W/U} = \mathrm{rest}_{W/V} \circ \mathrm{rest}_{V/U}$$

for every double inclusion $W \subseteq V \subseteq U$, and that $\mathrm{rest}_{U/U} = \mathrm{id}$ for every U. We often use the abbreviation

$$m|_V := \mathrm{rest}_{V/U}(m) \in \Gamma(V, \mathcal{M})$$

for a section $m \in \Gamma(U, \mathcal{M})$.

The presehaves of $\mathbb{K}$-modules on X form a category. The morphisms are the obvious ones. We denote it by $\mathsf{Mod}^{\mathrm{pre}}\,\mathbb{K}_X$. Here $\mathbb{K}_X$ is the constant sheaf on X with values in $\mathbb{K}$ (but we didn't define sheaves yet...)

Given a presheaf $\mathcal{M}$ and a point $x \in X$, we have the *stalk* $\mathcal{M}_x$ of $\mathcal{M}$ at x. This is a $\mathbb{K}$-module. Recall the formula:

$$\mathcal{M}_x = \lim_{U \to} \Gamma(U, \mathcal{M}),$$

where the direct limit is on the open neighborhoods U of x. Taking the stalk at x is a functor

$$\mathsf{Mod}^{\mathrm{pre}}\,\mathbb{K}_X \to \mathsf{Mod}\,\mathbb{K}.$$

A presheaf $\mathcal{M}$ is a *sheaf* if it satisfies the two sheaf axioms. These can be encoded as follows: for every open set $U \subseteq X$ and every open covering $U = \bigcup_{i \in I} U_i$, the sequence of $\mathbb{K}$-modules

$$0 \to \Gamma(U, \mathcal{M}) \xrightarrow{\epsilon} \prod_{i_0 \in I} \Gamma(U_{i_0}, \mathcal{M}) \xrightarrow{\delta^0 - \delta^1} \prod_{i_0, i_1 \in I} \Gamma(U_{i_0} \cap U_{i_1}, \mathcal{M})$$

is exact. Here the homomorphisms ϵ, δ^i are induced by the restrictions. For more details see [Ye4, Sec 3 and Prop 7.10].

The sheaves of $\mathbb{K}$-modules on X, also called $\mathbb{K}_X$*-modules*, form a full subcategory of $\mathsf{Mod}^{\mathrm{pre}}\,\mathbb{K}_X$, that we denote by $\mathsf{Mod}\,\mathbb{K}_X$.

The *sheafification functor* assigns to each presheaf $\mathcal{M}$ a sheaf $\mathrm{Sh}(\mathcal{M})$, and a homomorphism of presheaves

$$\tau_{\mathcal{M}} : \mathcal{M} \to \mathrm{Sh}(\mathcal{M}),$$

which is universal for homomorphisms into sheaves. Namely: if $\mathcal{N}$ is a sheaf and $\phi : \mathcal{M} \to \mathcal{N}$ is a homomorphism of presehaves, then there is a unique homomorphism of sheaves $\phi' : \mathrm{Sh}(\mathcal{M}) \to \mathcal{N}$ such that $\phi = \phi' \circ \tau_{\mathcal{M}}$.

Exercise 1.1. Read the proof of the sheafication, including an understanding of the Godement sheaf $\mathrm{GSh}(\mathcal{M})$. This is [Ye4, Thm 6.1]. We will need this next week when we define the structure sheaf of an affine scheme.

Exercise 1.2. State the categorical property of the functor Sh, as an adjoint (from which side?) to the inclusion functor $\mathsf{Mod}\,\mathbb{K}_X \to \mathsf{Mod}^{\mathrm{pre}}\,\mathbb{K}_X$.

The sheafication does not change the stalks: for every $x \in X$ the homomorphism:

$$\tau_{\mathcal{M},x} : \mathcal{M}_x \xrightarrow{\simeq} \mathrm{Sh}(\mathcal{M})_x$$

is bijective.

A *ringed space* over $\mathbb{K}$ is a pair $(X, \mathcal{O}_X)$, consisting of a topological space X, and a sheaf of $\mathbb{K}$-rings $\mathcal{O}_X$ on X.

Let $(X, \mathcal{O}_X)$ be such a ringed space. A sheaf of $\mathcal{O}_X$-modules, also called an $\mathcal{O}_X$*-module*, is a sheaf of $\mathbb{K}$-modules on X, together with a structure of a $\Gamma(U, \mathcal{O}_X)$-module for every open set $U \subseteq X$, which respects restrictions to open subsets.

The category of $\mathcal{O}_X$-modules is denoted by $\mathsf{Mod}\,\mathcal{O}_X$. The morphisms are the obvious ones.

The sheafifcation functor respects the $\mathcal{O}_X$-module structure: if $\mathcal{M} \in \mathsf{Mod}^{\mathrm{pre}}\,\mathcal{O}_X$ then $\mathrm{Sh}(\mathcal{M}) \in \mathsf{Mod}\,\mathcal{O}_X$, and $\tau_{\mathcal{M}}$ is $\mathcal{O}_X$-linear.

Let $\phi : \mathcal{M} \to \mathcal{N}$ be a homomorphism in $\mathsf{Mod}\,\mathcal{O}_X$. Its *kernel* is the $\mathcal{O}_X$-module $\mathrm{Ker}(\phi)$ such that

$$\Gamma(U, \mathrm{Ker}(\phi)) = \mathrm{Ker}\Big(\Gamma(U, \phi) : \Gamma(U, \mathcal{M}) \to \Gamma(U, \mathcal{N})\Big).$$

The *image* of ϕ is the $\mathcal{O}_X$-module

$$\mathrm{Im}(\phi) := \mathrm{Sh}(\mathrm{Im}^{\mathrm{pre}}(\phi)),$$

where $\mathrm{Im}^{\mathrm{pre}}(\phi)$ is the presheaf defined by

$$\Gamma(U, \mathrm{Im}^{\mathrm{pre}}(\phi)) = \mathrm{Im}\Big(\Gamma(U, \phi) : \Gamma(U, \mathcal{M}) \to \Gamma(U, \mathcal{N})\Big).$$

Note that $\mathrm{Ker}(\phi)$ is a subsheaf of $\mathcal{M}$ and $\mathrm{Im}(\phi)$ is a subsheaf of $\mathcal{N}$.

A sequence of homomorphisms

$$\boldsymbol{S} = \Big(\cdots \mathcal{M}^i \xrightarrow{\phi^i} \mathcal{M}^{i+1} \xrightarrow{\phi^{i+1}} \mathcal{M}^{i+1} \cdots\Big)$$

in $\mathsf{Mod}\,\mathcal{O}_X$ is called *exact* if for every point $x \in X$ the sequence of homomorphisms

$$\cdots \mathcal{M}^i_x \xrightarrow{\phi^i_x} \mathcal{M}^{i+1}_x \xrightarrow{\phi^{i+1}_x} \mathcal{M}^{i+1}_x \cdots$$

in $\mathsf{Mod}\,\mathcal{O}_{X,x}$ is exact. We know that $\boldsymbol{S}$ is exact iff for every i there is equality

$$\mathrm{Im}(\phi^{i-1}) = \mathrm{Ker}(\phi^i)$$

of these subsheaves of $\mathcal{M}^i$.

Given an open set $U \subseteq X$ we write $\mathcal{O}_U := \mathcal{O}_X|_U$. Thus $(U, \mathcal{O}_U)$ is also a ringed space. There is a restriction functor

$$\mathsf{Mod}\,\mathcal{O}_X \to \mathsf{Mod}\,\mathcal{O}_U,\ \mathcal{M} \mapsto \mathcal{M}|_U.$$

Sheaves, and homomorphisms between sheaves, can be glued.

Notation: given an open covering $X = \bigcup_{i \in I} U_i$, and indices $i, j, \ldots \in I$, we often write

$$U_{i,j,\ldots} := U_i \cap U_j \cap \cdots . \tag{1.3}$$

Theorem 1.4 (Gluing Sheaf Homomorphisms)**.** *Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{O}_X$-modules, let $X = \bigcup_{i \in I} U_i$ be an open covering, and let*

$$\phi_i : \mathcal{M}|_{U_i} \to \mathcal{N}|_{U_i}$$

be homomorphisms of $\mathcal{O}_{U_i}$-modules satisfying the condition

$$\phi_i|_{U_{i,j}} = \phi_j|_{U_{i,j}} : \mathcal{M}|_{U_{i,j}} \to \mathcal{N}|_{U_{i,j}}.$$

(*This is the* 0*-cocycle condition.*)

Then there is a unique homomorphism of $\mathcal{O}_X$-modules

$$\phi : \mathcal{M} \to \mathcal{N}$$

such that

$$\phi|_{U_i} = \phi_i : \mathcal{M}|_{U_i} \to \mathcal{N}|_{U_i}$$

for all i.

Theorem 1.5 (Gluing Sheaves)**.** *Suppose $X = \bigcup_{i \in I} U_i$ is an open covering. For every i let $\mathcal{M}_i$ be an $\mathcal{O}_{U_i}$-module, and for every i, j let*

$$\phi_{i,j} : \mathcal{M}_i|_{U_{i,j}} \xrightarrow{\simeq} \mathcal{M}_j|_{U_{i,j}}$$

be an isomorphism of $\mathcal{O}_{U_{i,j}}$-modules. The condition is that

$$\phi_{j,k}|_{U_{i,j,k}} \circ \phi_{i,j}|_{U_{i,j,k}} = \phi_{i,k}|_{U_{i,j,k}}$$

as isomorphisms

$$\mathcal{M}_i|_{U_{i,j,k}} \xrightarrow{\simeq} \mathcal{M}_k|_{U_{i,j,k}}$$

for all i, j, k. (This is the 1-cocycle condition.)

Then there is an $\mathcal{O}_X$-module $\mathcal{M}$, together with isomorphisms

$$\phi_i : \mathcal{M}|_{U_i} \xrightarrow{\simeq} \mathcal{M}_i$$

of $\mathcal{O}_X|_{U_i}$-modules, such that

$$\phi_{i,j} \circ \phi_i|_{U_{i,j}} = \phi_j|_{U_{i,j}} : \mathcal{M}|_{U_{i,j}} \xrightarrow{\simeq} \mathcal{M}_j|_{U_{i,j}}.$$

Moreover, the $\mathcal{O}_X$-module $\mathcal{M}$, with the collection of isomorphisms $\{\phi_i\}$, is unique up to a unique isomorphism.

Exercise 1.6. Read and make sure you understand these last two thms. Proofs can be found here: [Ye4, Thm 7.3] and [Ye4, Thm 7.4].

Let $\mathcal{M}, \mathcal{N} \in \mathsf{Mod}\, \mathcal{O}_X$. The $\mathcal{O}_X$-module $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ is defined by

$$\Gamma(U, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})) := \mathrm{Hom}_{\mathsf{Mod}\, \mathcal{O}_X|_U}(\mathcal{M}|_U, \mathcal{N}|_U).$$

The $\mathcal{O}_X$-module $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$ is the sheaf associated to the presheaf

$$U \mapsto \Gamma(U, \mathcal{M}) \otimes_{\Gamma(U, \mathcal{O}_X)} \Gamma(U, \mathcal{N}). \tag{1.7}$$

Exercise 1.8. Find an example of $\mathcal{M}$ and $\mathcal{N}$ such that the presheaf tensor product (1.7) is not a sheaf. (Hint: line bundles on $\mathbf{P}^1$)

Let $f : Y \to X$ be a map of topological spaces. For a $\mathbb{K}_Y$-module $\mathcal{N}$, its pushforward, or *direct image*, is the $\mathbb{K}_X$-module $f_*(\mathcal{N})$ defined by

$$\Gamma(U, f_*(\mathcal{N})) := \Gamma(f^{-1}(U), \mathcal{N})$$

for open sets $U \subseteq X$. We get a functor

$$f_* : \mathsf{Mod}\, \mathbb{K}_Y \to \mathsf{Mod}\, \mathbb{K}_X. \tag{1.9}$$

For a $\mathbb{K}_X$-module $\mathcal{M}$, the pullback, or *inverse image*, is the $\mathbb{K}_Y$-module $f^{-1}(\mathcal{M})$ defined by

$$\Gamma(V, f^{-1}(\mathcal{M})) := \lim_{U \to} \Gamma(U, \mathcal{M}),$$

where $V \subseteq Y$ is open, and U runs over the open sets in X that contain $f(V)$. We get a functor

$$f^{-1} : \mathsf{Mod}\, \mathbb{K}_X \to \mathsf{Mod}\, \mathbb{K}_Y. \tag{1.10}$$

There is adjunction: an isomorphism of $\mathbb{K}$-modules

$$\mathrm{Hom}_{\mathsf{Mod}\, \mathbb{K}_X}(\mathcal{M}, f_*(\mathcal{N})) \cong \mathrm{Hom}_{\mathsf{Mod}\, \mathbb{K}_Y}(f^{-1}(\mathcal{M}), \mathcal{N}) \tag{1.11}$$

which is functorial in $\mathcal{M}$ and $\mathcal{N}$.

Exercise 1.12. Prove the adjunction formula (1.11).

Exercise 1.13. Prove that the functor f^{-1} in (1.10) is exact.

Exercise 1.14. Prove that the functor f_* in (1.9) is left exact. Find and example showing that it is not exact. (Hint: line bundles on $\mathbf{P}^1$)

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be ringed spaces. A map of ringed spaces

$$(f, \psi) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X) \tag{1.15}$$

consists of a map of topological spaces $f : Y \to X$, together with a homomorphism of $\mathbb{K}_X$-rings

$$\psi : \mathcal{O}_X \to f_*(\mathcal{O}_Y). \tag{1.16}$$

We shall often use the notation (f, f^*) instead of (f, ψ). The notation (f, ψ) is common in most textbooks, but it is a bit redundant, so we will only use it when it is need to clarify matters. Note that in many cases (see examples below) the homomorphism f^* is literally pullback of functions; and this makes the notation (f, f^*) good heuristically. On the other hand, we sometimes omit all mention of the structure sheaves, and just talk about a map $f : Y \to X$ of locally ringed spaces.

Lecture 2, 6 Mar 2019

Given a maps (1.15) of ringed spaces, the direct image is a functor

$$f_* : \mathsf{Mod}\, \mathcal{O}_Y \to \mathsf{Mod}\, \mathcal{O}_X.$$

There is another kind of inverse image here: for $\mathcal{M} \in \mathsf{Mod}\, \mathcal{O}_X$ we define $f^*(\mathcal{M}) \in \mathsf{Mod}\, \mathcal{O}_Y$ by

$$f^*(\mathcal{M}) := \mathcal{O}_Y \otimes_{f^{-1}(\mathcal{O}_X)} f^{-1}(\mathcal{M}).$$

Again there is adjunction:

$$\mathrm{Hom}_{\mathsf{Mod}\, \mathcal{O}_X}(\mathcal{M}, f_*(\mathcal{N})) \cong \mathrm{Hom}_{\mathsf{Mod}\, \mathcal{O}_Y}(f^*(\mathcal{M}), \mathcal{N}).$$

Definition 1.17. A *locally ringed* $\mathbb{K}$*-space* is a ringed $\mathbb{K}$-space $(X, \mathcal{O}_X)$, such that for every point $x \in X$ the stalk $\mathcal{O}_{X,x}$ is a local ring. The maximal ideal of $\mathcal{O}_{X,x}$ is denoted by $\mathfrak{m}_x$, and the residue field is denoted by $\boldsymbol{k}(x)$.

Definition 1.18. If $(Y, \mathcal{O}_Y)$ is another locally ringed space, then a map of locally ringed spaces

$$(f, \psi) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$$

is a map of ringed spaces, such that for every point $y \in Y$, with image $x := f(y) \in X$, the induced $\mathbb{K}$-ring homomorphism

$$\psi_x : \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$$

is a local homomorphism, namely $\psi_x(\mathfrak{m}_x) \subseteq \mathfrak{m}_y$.

The category of locally ringed $\mathbb{K}$-spaces is denoted by $\mathsf{LRSp}/\mathbb{K}$.

Here are three types of locally ringed spaces.

Example 1.19. The category Top of topological spaces. The base ring is $\mathbb{K} = \mathbb{R}$. A space $X \in \mathsf{Top}$ is made into a ringed space by putting on it the sheaf $\mathcal{O}_X$ of continous $\mathbb{R}$-valued functions (for the metric topology of $\mathbb{R}$). Then $(X, \mathcal{O}_X)$ belongs to $\mathsf{LRSp}/\mathbb{R}$. Moreover, the functor $\mathsf{Top} \to \mathsf{LRSp}/\mathbb{R}$ is fully faithful.

Example 1.20. The category Mfld of differentiable (of type C^∞) real manifolds. The base ring is $\mathbb{K} = \mathbb{R}$. A manifold $X \in \mathsf{Mfld}$ is made into a ringed space by putting on it the sheaf $\mathcal{O}_X$ of differentiable $\mathbb{R}$-valued functions. Then $(X, \mathcal{O}_X)$ belongs to $\mathsf{LRSp}/\mathbb{R}$. Moreover, the functor $\mathsf{Mfld} \to \mathsf{LRSp}/\mathbb{R}$ is fully faithful.

Example 1.21. Let $\mathbb{K}$ be an algebraically closed field, and let Var be the category of algebraic varieties over $\mathbb{K}$. Here $\mathcal{O}_X$ is the sheaf of algebraic $\mathbb{K}$-valued functions. Then $(X, \mathcal{O}_X)$ belongs to $\mathsf{LRSp}/\mathbb{K}$. Moreover, the functor $\mathsf{Var} \to \mathsf{LRSp}/\mathbb{K}$ is fully faithful.

Given a locally ringed space $(X, \mathcal{O}_X)$ and an open subset $U \subseteq X$, the pair $(U, \mathcal{O}_X|_U)$ is a locally ringed space. We call it an *open subspace* of X. The inclusion map

$$(U, \mathcal{O}_X|_U) \to (X, \mathcal{O}_X)$$

is called an *open embedding*.

Just like sheaves, locally ringed spaces and maps between them can be glued.

Theorem 1.22 (Gluing Maps of LR Spaces)**.** *Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be objects of $\mathsf{LRSp}/\mathbb{K}$, let $Y = \bigcup_{i \in I} V_i$ be an open covering, and for every i let*

$$(f_i, \psi_i) : (V_i, \mathcal{O}_Y|_{V_i}) \to (X, \mathcal{O}_X)$$

be a map in $\mathsf{LRSp}/\mathbb{K}$.

We assume that this condition holds: for every $i, j \in I$ there is equality

$$(f_i, \psi_i)|_{V_{i,j}} = (f_j, \psi_j)|_{V_{i,j}}$$

of maps

$$(V_{i,j}, \mathcal{O}_Y|_{V_{i,j}}) \to (X, \mathcal{O}_X)$$

in $\mathsf{LRSp}/\mathbb{K}$.

Then there is a unique map

$$(f, \psi) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$$

in $\mathsf{LRSp}/\mathbb{K}$ such that

$$(f, \psi)|_{V_i} = (f_i, \psi_i)$$

for every i.

See picture in figure 1.

Proof. The existence and uniqueness of a map of topological spaces $f : Y \to X$ satisfying $f|_{V_i} = f_i$ is clear.

We need to produce the homomorphism of sheaves of rings

$$\psi : \mathcal{O}_X \to f_*(\mathcal{O}_Y)$$

on X. By the adjunction formula (1.11), this amounts to producing a homomorphism of sheaves of rings

$$\psi : f^{-1}(\mathcal{O}_X) \to \mathcal{O}_Y \tag{1.23}$$

on Y. Now we are given homomorphisms

$$\psi_i : f^{-1}(\mathcal{O}_X)|_{V_i} \to \mathcal{O}_Y|_{V_i}$$

that agree on double intersections. According to Theorem 1.4 these can be glued uniquely to a homomorphism of sheaves of $\mathbb{K}$-modules ψ as in (1.23), such that $\psi|_{V_i} = \psi_i$. This is a homomorphism of sheaves of rings, because this propeorty can be checked locally. Also on stalks it is a local homomorphism. So (f, ψ) is a map of locally ringed spaces. □

fig-1.jpg

FIGURE 1. Gluing maps of spaces

Theorem 1.24 (Gluing LR Spaces)**.** *Let* $\{(U_i, \mathcal{O}_{U_i})\}_{i \in I}$ *be a collection of objects of* $\mathsf{LRSp}/\mathbb{K}$. *For every* $i, j \in I$ *there is an open subset* $U_{i,j} \subseteq U_i$, *and an isomorphism*

$$(f_{i,j}, \psi_{i,j}) : (U_{i,j}, \mathcal{O}_{U_i}|_{U_{i,j}}) \xrightarrow{\simeq} (U_{j,i}, \mathcal{O}_{U_j}|_{U_{j,i}})$$

in $\mathsf{LRSp}/\mathbb{K}$.

These conditions hold:

(a) *For every* i *there are equalities* $U_{i,i} = U_i$ *and* $(f_{i,i}, \psi_{i,i}) = \mathrm{id}$.

(b) *For every* i, j, k *there are equalities*

$$f_{i,j}(U_{i,j} \cap U_{i,k}) = U_{j,i} \cap U_{j,k}$$

of subsets of U_i, *and*

$$(f_{j,k}, \psi_{j,k}) \circ (f_{i,j}, \psi_{i,j}) = (f_{i,k}, \psi_{i,k})$$

of isomorphisms

$$(U_{i,j} \cap U_{i,k}, \mathcal{O}_{U_i}|_{U_{i,j} \cap U_{i,k}}) \to (U_{k,i} \cap U_{k,j}, \mathcal{O}_{U_k}|_{U_{k,i} \cap U_{k,j}})$$

in $\mathsf{LRSp}/\mathbb{K}$.

Then there is an object $(X, \mathcal{O}_X)$ *in* $\mathsf{LRSp}/\mathbb{K}$, *with open embeddings*

$$(f_i, \psi_i) : (U_i, \mathcal{O}_{U_i}) \to (X, \mathcal{O}_X),$$

such that

$$(f_j, \psi_j) \circ (f_{i,j}, \psi_{i,j}) = (f_i, \psi_i)$$

as morphisms

$$(U_{i,j}, \mathcal{O}_{U_i}|_{U_{i,j}}) \to (X, \mathcal{O}_X),$$

and such that

$$X = \bigcup_{i \in I} f_i(U_i).$$

Moreover, the space $(X, \mathcal{O}_X)$, *with the collection of morphisms* $\{(f_i, \psi_i)\}_{i \in I}$, *are unique up to a unique isomorphism.*

fig-4.jpg

Figure 2. Gluing spaces: the input

fig-5.jpg

Figure 3. Gluing spaces: the output

See figures 2 and 3 for an illustration.

The data

$$\Big(\big\{(U_i, \mathcal{O}_{U_i})\big\}_{i \in I},\ \big\{(f_{i,j}, \psi_{i,j})\big\}_{i,j \in I}\Big) \tag{1.25}$$

is called *gluing data* or *descent data*..

Proof. This is done in a few steps.

Step 1. We define the set X. Consider the disjoint union $U := \coprod_{i \in I} U_i$. We define a relation $\sim$ on this set as follows: two points $x, y \in U$ are in the relation $x \sim y$ if there are $i, j \in I$

such that $x \in U_{i,j} \subseteq U_i$, $y \in U_{j,i} \subseteq U_j$, and $f_{i,j}(x) = y$. This is an equivalence relation (exercise). We let $X := U/{\sim}$, the quotient set. The canonical surjection is $\pi : U \to X$.

For each i there a map $f_i : U_i \to X$, gotten by composing the inclusion $U_i \subseteq U$ with π. The map f_i is injective (exercise). We identify U_i with its image $f_i(U_i) \subseteq X$. With this identification, there is equality $U_{i,j} = U_{j,i}$ of subsets of X.

Step 2. We put a topology on X. The disjoint union U gets the disjoint union topology. Then we put on X the quotient topology relative to the surjection $\pi : U \to X$.

Each U_i is an open set of X, and so $X = \bigcup_i U_i$ is an open covering (exercise).

Step 3. Now we construct the sheaf of rings $\mathcal{O}_X$. On each open set $U_i \subseteq X$ we have a sheaf of rings $\mathcal{O}_{U_i}$. On double intersections we have isomorphisms of sheaves of rings

$$\psi_{i,j} : \mathcal{O}_{U_i}|_{U_{i,j}} \xrightarrow{\simeq} \mathcal{O}_{U_j}|_{U_{i,j}},$$

and these agree on triple intersections. By Theorem 1.5 we get a sheaf $\mathcal{O}_X$ on X, with isomorphisms of sheaves

$$\psi_i : \mathcal{O}_{U_i} \xrightarrow{\simeq} \mathcal{O}_X|_{U_i}$$

such that $\psi_j \circ \psi_{i,j} = \psi_i$ on $U_{i,j}$. We see that $\mathcal{O}_X$ is a sheaf of rings, and the stalks $\mathcal{O}_{X,x}$ are local rings. So $(X, \mathcal{O}_X)$ is a locally ringed $\mathbb{K}$-space.

By construction there are open embeddings

$$(f_i, \psi_i) : (U_i, \mathcal{O}_{U_i}) \to (X, \mathcal{O}_X)$$

satisfying the required compatibility.

Step 4. The uniqueness is due to Theorem 1.22. □

Exercise 1.26. Finish the proof of the theorem.

Exercise 1.27. Find an example of a collection of spaces as in the theorem, such that all the spaces U_i are separated topological spaces (i.e. Hausdorff), yet X is not separetd.

Exercise 1.28. Suppose that the spaces and the gluing data are in Mfld (Example 1.20), the indexing set I is countable, and the topological space X is Hausdorff. Prove that $(X, \mathcal{O}_X)$ is an object of Mfld.

Remark 1.29. The gluing in Thm 1.24 will be used to construct fiber products of schemes. Then the fiber products will allow us to define the notion of a *separated map of schemes*. This is the relative and generalized variant of the Hausdorff condition.

2. Affine Schemes

Let A be a $\mathbb{K}$-ring. The *prime spectrum* of A is the set $\operatorname{Spec}(A)$ of prime ideals of A. As we all know, this is a topological space with the *Zariski topology*. By definition the closed sets of $\operatorname{Spec}(A)$ are the sets

$$\mathrm{Z}(\mathfrak{a}) := \{\mathfrak{p} \in \operatorname{Spec}(A) \mid \mathfrak{a} \subseteq \mathfrak{p}\},$$

where $\mathfrak{a}$ is some ideal in A.

We sometimes refer to $\mathrm{Z}(\mathfrak{a})$ as the *zero locus* of the ideal $\mathfrak{a}$. Here is the reason: to a prime ideal $\mathfrak{p}$ we associate the local ring $A_{\mathfrak{p}}$ and the residue field

$$\boldsymbol{k}(\mathfrak{p}) := A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}.$$

An element $a \in A$ has a class $a(\mathfrak{p}) \in \boldsymbol{k}(\mathfrak{p})$, coming from the canonical ring homomorphism $A \to \boldsymbol{k}(\mathfrak{p})$. Then

$$\mathrm{Z}(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid a(\mathfrak{p}) = 0 \text{ for all } a \in \mathfrak{a}\}. \tag{2.1}$$

Exercise 2.2. Prove the formula above.

For an element $s \in A$ we define

$$\mathrm{D}(s) = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid s \notin \mathfrak{p}\}. \tag{2.3}$$

This is an open set: it it the complement of the closed set $\mathrm{Z}(\mathfrak{a})$, where $\mathfrak{a} := (s)$, the principal ideal generated by s. We call such an open set a *principal open set*. Analogously to (2.1) we have

$$\mathrm{D}(s) = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid s(\mathfrak{p}) \neq 0\}. \tag{2.4}$$

Proposition 2.5. *The principal open sets are a basis of the topology of* $\operatorname{Spec}(A)$. *Namely every open set set U is a union $U = \bigcup_i \mathrm{D}(s_i)$ for a suitable collection $\{s_i\}$ of elements of A.*

Exercise 2.6. Prove the proposition above.

Definition 2.7. Let A be a ring, and write $X := \operatorname{Spec}(A)$ for this topological space. For an open set $U \subseteq X$ we let $\mathrm{S}(U) \subseteq A$ be the multiplicatively closed set

$$\mathrm{S}(U) := \{s \in A \mid s(\mathfrak{p}) \neq 0 \ \text{ for all } \ \mathfrak{p} \in U\}.$$

Let $A_{\mathrm{S}(U)}$ be the localization of A w.r.t. $\mathrm{S}(U)$.

Lemma 2.8. *The assignment $U \mapsto \mathrm{S}(U)$ is a presheaf of rings on $X = \operatorname{Spec}(A)$, that we denote by $\mathcal{O}^{\mathrm{pre}}$.*

Proof. This is easy: if $V \subseteq U$ is a smaller open set, then $\mathrm{S}(U) \subseteq \mathrm{S}(V)$, so by the universal property of localization there is a unique A-ring homomorphism $A_{\mathrm{S}(U)} \to A_{\mathrm{S}(V)}$. □

Definition 2.9. Let A be a ring. The *structure sheaf* of $\operatorname{Spec}(A)$ is the sheaf of rings

$$\mathcal{O}_{\operatorname{Spec}(A)} := \operatorname{Sh}(\mathcal{O}^{\mathrm{pre}}).$$

Lecture 3, 13 Mar 2019

By construction, $\mathcal{O}^{\mathrm{pre}}$ is a presheafs of A-rings on $X := \operatorname{Spec}(A)$, and $\mathcal{O}_X$ is a sheaf of A-rings on X.

Proposition 2.10. *Let A be a ring and write $X := \operatorname{Spec}(A)$. For every point $x = \mathfrak{p} \in X$ the stalk of $\mathcal{O}_X$ at x is $\mathcal{O}_{X,x} = A_{\mathfrak{p}}$, the local ring at $\mathfrak{p}$. More precisely, there is a unique A-ring isomorphism $\mathcal{O}_{X,x} \cong A_{\mathfrak{p}}$.*

Proof. Let $\mathrm{S}(x) := A - \mathfrak{p}$, the complement of $\mathfrak{p}$. By definition we have $A_{\mathfrak{p}} = A_{\mathrm{S}(x)}$. The universal property of localization says that there is at most one A-ring isomorphism $A_{\mathfrak{p}} \xrightarrow{\simeq} \mathcal{O}_{X,x}$. We will produce it.

Let us denote by $\boldsymbol{U}_x$ the set of principal open neighborhoods of x. These are the open sets $U = \mathrm{D}(s)$ for some $s \in \mathrm{S}(x)$. By Proposition 2.5, $\boldsymbol{U}_x$ is a basis of open neighborhoods of x. Because the stalks of the presheaf $\mathcal{O}^{\mathrm{pre}}$ and its associated sheaf $\mathcal{O}$ are the same, we have

$$\mathcal{O}_{X,x} = \lim_{U \to} \Gamma(U, \mathcal{O}^{\mathrm{pre}}) = \lim_{U \to} A_{\mathrm{S}(U)} \tag{2.11}$$

where U runs over $\boldsymbol{U}_x$.

For each $U = \mathrm{D}(s) \in \boldsymbol{U}_x$ we have $\mathrm{S}(U) \subseteq \mathrm{S}(x)$, so there is a unique A-ring homomorphism $A_{\mathrm{S}(U)} \to A_{\mathrm{S}(x)} = A_{\mathfrak{p}}$. Going to the limit in U we get an A-ring homomorphism

$$\psi : \mathcal{O}_{X,x} \to A_{\mathfrak{p}}.$$

In the other direction, every element $s \in \mathrm{S}(x)$ belongs to $\mathrm{S}(U)$ for $U := \mathrm{D}(s) \in \boldsymbol{U}_x$, and hence s is invertible in $A_{\mathrm{S}(U)}$. This means that every $s \in \mathrm{S}(x)$ is invertible in the A-ring $\mathcal{O}_{X,x}$. Hence there is a unique A-ring homomorphism

$$\phi : A_{\mathfrak{p}} = A_{\mathrm{S}(x)} \to \mathcal{O}_{X,x}.$$

The homomorphism ϕ is surjective, because every $f \in \mathcal{O}_{X,x}$ is the image of some fraction $a/s \in A_{\mathrm{S}(U)}$, see (2.11). But $s \in \mathrm{S}(x)$, so $a/s \in A_{\mathfrak{p}}$ and $f = \phi(a/s)$.

Finally consider the A-ring homomorphism $\psi \circ \phi$ from $A_{\mathfrak{p}}$ to itself. By uniqueness there is equality $\psi \circ \phi = \mathrm{id}_{A_{\mathfrak{p}}}$. This shows that ϕ is injective. In conclusion, ϕ is an isomorphism of A-rings. □

Corollary 2.12. $\big(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)}\big)$ *is a locally ringed space.*

Definition 2.13. An *affine* $\mathbb{K}$*-scheme* is a locally ringed space $(X, \mathcal{O}_X) \in \mathsf{LRSp}/\mathbb{K}$ which is isomorphic to $\big(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)}\big)$ for some $\mathbb{K}$-ring A.

We now provide a more explicit description of the structur sheaf $\mathcal{O}_{\mathrm{Spec}(A)}$ in terms of its *Godement sheaf* from [Ye4, Sec 6]. Recall that for a presheaf of $\mathbb{K}$-modules $\mathcal{M}$ on a space X, its Godement sheaf $\mathrm{GSh}(\mathcal{M})$ is defined by

$$\Gamma(U, \mathrm{GSh}(\mathcal{M})) := \prod_{x \in U} \mathcal{M}_x$$

for an open set $U \subseteq X$. Then $\mathrm{Sh}(\mathcal{M}) \subseteq \mathrm{GSh}(\mathcal{M})$ is the subsheaf of *geometric sections*. Here is what this means. There is a canonical homomorphism of presehaves $\mathcal{M} \to \mathrm{GSh}(\mathcal{M})$. For an open set $U \subseteq X$, a section

$$m \in \Gamma(U, \mathrm{GSh}(\mathcal{M}))$$

is called geometric if for every point $x \in U$ there is an open set V such that $x \in V \subseteq U$, and a section $m' \in \Gamma(V, \mathcal{M})$, such that

$$m|_V = m' \in \Gamma(V, \mathrm{GSh}(\mathcal{M})).$$

Specializing to our case it says that following:

Proposition 2.14. *Let* $(X, \mathcal{O}_X) := \big(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)}\big)$. *Then* $\mathcal{O}_X$ *is the subsheaf of* $\mathrm{GSh}(\mathcal{O}^{\mathrm{pre}})$ *consisting of the geometric sections. Specifically, let* $U \subseteq X$ *be an open set, and let*

$$f \in \Gamma(U, \mathrm{GSh}(\mathcal{O}^{\mathrm{pre}})) = \prod_{\mathfrak{p} \in U} A_{\mathfrak{p}}.$$

The section f *belongs to* $\Gamma(U, \mathcal{O}_X)$ *iff for every point* $x = \mathfrak{p} \in U$ *there is an open set* V *such that* $x \in V \subseteq U$*, and elements* $a \in A$ *and* $s \in \mathrm{S}(V)$*, such that*

$$f|_V = a/s \in \Gamma(V, \mathrm{GSh}(\mathcal{O}^{\mathrm{pre}})) = \prod_{\mathfrak{q} \in V} A_{\mathfrak{q}}.$$

See Figure 4.

Here are some example of affine schemes.

Example 2.15. Consider the ring $A = \mathbb{Z}$. The affine scheme $X := \mathrm{Spec}(\mathbb{Z})$ has these points: for every (positive) prime number p there is a maximal ideal $\mathfrak{m} := (p)$. These are closed points of X, since

$$\mathrm{Z}(p) = \{\mathfrak{m}\}.$$

The local ring is

$$\mathbb{Z}_{(p)} = \{a/s \mid s \notin (p)\} \subseteq \mathbb{Q}.$$

The residue field is $\mathbb{F}_p$.

The zero ideal $\mathfrak{p} := (0)$ is also prime. Is is the *generic point* of X; namely its topological closure is X. The local ring and the residue field at $\mathfrak{p}$ are $\mathbb{Q}$.

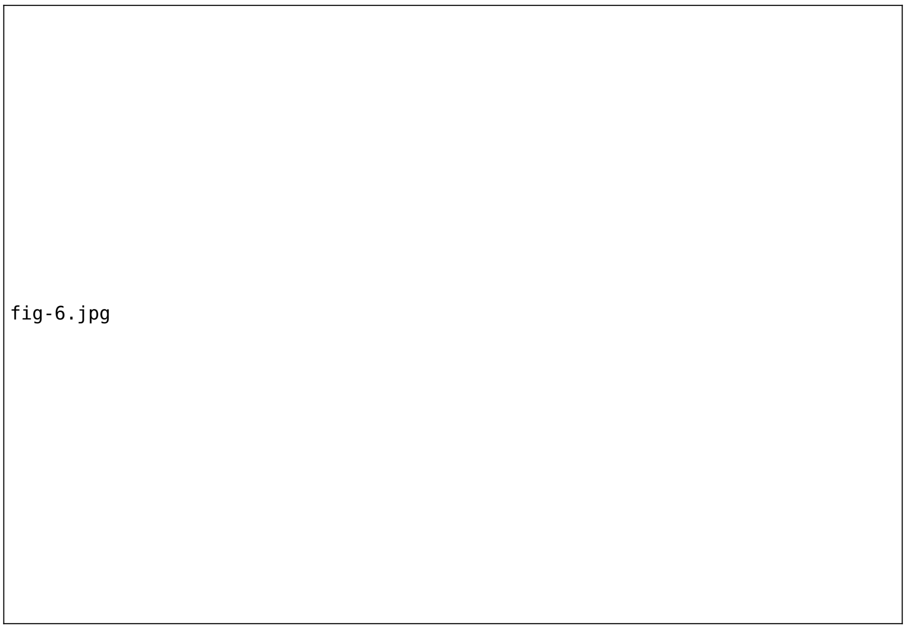

FIGURE 4. Picture for Prop 2.14

Exercise 2.16. Analyze the affine scheme $\operatorname{Spec}(A)$ for the ring $A := \mathbb{K}[t]$, the polynomial ring in one variable over a field $\mathbb{K}$.

Example 2.17. Suppose $\mathbb{K}$ is a nonzero ring. For every $n \geq 0$ there is an affine scheme

$$\mathbf{A}^n_{\mathbb{K}} := \operatorname{Spec}(\mathbb{K}[t_1, \ldots, t_n])$$

called the *n dimensional affine space over* $\mathbb{K}$.

Recall that for an ideal $\mathfrak{a} \subseteq A$, its zero locus is

$$\mathrm{Z}(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid \mathfrak{a} \subseteq \mathfrak{p}\}.$$

The *radical* of $\mathfrak{a}$ is the ideal

$$\sqrt{\mathfrak{a}} := \{a \in A \mid a^i \in \mathfrak{a} \text{ for some } i > 0\} \subseteq A.$$

Lemma 2.18. *Let $\mathfrak{a} \subseteq A$ be an ideal. Then*

$$\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{p} \in \mathrm{Z}(\mathfrak{a})} \mathfrak{p}$$

and

$$\mathrm{Z}(\mathfrak{a}) = \mathrm{Z}(\sqrt{\mathfrak{a}}).$$

Exercise 2.19. Prove the lemma. (Hint: do not use the Nullstellensatz.)

Lemma 2.20. *For ideals $\mathfrak{a}, \mathfrak{b} \subseteq A$ the following are equivalent*:

(i) $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$.
(ii) $\mathrm{Z}(\mathfrak{a}) = \mathrm{Z}(\mathfrak{b})$.

Proof. Clear from Lemma 2.18. □

Definition 2.21. A topological space X is called *quasi-compact* if every open covering of X has a finite subcovering.

Remark 2.22. The term "compact" is usually reserved for spaces that are Hausdorff and quasi-compact. Schemes are almost never Hausdorff. There is an analogous notion of separation, that we will study later. Cf. Exercise 1.27 and Rem 1.29.

Proposition 2.23. *Let A be a ring. The topological space $X := \operatorname{Spec}(A)$ is quasi-compact.*

Proof. Let $X = \bigcup_{i \in I} U_i$ be an open covering. For each i there is an ideal $\mathfrak{a}_i$ such that $U_i = X - \mathrm{Z}(\mathfrak{a}_i)$. Write $\mathfrak{a} := \sum_i \mathfrak{a}_i$. Then there is equality

$$\mathrm{Z}(A) = \varnothing = \bigcap_{i \in I} \mathrm{Z}(\mathfrak{a}_i) = \mathrm{Z}(\mathfrak{a})$$

of subsets of X. By Lemma 2.20 we know that $\sqrt{A} = \sqrt{\mathfrak{a}}$. Since $1 \in A$, we see that $1 \in \sqrt{\mathfrak{a}}$, and hence $1 \in \mathfrak{a}$. This says that we can express 1 as a finite sum: $1 = \sum_{i \in I'} a_i$ with $I' \subseteq I$ a finite subset and $a_i \in \mathfrak{a}_i$. We see that $A = \sum_{i \in I'} \mathfrak{a}_i$, and therefore

$$\varnothing = \mathrm{Z}(A) = \bigcap_{i \in I'} \mathrm{Z}(\mathfrak{a}_i)$$

and $X = \bigcup_{i \in I'} U_i$. □

Lemma 2.24. *Let $s_1, \ldots, s_m \in A$. TFAE:*

(i) $\operatorname{Spec}(A) = \bigcup_i \mathrm{D}(s_i)$.

(ii) *There exists $a_1, \ldots, a_m \in A$ s.t. $1_A = \sum_i a_i \cdot s_i$.*

Proof.

(i) ⇒ (ii): As in the proof of the proposition, $A = \sum_i \mathfrak{a}_i$, where $\mathfrak{a}_i := (s_i)$. So the element 1_A is a linear combination $1_A = \sum_i a_i \cdot s_i$ for some $a_i \in A$.

(ii) ⇒ (i): Here $1_A \in \sum_i \mathfrak{a}_i$, so $A = \sum_i \mathfrak{a}_i$, and, as in the proof of the proposition, $\operatorname{Spec}(A) = \bigcup_i \mathrm{D}(s_i)$. □

By construction, for every $s \in A$ the element s is invertible in the rings $\Gamma(\mathrm{D}(s), \mathcal{O}^{\mathrm{pre}})$ and $\Gamma(\mathrm{D}(s), \mathcal{O}_X)$.

Lemma 2.25. *Let $(X, \mathcal{O}_X) := \big(\operatorname{Spec}(A), \mathcal{O}_{\operatorname{Spec}(A)}\big)$. Suppose $U \subseteq X$ is an open set, $f \in \Gamma(U, \mathcal{O}_X)$, and $x = \mathfrak{p} \in U$. Then there are elements $a, s \in A$ such that $x \in \mathrm{D}(s) \subseteq U$ and*

$$f|_{\mathrm{D}(s)} = a/s \in \Gamma(\mathrm{D}(s), \mathcal{O}_X).$$

Proof. By Proposition 2.14 the point $x = \mathfrak{p} \in U$ has an open neighborhood $V \subseteq U$, and elements $b, t \in A$, such that $t \in \mathrm{S}(V)$ and $f|_V = b/t \in \Gamma(V, \mathcal{O}_X)$.

According to Proposition 2.5 we can replace V with a smaller open neighborhood $\mathrm{D}(r)$ for some $r \in A$, i.e. $x \in \mathrm{D}(r) \subseteq V$.

Define the elements $a := b \cdot r \in A$ and $s := t \cdot r \in A$, and the open set $W := \mathrm{D}(s)$. Since $\mathrm{D}(r) \subseteq V \subseteq \mathrm{D}(t)$, it follows that

$$\mathrm{D}(r) = \mathrm{D}(t \cdot r) = \mathrm{D}(s) = W.$$

Also

$$a/s = (b \cdot r)/(t \cdot r) = f|_W \in \Gamma(W, \mathcal{O}_X).$$

□

Lemma 2.26. *Let $(X, \mathcal{O}_X) := \big(\operatorname{Spec}(A), \mathcal{O}_{\operatorname{Spec}(A)}\big)$. Suppose $V \subseteq X$ is a quasi-compact open set, and $f \in \Gamma(V, \mathcal{O}_X)$. Then there are finitely many elements $a_1, \ldots, a_m, s_1, \ldots, s_m \in A$ such that $V = \bigcup_{i=1}^m \mathrm{D}(s_i)$, and*

$$f|_{\mathrm{D}(s_i)} = a_i/s_i \in \Gamma(\mathrm{D}(s_i), \mathcal{O}_X)$$

for every i.

Proof. By Lemma 2.25, for every point $x = \mathfrak{p} \in V$ there are elements $a_x, s_x \in A$ such that $x \in \mathrm{D}(s_x) \subseteq V$ and

$$f|_{\mathrm{D}(s_x)} = a_x / s_x \in \Gamma(\mathrm{D}(s_x), \mathcal{O}_X).$$

We have an open covering $V = \bigcup_{x \in V} \mathrm{D}(s_x)$. Because of quasi-compactness we can pass to a finite subcovering, that is to a finite subset $\{x_1, \ldots, x_m\}$ of V. Finally, by letting $a_i := a_{x_i}$ and $s_i := s_{x_i}$ we are done. □

A ring homomorphism $\psi : A \to B$ induces a map of sets

$$\mathrm{Spec}(\psi) : \mathrm{Spec}(B) \to \mathrm{Spec}(A) \tag{2.27}$$

with formula

$$\mathrm{Spec}(\psi)(\mathfrak{q}) := \psi^{-1}(\mathfrak{q}).$$

Lemma 2.28. *Let $\psi : A \to B$ be a ring homomorphism.*

(1) *The map* $\mathrm{Spec}(\psi)$ *is continuous.*
(2) *Suppose $B = A_s$ for some $s \in A$. Then the image of* $\mathrm{Spec}(\psi)$ *is* $\mathrm{D}(s)$*, and*

$$\mathrm{Spec}(\psi) : \mathrm{Spec}(A_s) \to \mathrm{D}(s)$$

is a homeomorphism.

Exercise 2.29. Prove the last lemma.

Lecture 4, 27 Mar 2019

Theorem 2.30. *Let A be a ring and $s \in A$ an element. Consider the affine scheme $(X, \mathcal{O}_X) := \big(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)}\big)$. There is a unique A-ring isomorphism*

$$A_s \cong \Gamma(\mathrm{D}(s), \mathcal{O}_X).$$

Proof.

Step 1. Since the element s is invertible in the ring $\Gamma(\mathrm{D}(s), \mathcal{O}_X)$, there is a unique A-ring homomorphism

$$\phi : A_s \to \Gamma(\mathrm{D}(s), \mathcal{O}_X). \tag{2.31}$$

We need to prove that ϕ is bijective.

Step 2. In this step we we will prove that ϕ is injective. Let's write $U := \mathrm{D}(s)$. Since $\mathcal{O}_X$ is a subsheaf of the Godement sheaf $\mathrm{GSh}(\mathcal{O}_X)$, there is an embedding of A-rings

$$\Gamma(U, \mathcal{O}_X) \rightarrowtail \Gamma\big(U, \mathrm{GSh}(\mathcal{O}_X)\big) = \prod_{\mathfrak{p} \in U} A_{\mathfrak{p}}.$$

It thus suffices to prove that the homomorphism

$$\phi' : A_s \to \prod_{\mathfrak{p} \in U} A_{\mathfrak{p}} \tag{2.32}$$

is injective.

Let's write $B := A_s$. By Lemma 2.28 we know that $\mathrm{Spec}(B) = U$ as topological subspaces of X. For every $\mathfrak{p} \in U$ the element s is invertible in $A_{\mathfrak{p}}$, and hence the homomorphism $A_{\mathfrak{p}} \to B_{\mathfrak{p}}$ is bijective. So we can rewrite (2.32) as $\phi' : B \to \prod_{\mathfrak{p} \in U} B_{\mathfrak{p}}$, and we need to prove it is injective.

Suppose $b \in B$ is such that $\phi'(b) = 0$. Then $\phi'_{\mathfrak{p}}(b) = 0$ in $B_{\mathfrak{p}}$ for all $\mathfrak{p} \in U$. This means that there is some element $t_{\mathfrak{p}} \in B$ such that $\mathfrak{p} \in \mathrm{D}(t_{\mathfrak{p}})$ and $t_{\mathfrak{p}} \cdot b = 0$ in B. Now $U = \mathrm{Spec}(B) = \bigcup_{\mathfrak{p} \in U} \mathrm{D}(t_{\mathfrak{p}})$.

By quasi-compactness of U we can pass to a finite subcovering, indexed by $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}$. Let $t_i := t_{\mathfrak{p}_i} \in B$. Then $t_i \cdot b = 0$ in B, and $\operatorname{Spec}(B) = \bigcup_{i=1}^m \mathrm{D}(t_i)$. According to Lemma 2.24 as in the proof of Prop 2.23, there are elements $c_1, \ldots, c_m \in B$ such that $1_B = \sum_i c_i \cdot t_i$. Then $b = 1_B \cdot b = \sum_i c_i \cdot t_i \cdot b = 0$.

Step 3. We now prove that the homomorphism ϕ from (2.31) is surjective. As before let $U := \mathrm{D}(s)$. Take an element $f \in \Gamma(U, \mathcal{O}_X)$. We know that $U = \operatorname{Spec}(B)$ is a quasi-compact topological space. By Lem 2.26 there are finitely many elements $a_i, s_i \in A$, $1 \leq i \leq m$, such that $U = \bigcup_{i=1}^m \mathrm{D}(s_i)$, and $f|_{\mathrm{D}(s_i)} = a_i / s_i \in \Gamma(\mathrm{D}(s_i), \mathcal{O}_X)$ for every i.

Step 2, when applied to the element $s_i \cdot s_j \cdot s \in A$ instead of to s, shows that for every i, j the A-ring homomorphism

$$\phi_{i,j} : A_{s_i \cdot s_j \cdot s} \to \Gamma\big(\mathrm{D}(s_i \cdot s_j \cdot s), \mathcal{O}_X\big)$$

is injective. Recall that $B = A_s$, so that $A_{s_i \cdot s_j \cdot s} = B_{s_i \cdot s_j}$. We know that

$$f|_{\mathrm{D}(s_i \cdot s_j \cdot s)} = a_i \cdot s_i^{-1} = a_j \cdot s_j^{-1} \in \Gamma\big(\mathrm{D}(s_i \cdot s_j \cdot s), \mathcal{O}_X\big).$$

Therefore $a_i \cdot s_i^{-1} = a_j \cdot s_j^{-1}$ in $B_{s_i \cdot s_j}$. The kernel of the localization homomorphism $B \to B_{s_i \cdot s_j}$ is known: there is a positive integer $l_{i,j}$ such that

$$(s_i \cdot s_j)^{l_{i,j}} \cdot (a_i \cdot s_j - a_j \cdot s_i) = 0$$

in B. Taking $l := \max(\{l_{i,j}\})$ we obtain

$$(s_i \cdot s_j)^{l} \cdot (a_i \cdot s_j - a_j \cdot s_i) = 0 \tag{2.33}$$

in B.

Define $b_i := a_i \cdot s_i^l$ and $t_i := s_i^{l+1}$. Then $\mathrm{D}(t_i) = \mathrm{D}(s_i)$ and

$$f|_{\mathrm{D}(t_i)} = a_i \cdot s_i^{-1} = b_i \cdot t_i^{-1} \in \Gamma(\mathrm{D}(t_i), \mathcal{O}_X). \tag{2.34}$$

Also, from (2.33) we have

$$t_i \cdot b_j = t_j \cdot b_i \tag{2.35}$$

in B.

Since

$$\operatorname{Spec}(B) = \bigcup_{i=1}^m \mathrm{D}(t_i), \tag{2.36}$$

by Lem 2.24 we can find elements $c_1, \ldots, c_m \in B$ such that $1_B = \sum_i c_i \cdot t_i$. Let

$$b := \sum_i c_i \cdot b_i \in B.$$

For every i we have – using (2.35) and (2.34) –

$$\phi(b)|_{\mathrm{D}(t_i)} = \sum_j c_j \cdot b_j = t_i^{-1} \cdot \Big(\sum_j c_j \cdot t_i \cdot b_j\Big) = t_i^{-1} \cdot \Big(\sum_j c_j \cdot t_j\Big) \cdot b_i = t_i^{-1} \cdot b_i = f|_{\mathrm{D}(t_i)}$$

in $\Gamma(\mathrm{D}(t_i), \mathcal{O}_X)$. But by (2.36) and the sheaf axioms this implies that $\phi(b) = f$. □

Definition 2.37. The category of *affine* $\mathbb{K}$*-schemes*, denoted by $\mathsf{Sch}_{\mathrm{aff}}/\mathbb{K}$, is the full subcategory of $\mathsf{LRSp}/\mathbb{K}$ on the affine schemes (see Definition 2.13).

Let $\phi : A \to B$ be a ring homomorphism. For a prime ideal $\mathfrak{q} \in B$, the ideal $\mathfrak{p} := \phi^{-1}(\mathfrak{q}) \subseteq A$ is prime, and there is an induced local homomorphism of local rings $\phi_{\mathfrak{q}} : A_{\mathfrak{p}} \to B_{\mathfrak{q}}$.

Theorem 2.38. *A ring homomorphism $\phi : A \to B$ induces a unique map of affine schemes*

$$(f, \tilde{\phi}) : \big(\mathrm{Spec}(B), \mathcal{O}_{\mathrm{Spec}(B)}\big) \to \big(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)}\big)$$

such that on points $f(\mathfrak{q}) = \phi^{-1}(\mathfrak{q})$, and on local rings $\tilde{\phi}_{\mathfrak{q}} = \phi_{\mathfrak{q}} : A_{f(\mathfrak{q})} \to B_{\mathfrak{q}}$.

The proof will be done next week.

Recall that $\mathsf{Rng}/\mathbb{K}$ is the catgory of (commutative) $\mathbb{K}$-rings.

Proposition 2.39. *The assignment that sends a ring A to the affine scheme $\big(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)}\big)$, and a ring homomorphism $\phi : A \to B$ to the map of affine schemes $(f, \tilde{\phi})$ in the theorem above, is a functor*

$$\mathrm{Spec} : \mathsf{Rng}/\mathbb{K} \to \mathsf{Sch}_{\mathrm{aff}}/\mathbb{K}$$

Proposition 2.40. *The assignment that sends an affine scheme $(X, \mathcal{O}_X)$ to the ring $\Gamma(X, \mathcal{O}_X)$, and a map of affine schemes $(f, \tilde{\phi})$ to the ring homomorphism $\Gamma(X, \phi)$, is a functor*

$$\Gamma : \mathsf{Sch}_{\mathrm{aff}}/\mathbb{K} \to \mathsf{Rng}/\mathbb{K}.$$

Exercise 2.41. Prove propositions 2.39 and 2.40.

The global sections functor makes sense for any locally ringed space $(X, \mathcal{O}_X)$.

Theorem 2.42. *The functor*

$$\mathrm{Spec} : \mathsf{Rng}/\mathbb{K} \to \mathsf{LRS}/\mathbb{K}$$

is right adjoint to the functor

$$\Gamma : \mathsf{LRS}/\mathbb{K} \to \mathsf{Rng}/\mathbb{K}.$$

The proof will be done next week.

Corollary 2.43. *The functor*

$$\mathrm{Spec} : \mathsf{Rng}/\mathbb{K} \to \mathsf{Sch}_{\mathrm{aff}}/\mathbb{K}$$

is an equivalence of categories, with quasi-inverse Γ.

References

[Gro] A. Grothendieck, Sur quelques points d'algèbre homologique, Tôhoku Math. J. **9** (1957), 119-221.
[Har] R. Hartshorne, "Algebraic Geometry", Springer-Verlag, New-York, 1977.
[HltSt] P.J. Hilton and U. Stammbach, "A Course in Homological Algebra", Springer, 1971.
[Lee] John M. Lee, "Introduction to Smooth Manifolds", LNM **218**, Springer, 2013.
[KaSc] M. Kashiwara and P. Schapira, "Sheaves on manifolds", Springer-Verlag, 1990.
[Mac1] S. Maclane, "Homology", Springer, 1994 (reprint).
[Mac2] S. Maclane, "Categories for the Working Mathematician", Springer, 1978.
[Rot] J. Rotman, "An Introduction to Homological Algebra", Academic Press, 1979.
[Row] L.R. Rowen, "Ring Theory" (Student Edition), Academic Press, 1991.
[We] C. Weibel, "An introduction to homological algebra", Cambridge Studies in Advanced Math. **38**, 1994.
[Ye1] A. Yekutieli, "Derived Categories", prepublication, eprint `https://arxiv.org/abs/1610.09640`.
[Ye2] A. Yekutieli, "Commutative Algebra", Course Notes, `http://www.math.bgu.ac.il/~amyekut/teaching/2017-18/comm-alg/course_page.html`.
[Ye3] A. Yekutieli, "Homological Algebra", Course Notes, `http://www.math.bgu.ac.il/~amyekut/teaching/2017-18/hom-alg/course_page.html`.
[Ye4] A. Yekutieli, "Algebraic Geoemtry – Schemes 1", Course Notes, `https://www.math.bgu.ac.il/~amyekut/teaching/2018-19/schemes-1/course_page.html` .

Department of Mathematics, Ben Gurion University, Be'er Sheva 84105, Israel.
Email: `amyekut@math.bgu.ac.il`, *Web*: `http://www.math.bgu.ac.il/~amyekut`.