Course Notes:

Algebraic Geometry – Schemes 2

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CONTENTS

1. Review of Prior Material

Lecture 1, 27 Feb 2019

We fix a nonzero commutative base ring $\mathbb{K}$. All rings will be commutative $\mathbb{K}$-rings by default.

Let X be a topological space. Recall that a *presheaf* of $\mathbb{K}$-modules on X is a functor

$$\mathcal{M} : \mathsf{Open}(X)^{\mathrm{op}} \to \mathsf{Mod}\, \mathbb{K},$$

where $\mathsf{Open}(X)$ is the category of open sets of X, and $\mathsf{Mod}\, \mathbb{K}$ is the category of $\mathbb{K}$-modules.

More concretely, the presheaf $\mathcal{M}$ is the data of a $\mathbb{K}$-module $\Gamma(U, \mathcal{M})$ for every open set $U \subseteq X$, and a module homomorphism

$$\mathrm{rest}_{V/U} : \Gamma(U, \mathcal{M}) \to \Gamma(V, \mathcal{M})$$

for every inclusion $V \subseteq U$. The conditions are that

$$\mathrm{rest}_{W/U} = \mathrm{rest}_{W/V} \circ \mathrm{rest}_{V/U}$$

for every double inclusion $W \subseteq V \subseteq U$, and that $\mathrm{rest}_{U/U} = \mathrm{id}$ for every U. We often use the abbreviation

$$m|_V := \mathrm{rest}_{V/U}(m) \in \Gamma(V, \mathcal{M})$$

for a section $m \in \Gamma(U, \mathcal{M})$.

The presehaves of $\mathbb{K}$-modules on X form a category. The morphisms are the obvious ones. We denote it by $\mathsf{Mod}^{\mathrm{pre}}\, \mathbb{K}_X$. Here $\mathbb{K}_X$ is the constant sheaf on X with values in $\mathbb{K}$ (but we didn't define sheaves yet...)

Given a presheaf $\mathcal{M}$ and a point $x \in X$, we have the *stalk* $\mathcal{M}_x$ of $\mathcal{M}$ at x. This is a $\mathbb{K}$-module. Recall the formula:

$$\mathcal{M}_x = \lim_{U \to} \Gamma(U, \mathcal{M}),$$

where the direct limit is on the open neighborhoods U of x. Taking the stalk at x is a functor

$$\mathsf{Mod}^{\mathrm{pre}}\, \mathbb{K}_X \to \mathsf{Mod}\, \mathbb{K}.$$

A presheaf $\mathcal{M}$ is a *sheaf* if it satisfies the two sheaf axioms. These can be encoded as follows: for every open set $U \subseteq X$ and every open covering $U = \bigcup_{i \in I} U_i$, the sequence of $\mathbb{K}$-modules

$$0 \to \Gamma(U, \mathcal{M}) \xrightarrow{\epsilon} \prod_{i_0 \in I} \Gamma(U_{i_0}, \mathcal{M}) \xrightarrow{\delta^0 - \delta^1} \prod_{i_0, i_1 \in I} \Gamma(U_{i_0} \cap U_{i_1}, \mathcal{M})$$

is exact. Here the homomorphisms ϵ, δ^i are induced by the restrictions. For more details see [Ye4, Sec 3 and Prop 7.10].

The sheaves of $\mathbb{K}$-modules on X, also called *$\mathbb{K}_X$-modules*, form a full subcategory of $\mathsf{Mod}^{\mathrm{pre}}\, \mathbb{K}_X$, that we denote by $\mathsf{Mod}\, \mathbb{K}_X$.

The *sheafification functor* assigns to each presheaf $\mathcal{M}$ a sheaf $\mathrm{Sh}(\mathcal{M})$, and a homomorphism of presheaves

$$\tau_{\mathcal{M}} : \mathcal{M} \to \mathrm{Sh}(\mathcal{M}),$$

which is universal for homomorphisms into sheaves. Namely: if $\mathcal{N}$ is a sheaf and $\phi : \mathcal{M} \to \mathcal{N}$ is a homomorphism of presehaves, then there is a unique homomorphism of sheaves $\phi' : \mathrm{Sh}(\mathcal{M}) \to \mathcal{N}$ such that $\phi = \phi' \circ \tau_{\mathcal{M}}$.

Exercise 1.1. Read the proof of the sheafication, including an understanding of the Godement sheaf $\mathrm{GSh}(\mathcal{M})$. This is [Ye4, Thm 6.1]. We will need this next week when we define the structure sheaf of an affine scheme.

Exercise 1.2. State the categorical property of the functor Sh, as an adjoint (from which side?) to the inclusion functor $\mathsf{Mod}\,\mathbb{K}_X \to \mathsf{Mod}^{\mathrm{pre}}\,\mathbb{K}_X$.

The sheafication does not change the stalks: for every $x \in X$ the homomorphism:

$$\tau_{\mathcal{M},x} : \mathcal{M}_x \xrightarrow{\simeq} \mathrm{Sh}(\mathcal{M})_x$$

is bijective.

A *ringed space* over $\mathbb{K}$ is a pair $(X, \mathcal{O}_X)$, consisting of a topological space X, and a sheaf of $\mathbb{K}$-rings $\mathcal{O}_X$ on X.

Let $(X, \mathcal{O}_X)$ be such a ringed space. A sheaf of $\mathcal{O}_X$-modules, also called an $\mathcal{O}_X$*-module*, is a sheaf of $\mathbb{K}$-modules on X, together with a structure of a $\Gamma(U, \mathcal{O}_X)$-module for every open set $U \subseteq X$, which respects restrictions to open subsets.

The category of $\mathcal{O}_X$-modules is denoted by $\mathsf{Mod}\,\mathcal{O}_X$. The morphisms are the obvious ones.

The sheafifcation functor respects the $\mathcal{O}_X$-module structure: if $\mathcal{M} \in \mathsf{Mod}^{\mathrm{pre}}\,\mathcal{O}_X$ then $\mathrm{Sh}(\mathcal{M}) \in \mathsf{Mod}\,\mathcal{O}_X$, and $\tau_{\mathcal{M}}$ is $\mathcal{O}_X$-linear.

Let $\phi : \mathcal{M} \to \mathcal{N}$ be a homomorphism in $\mathsf{Mod}\,\mathcal{O}_X$. Its *kernel* is the $\mathcal{O}_X$-module $\mathrm{Ker}(\phi)$ such that

$$\Gamma(U, \mathrm{Ker}(\phi)) = \mathrm{Ker}\Big(\Gamma(U, \phi) : \Gamma(U, \mathcal{M}) \to \Gamma(U, \mathcal{N})\Big).$$

The *image* of ϕ is the $\mathcal{O}_X$-module

$$\mathrm{Im}(\phi) := \mathrm{Sh}(\mathrm{Im}^{\mathrm{pre}}(\phi)),$$

where $\mathrm{Im}^{\mathrm{pre}}(\phi)$ is the presheaf defined by

$$\Gamma(U, \mathrm{Im}^{\mathrm{pre}}(\phi)) = \mathrm{Im}\Big(\Gamma(U, \phi) : \Gamma(U, \mathcal{M}) \to \Gamma(U, \mathcal{N})\Big).$$

Note that $\mathrm{Ker}(\phi)$ is a subsheaf of $\mathcal{M}$ and $\mathrm{Im}(\phi)$ is a subsheaf of $\mathcal{N}$.

A sequence of homomorphisms

$$\boldsymbol{S} = \Big(\cdots \mathcal{M}^i \xrightarrow{\phi^i} \mathcal{M}^{i+1} \xrightarrow{\phi^{i+1}} \mathcal{M}^{i+1} \cdots\Big)$$

in $\mathsf{Mod}\,\mathcal{O}_X$ is called *exact* if for every point $x \in X$ the sequence of homomorphisms

$$\cdots \mathcal{M}^i_x \xrightarrow{\phi^i_x} \mathcal{M}^{i+1}_x \xrightarrow{\phi^{i+1}_x} \mathcal{M}^{i+1}_x \cdots$$

in $\mathsf{Mod}\,\mathcal{O}_{X,x}$ is exact. We know that $\boldsymbol{S}$ is exact iff for every i there is equality

$$\mathrm{Im}(\phi^{i-1}) = \mathrm{Ker}(\phi^i)$$

of these subsheaves of $\mathcal{M}^i$.

Given an open set $U \subseteq X$ we write $\mathcal{O}_U := \mathcal{O}_X|_U$. Thus $(U, \mathcal{O}_U)$ is also a ringed space. There is a restriction functor

$$\mathsf{Mod}\,\mathcal{O}_X \to \mathsf{Mod}\,\mathcal{O}_U,\ \mathcal{M} \mapsto \mathcal{M}|_U.$$

Sheaves, and homomorphisms between sheaves, can be glued.

Notation: given an open covering $X = \bigcup_{i \in I} U_i$, and indices $i, j, \ldots \in I$, we often write

$$U_{i,j,\ldots} := U_i \cap U_j \cap \cdots . \tag{1.3}$$

Theorem 1.4 (Gluing Sheaf Homomorphisms)**.** *Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{O}_X$-modules, let $X = \bigcup_{i \in I} U_i$ be an open covering, and let*

$$\phi_i : \mathcal{M}|_{U_i} \to \mathcal{N}|_{U_i}$$

be homomorphisms of $\mathcal{O}_{U_i}$-modules satisfying the condition

$$\phi_i|_{U_{i,j}} = \phi_j|_{U_{i,j}} : \mathcal{M}|_{U_{i,j}} \to \mathcal{N}|_{U_{i,j}}.$$

(This is the 0*-cocycle condition.)*

Then there is a unique homomorphism of $\mathcal{O}_X$*-modules*

$$\phi : \mathcal{M} \to \mathcal{N}$$

such that

$$\phi|_{U_i} = \phi_i : \mathcal{M}|_{U_i} \to \mathcal{N}|_{U_i}$$

for all i*.*

Theorem 1.5 (Gluing Sheaves)**.** *Suppose* $X = \bigcup_{i \in I} U_i$ *is an open covering. For every* i *let* $\mathcal{M}_i$ *be an* $\mathcal{O}_{U_i}$*-module, and for every* i, j *let*

$$\phi_{i,j} : \mathcal{M}_i|_{U_{i,j}} \xrightarrow{\simeq} \mathcal{M}_j|_{U_{i,j}}$$

be an isomorphism of $\mathcal{O}_{U_{i,j}}$*-modules. The condition is that*

$$\phi_{j,k}|_{U_{i,j,k}} \circ \phi_{i,j}|_{U_{i,j,k}} = \phi_{i,k}|_{U_{i,j,k}}$$

as isomorphisms

$$\mathcal{M}_i|_{U_{i,j,k}} \xrightarrow{\simeq} \mathcal{M}_k|_{U_{i,j,k}}$$

for all i, j, k*. (This is the* 1*-cocycle condition.)*

Then there is an $\mathcal{O}_X$*-module* $\mathcal{M}$*, together with isomorphisms*

$$\phi_i : \mathcal{M}|_{U_i} \xrightarrow{\simeq} \mathcal{M}_i$$

of $\mathcal{O}_X|_{U_i}$*-modules, such that*

$$\phi_{i,j} \circ \phi_i|_{U_{i,j}} = \phi_j|_{U_{i,j}} : \mathcal{M}|_{U_{i,j}} \xrightarrow{\simeq} \mathcal{M}_j|_{U_{i,j}}.$$

Moreover, the $\mathcal{O}_X$*-module* $\mathcal{M}$*, with the collection of isomorphisms* $\{\phi_i\}$*, is unique up to a unique isomorphism.*

Exercise 1.6. Read and make sure you understand these last two thms. Proofs can be found here: [Ye4, Thm 7.3] and [Ye4, Thm 7.4].

Let $\mathcal{M}, \mathcal{N} \in \mathsf{Mod}\, \mathcal{O}_X$. The $\mathcal{O}_X$-module $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ is defined by

$$\Gamma(U, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})) := \mathrm{Hom}_{\mathsf{Mod}\, \mathcal{O}_X|_U}(\mathcal{M}|_U, \mathcal{N}|_U).$$

The $\mathcal{O}_X$-module $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$ is the sheaf associated to the presheaf

$$U \mapsto \Gamma(U, \mathcal{M}) \otimes_{\Gamma(U, \mathcal{O}_X)} \Gamma(U, \mathcal{N}). \tag{1.7}$$

Exercise 1.8. Find an example of $\mathcal{M}$ and $\mathcal{N}$ such that the presheaf tensor product (1.7) is not a sheaf. (Hint: line bundles on $\mathbf{P}^1$)

Let $f : Y \to X$ be a map of topological spaces. For a $\mathbb{K}_Y$-module $\mathcal{N}$, its pushforward, or *direct image*, is the $\mathbb{K}_X$-module $f_*(\mathcal{N})$ defined by

$$\Gamma(U, f_*(\mathcal{N})) := \Gamma(f^{-1}(U), \mathcal{N})$$

for open sets $U \subseteq X$. We get a functor

$$f_* : \mathsf{Mod}\, \mathbb{K}_Y \to \mathsf{Mod}\, \mathbb{K}_X. \tag{1.9}$$

For a $\mathbb{K}_X$-module $\mathcal{M}$, the pullback, or *inverse image*, is the $\mathbb{K}_Y$-module $f^{-1}(\mathcal{M})$ defined by

$$\Gamma(V, f^{-1}(\mathcal{M})) := \lim_{U \to} \Gamma(U, \mathcal{M}),$$

where $V \subseteq Y$ is open, and U runs over the open sets in X that contain $f(V)$. We get a functor

$$f^{-1} : \mathsf{Mod}\, \mathbb{K}_X \to \mathsf{Mod}\, \mathbb{K}_Y. \tag{1.10}$$

There is adjunction: an isomorphism of $\mathbb{K}$-modules

$$\mathrm{Hom}_{\mathsf{Mod}\, \mathbb{K}_X}(\mathcal{M}, f_*(\mathcal{N})) \cong \mathrm{Hom}_{\mathsf{Mod}\, \mathbb{K}_Y}(f^{-1}(\mathcal{M}), \mathcal{N}) \tag{1.11}$$

which is functorial in $\mathcal{M}$ and $\mathcal{N}$.

Exercise 1.12. Prove the adjunction formula (1.11).

Exercise 1.13. Prove that the functor f^{-1} in (1.10) is exact.

Exercise 1.14. Prove that the functor f_* in (1.9) is left exact. Find and example showing that it is not exact. (Hint: line bundles on $\mathbf{P}^1$)

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be ringed spaces. A map of ringed spaces

$$(f, \psi) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X) \tag{1.15}$$

consists of a map of topological spaces $f : Y \to X$, together with a homomorphism of $\mathbb{K}_X$-rings

$$\psi : \mathcal{O}_X \to f_*(\mathcal{O}_Y). \tag{1.16}$$

We shall often use the notation (f, f^*) instead of (f, ψ). The notation (f, ψ) is common in most textbooks, but it is a bit redundant, so we will only use it when it is need to clarify matters. Note that in many cases (see examples below) the homomorphism f^* is literally pullback of functions; and this makes the notation (f, f^*) good heuristically. On the other hand, we sometimes omit all mention of the structure sheaves, and just talk about a map $f : Y \to X$ of locally ringed spaces.

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Given a maps (1.15) of ringed spaces, the direct image is a functor

$$f_* : \mathsf{Mod}\, \mathcal{O}_Y \to \mathsf{Mod}\, \mathcal{O}_X .$$

There is another kind of inverse image here: for $\mathcal{M} \in \mathsf{Mod}\, \mathcal{O}_X$ we define $f^*(\mathcal{M}) \in \mathsf{Mod}\, \mathcal{O}_Y$ by

$$f^*(\mathcal{M}) := \mathcal{O}_Y \otimes_{f^{-1}(\mathcal{O}_X)} f^{-1}(\mathcal{M}).$$

Again there is adjunction:

$$\mathrm{Hom}_{\mathsf{Mod}\, \mathcal{O}_X}(\mathcal{M}, f_*(\mathcal{N})) \cong \mathrm{Hom}_{\mathsf{Mod}\, \mathcal{O}_Y}(f^*(\mathcal{M}), \mathcal{N}).$$

Definition 1.17. A *locally ringed* $\mathbb{K}$*-space* is a ringed $\mathbb{K}$-space $(X, \mathcal{O}_X)$, such that for every point $x \in X$ the stalk $\mathcal{O}_{X,x}$ is a local ring. The maximal ideal of $\mathcal{O}_{X,x}$ is denoted by $\mathfrak{m}_x$, and the residue field is denoted by $\boldsymbol{k}(x)$.

Definition 1.18. If $(Y, \mathcal{O}_Y)$ is another locally ringed space, then a map of locally ringed spaces

$$(f, \psi) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$$

is a map of ringed spaces, such that for every point $y \in Y$, with image $x := f(y) \in X$, the induced $\mathbb{K}$-ring homomorphism

$$\psi_x : \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$$

is a local homomorphism, namely $\psi_x(\mathfrak{m}_x) \subseteq \mathfrak{m}_y$.

The category of locally ringed $\mathbb{K}$-spaces is denoted by $\mathsf{LRSp}/\mathbb{K}$.

Here are three types of locally ringed spaces.

Example 1.19. The category Top of topological spaces. The base ring is $\mathbb{K} = \mathbb{R}$. A space $X \in \mathsf{Top}$ is made into a ringed space by putting on it the sheaf $\mathcal{O}_X$ of continous $\mathbb{R}$-valued functions (for the metric topology of $\mathbb{R}$). Then $(X, \mathcal{O}_X)$ belongs to $\mathsf{LRSp}/\mathbb{R}$. Moreover, the functor $\mathsf{Top} \to \mathsf{LRSp}/\mathbb{R}$ is fully faithful.

Example 1.20. The category Mfld of differentiable (of type C^{∞}) real manifolds. The base ring is $\mathbb{K} = \mathbb{R}$. A manifold $X \in \mathsf{Mfld}$ is made into a ringed space by putting on it the sheaf $\mathcal{O}_X$ of differentiable $\mathbb{R}$-valued functions. Then $(X, \mathcal{O}_X)$ belongs to $\mathsf{LRSp}/\mathbb{R}$. Moreover, the functor $\mathsf{Mfld} \to \mathsf{LRSp}/\mathbb{R}$ is fully faithful.

Example 1.21. Let $\mathbb{K}$ be an algebraically closed field, and let Var be the category of algebraic varieties over $\mathbb{K}$. Here $\mathcal{O}_X$ is the sheaf of algebraic $\mathbb{K}$-valued functions. Then $(X, \mathcal{O}_X)$ belongs to $\mathsf{LRSp}/\mathbb{K}$. Moreover, the functor $\mathsf{Var} \to \mathsf{LRSp}/\mathbb{K}$ is fully faithful.

Given a locally ringed space $(X, \mathcal{O}_X)$ and an open subset $U \subseteq X$, the pair $(U, \mathcal{O}_X|_U)$ is a locally ringed space. We call it an *open subspace* of X. The inclusion map

$$(U, \mathcal{O}_X|_U) \to (X, \mathcal{O}_X)$$

is called an *open embedding*.

Just like sheaves, locally ringed spaces and maps between them can be glued.

Theorem 1.22 (Gluing Maps of LR Spaces)**.** *Let* $(X, \mathcal{O}_X)$ *and* $(Y, \mathcal{O}_Y)$ *be objects of* $\mathsf{LRSp}/\mathbb{K}$, *let* $Y = \bigcup_{i \in I} V_i$ *be an open covering, and for every* i *let*

$$(f_i, \psi_i) : (V_i, \mathcal{O}_Y|_{V_i}) \to (X, \mathcal{O}_X)$$

be a map in $\mathsf{LRSp}/\mathbb{K}$.

We assume that this condition holds: for every $i, j \in I$ *there is equality*

$$(f_i, \psi_i)|_{V_{i,j}} = (f_j, \psi_j)|_{V_{i,j}}$$

of maps

$$(V_{i,j}, \mathcal{O}_Y|_{V_{i,j}}) \to (X, \mathcal{O}_X)$$

in $\mathsf{LRSp}/\mathbb{K}$.

Then there is a unique map

$$(f, \psi) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$$

in $\mathsf{LRSp}/\mathbb{K}$ *such that*

$$(f, \psi)|_{V_i} = (f_i, \psi_i)$$

for every i.

See picture in figure 1.

Proof. The existence and uniqueness of a map of topological spaces $f : Y \to X$ satisfying $f|_{V_i} = f_i$ is clear.

We need to produce the homomorphism of sheaves of rings

$$\psi : \mathcal{O}_X \to f_*(\mathcal{O}_Y)$$

on X. By the adjunction formula (1.11), this amounts to producing a homomorphism of sheaves of rings

$$\psi : f^{-1}(\mathcal{O}_X) \to \mathcal{O}_Y \tag{1.23}$$

on Y. Now we are given homomorphisms

$$\psi_i : f^{-1}(\mathcal{O}_X)|_{V_i} \to \mathcal{O}_Y|_{V_i}$$

that agree on double intersections. According to Theorem 1.4 these can be glued uniquely to a homomorphism of sheaves of $\mathbb{K}$-modules ψ as in (1.23), such that $\psi|_{V_i} = \psi_i$. This is a homomorphism of sheaves of rings, because this propeorty can be checked locally. Also on stalks it is a local homomorphism. So (f, ψ) is a map of locally ringed spaces. □

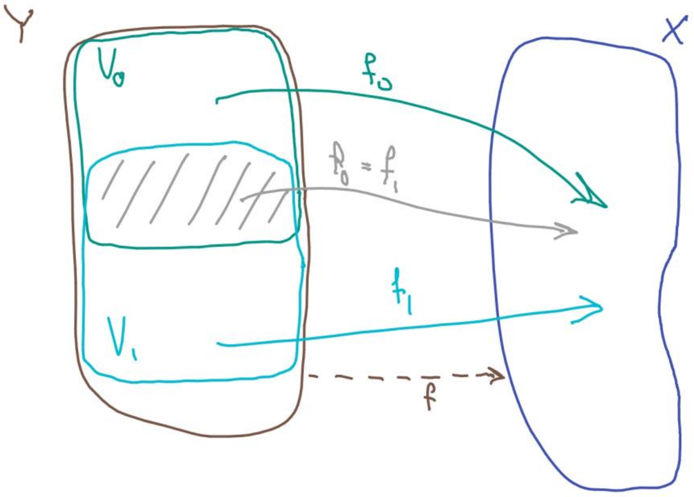

FIGURE 1. Gluing maps of spaces

Theorem 1.24 (Gluing LR Spaces). *Let $\{(U_i, \mathcal{O}_{U_i})\}_{i \in I}$ be a collection of objects of* $\mathsf{LRSp}/\mathbb{K}$. *For every $i, j \in I$ there is an open subset $U_{i,j} \subseteq U_i$, and an isomorphism*

$$(f_{i,j}, \psi_{i,j}) : (U_{i,j}, \mathcal{O}_{U_i}|_{U_{i,j}}) \xrightarrow{\simeq} (U_{j,i}, \mathcal{O}_{U_j}|_{U_{j,i}})$$

in $\mathsf{LRSp}/\mathbb{K}$.

These conditions hold:

(a) *For every i there are equalities $U_{i,i} = U_i$ and $(f_{i,i}, \psi_{i,i}) = \mathrm{id}$.*

(b) *For every i, j, k there are equalities*

$$f_{i,j}(U_{i,j} \cap U_{i,k}) = U_{j,i} \cap U_{j,k}$$

of subsets of U_i, and

$$(f_{j,k}, \psi_{j,k}) \circ (f_{i,j}, \psi_{i,j}) = (f_{i,k}, \psi_{i,k})$$

of isomorphisms

$$(U_{i,j} \cap U_{i,k}, \mathcal{O}_{U_i}|_{U_{i,j} \cap U_{i,k}}) \to (U_{k,i} \cap U_{k,j}, \mathcal{O}_{U_k}|_{U_{k,i} \cap U_{k,j}})$$

in $\mathsf{LRSp}/\mathbb{K}$.

Then there is an object $(X, \mathcal{O}_X)$ in $\mathsf{LRSp}/\mathbb{K}$, *with open embeddings*

$$(f_i, \psi_i) : (U_i, \mathcal{O}_{U_i}) \to (X, \mathcal{O}_X),$$

such that

$$(f_j, \psi_j) \circ (f_{i,j}, \psi_{i,j}) = (f_i, \psi_i)$$

as morphisms

$$(U_{i,j}, \mathcal{O}_{U_i}|_{U_{i,j}}) \to (X, \mathcal{O}_X),$$

and such that

$$X = \bigcup_{i \in I} f_i(U_i).$$

Moreover, the space $(X, \mathcal{O}_X)$, with the collection of morphisms $\{(f_i, \psi_i)\}_{i \in I}$, are unique up to a unique isomorphism.

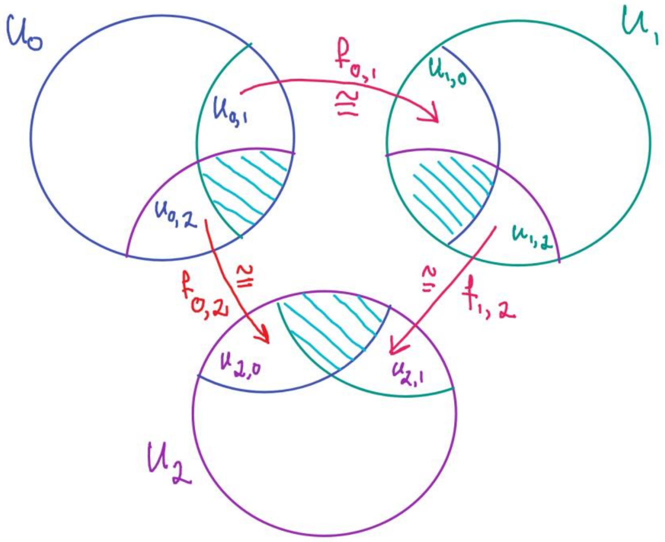

Figure 2. Gluing spaces: the input

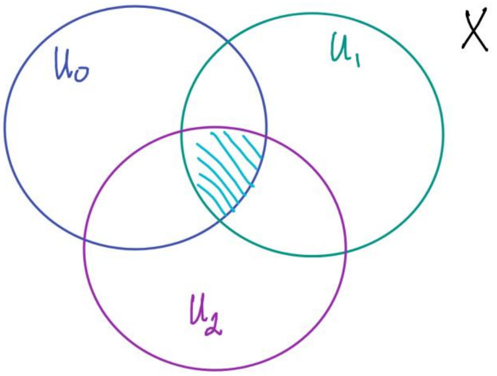

Figure 3. Gluing spaces: the output

See figures 2 and 3 for an illustration.

The data

$$\Big(\big\{ (U_i, \mathcal{O}_{U_i}) \big\}_{i \in I} ,\ \big\{ (f_{i,j}, \psi_{i,j}) \big\}_{i,j \in I} \Big) \tag{1.25}$$

is called *gluing data* or *descent data*..

Proof. This is done in a few steps.

Step 1. We define the set X. Consider the disjoint union $U := \coprod_{i \in I} U_i$. We define a relation $\sim$ on this set as follows: two points $x, y \in U$ are in the relation $x \sim y$ if there are $i, j \in I$

such that $x \in U_{i,j} \subseteq U_i$, $y \in U_{j,i} \subseteq U_j$, and $f_{i,j}(x) = y$. This is an equivalence relation (exercise). We let $X := U/{\sim}$, the quotient set. The canonical surjection is $\pi : U \to X$.

For each i there a map $f_i : U_i \to X$, gotten by composing the inclusion $U_i \subseteq U$ with π. The map f_i is injective (exercise). We identify U_i with its image $f_i(U_i) \subseteq X$. With this identification, there is equality $U_{i,j} = U_{j,i}$ of subsets of X.

Step 2. We put a topology on X. The disjoint union U gets the disjoint union topology. Then we put on X the quotient topology relative to the surjection $\pi : U \to X$.

Each U_i is an open set of X, and so $X = \bigcup_i U_i$ is an open covering (exercise).

Step 3. Now we construct the sheaf of rings $\mathcal{O}_X$. On each open set $U_i \subseteq X$ we have a sheaf of rings $\mathcal{O}_{U_i}$. On double intersections we have isomorphisms of sheaves of rings

$$\psi_{i,j} : \mathcal{O}_{U_i}|_{U_{i,j}} \xrightarrow{\simeq} \mathcal{O}_{U_j}|_{U_{i,j}},$$

and these agree on triple intersections. By Theorem 1.5 we get a sheaf $\mathcal{O}_X$ on X, with isomorphisms of sheaves

$$\psi_i : \mathcal{O}_{U_i} \xrightarrow{\simeq} \mathcal{O}_X|_{U_i}$$

such that $\psi_j \circ \psi_{i,j} = \psi_i$ on $U_{i,j}$. We see that $\mathcal{O}_X$ is a sheaf of rings, and the stalks $\mathcal{O}_{X,x}$ are local rings. So $(X, \mathcal{O}_X)$ is a locally ringed $\mathbb{K}$-space.

By construction there are open embeddings

$$(f_i, \psi_i) : (U_i, \mathcal{O}_{U_i}) \to (X, \mathcal{O}_X)$$

satisfying the required compatibility.

Step 4. The uniqueness is due to Theorem 1.22. □

Exercise 1.26. Finish the proof of the theorem.

Exercise 1.27. Find an example of a collection of spaces as in the theorem, such that all the spaces U_i are separated topological spaces (i.e. Hausdorff), yet X is not separetd.

Exercise 1.28. Suppose that the spaces and the gluing data are in Mfld (Example 1.20), the indexing set I is countable, and the topological space X is Hausdorff. Prove that $(X, \mathcal{O}_X)$ is an object of Mfld.

Remark 1.29. The gluing in Thm 1.24 will be used to construct fiber products of schemes. Then the fiber products will allow us to define the notion of a *separated map of schemes*. This is the relative and generalized variant of the Hausdorff condition.

2. Affine Schemes

Let A be a $\mathbb{K}$-ring. The *prime spectrum* of A is the set $\operatorname{Spec}(A)$ of prime ideals of A. As we all know, this is a topological space with the *Zariski topology*. By definition the closed sets of $\operatorname{Spec}(A)$ are the sets

$$\mathrm{Z}(\mathfrak{a}) := \{\mathfrak{p} \in \operatorname{Spec}(A) \mid \mathfrak{a} \subseteq \mathfrak{p}\},$$

where $\mathfrak{a}$ is some ideal in A.

We sometimes refer to $\mathrm{Z}(\mathfrak{a})$ as the *zero locus* of the ideal $\mathfrak{a}$. Here is the reason: to a prime ideal $\mathfrak{p}$ we associate the local ring $A_{\mathfrak{p}}$ and the residue field

$$\boldsymbol{k}(\mathfrak{p}) := A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}.$$

An element $a \in A$ has a class $a(\mathfrak{p}) \in \boldsymbol{k}(\mathfrak{p})$, coming from the canonical ring homomorphism $A \to \boldsymbol{k}(\mathfrak{p})$. Then

$$\mathrm{Z}(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid a(\mathfrak{p}) = 0 \ \text{ for all } \ a \in \mathfrak{a}\}. \tag{2.1}$$

Exercise 2.2. Prove the formula above.

For an element $s \in A$ we define

$$\mathrm{D}(s) = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid s \notin \mathfrak{p}\}. \tag{2.3}$$

This is an open set: it it the complement of the closed set $\mathrm{Z}(\mathfrak{a})$, where $\mathfrak{a} := (s)$, the principal ideal generated by s. We call such an open set a *principal open set*. Analogously to (2.1) we have

$$\mathrm{D}(s) = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid s(\mathfrak{p}) \neq 0\}. \tag{2.4}$$

Proposition 2.5. *The principal open sets are a basis of the topology of* $\operatorname{Spec}(A)$*. Namely every open set set U is a union $U = \bigcup_i \mathrm{D}(s_i)$ for a suitable collection $\{s_i\}$ of elements of A.*

Exercise 2.6. Prove the proposition above.

Definition 2.7. Let A be a ring, and write $X := \operatorname{Spec}(A)$ for this topological space. For an open set $U \subseteq X$ we let $\mathrm{S}(U) \subseteq A$ be the multiplicatively closed set

$$\mathrm{S}(U) := \{s \in A \mid s(\mathfrak{p}) \neq 0 \ \text{ for all } \ \mathfrak{p} \in U\}.$$

Let $A_{\mathrm{S}(U)}$ be the localization of A w.r.t. $\mathrm{S}(U)$.

Lemma 2.8. *The assignment $U \mapsto \mathrm{S}(U)$ is a presheaf of rings on $X = \operatorname{Spec}(A)$, that we denote by $\mathcal{O}^{\mathrm{pre}}$.*

Proof. This is easy: if $V \subseteq U$ is a smaller open set, then $\mathrm{S}(U) \subseteq \mathrm{S}(V)$, so by the universal property of localization there is a unique A-ring homomorphism $A_{\mathrm{S}(U)} \to A_{\mathrm{S}(V)}$. □

Definition 2.9. Let A be a ring. The *structure sheaf* of $\operatorname{Spec}(A)$ is the sheaf of rings

$$\mathcal{O}_{\operatorname{Spec}(A)} := \operatorname{Sh}(\mathcal{O}^{\mathrm{pre}}).$$

Lecture 3, 13 Mar 2019

By construction, $\mathcal{O}^{\mathrm{pre}}$ is a presheafs of A-rings on $X := \operatorname{Spec}(A)$, and $\mathcal{O}_X$ is a sheaf of A-rings on X.

Proposition 2.10. *Let A be a ring and write $X := \operatorname{Spec}(A)$. For every point $x = \mathfrak{p} \in X$ the stalk of $\mathcal{O}_X$ at x is $\mathcal{O}_{X,x} = A_{\mathfrak{p}}$, the local ring at $\mathfrak{p}$. More precisely, there is a unique A-ring isomorphism $\mathcal{O}_{X,x} \cong A_{\mathfrak{p}}$.*

Proof. Let $\mathrm{S}(x) := A - \mathfrak{p}$, the complement of $\mathfrak{p}$. By definition we have $A_{\mathfrak{p}} = A_{\mathrm{S}(x)}$. The universal property of localization says that there is at most one A-ring isomorphism $A_{\mathfrak{p}} \xrightarrow{\simeq} \mathcal{O}_{X,x}$. We will produce it.

Let us denote by $\boldsymbol{U}_x$ the set of principal open neighborhoods of x. These are the open sets $U = \mathrm{D}(s)$ for some $s \in \mathrm{S}(x)$. By Proposition 2.5, $\boldsymbol{U}_x$ is a basis of open neighborhoods of x. Because the stalks of the presheaf $\mathcal{O}^{\mathrm{pre}}$ and its associated sheaf $\mathcal{O}$ are the same, we have

$$\mathcal{O}_{X,x} = \lim_{U \to} \Gamma(U, \mathcal{O}^{\mathrm{pre}}) = \lim_{U \to} A_{\mathrm{S}(U)} \tag{2.11}$$

where U runs over $\boldsymbol{U}_x$.

For each $U = \mathrm{D}(s) \in \boldsymbol{U}_x$ we have $\mathrm{S}(U) \subseteq \mathrm{S}(x)$, so there is a unique A-ring homomorphism $A_{\mathrm{S}(U)} \to A_{\mathrm{S}(x)} = A_{\mathfrak{p}}$. Going to the limit in U we get an A-ring homomorphism

$$\psi : \mathcal{O}_{X,x} \to A_{\mathfrak{p}}.$$

In the other direction, every element $s \in \mathrm{S}(x)$ belongs to $\mathrm{S}(U)$ for $U := \mathrm{D}(s) \in \boldsymbol{U}_x$, and hence s is invertible in $A_{\mathrm{S}(U)}$. This means that every $s \in \mathrm{S}(x)$ is invertible in the A-ring $\mathcal{O}_{X,x}$. Hence there is a unique A-ring homomorphism

$$\phi : A_{\mathfrak{p}} = A_{\mathrm{S}(x)} \to \mathcal{O}_{X,x}.$$

The homomorphism ϕ is surjective, because every $f \in \mathcal{O}_{X,x}$ is the image of some fraction $a/s \in A_{\mathrm{S}(U)}$, see (2.11). But $s \in \mathrm{S}(x)$, so $a/s \in A_{\mathfrak{p}}$ and $f = \phi(a/s)$.

Finally consider the A-ring homomorphism $\psi \circ \phi$ from $A_{\mathfrak{p}}$ to itself. By uniqueness there is equality $\psi \circ \phi = \mathrm{id}_{A_{\mathfrak{p}}}$. This shows that ϕ is injective. In conclusion, ϕ is an isomorphism of A-rings. □

Corollary 2.12. $\big(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)}\big)$ *is a locally ringed space.*

Definition 2.13. An *affine* $\mathbb{K}$*-scheme* is a locally ringed space $(X, \mathcal{O}_X) \in \mathsf{LRSp}/\mathbb{K}$ which is isomorphic to $\big(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)}\big)$ for some $\mathbb{K}$-ring A.

We now provide a more explicit description of the structur sheaf $\mathcal{O}_{\mathrm{Spec}(A)}$ in terms of its *Godement sheaf* from [Ye4, Sec 6]. Recall that for a presheaf of $\mathbb{K}$-modules $\mathcal{M}$ on a space X, its Godement sheaf $\mathrm{GSh}(\mathcal{M})$ is defined by

$$\Gamma(U, \mathrm{GSh}(\mathcal{M})) := \prod_{x \in U} \mathcal{M}_x$$

for an open set $U \subseteq X$. Then $\mathrm{Sh}(\mathcal{M}) \subseteq \mathrm{GSh}(\mathcal{M})$ is the subsheaf of *geometric sections*. Here is what this means. There is a canonical homomorphism of presehaves $\mathcal{M} \to \mathrm{GSh}(\mathcal{M})$. For an open set $U \subseteq X$, a section

$$m \in \Gamma(U, \mathrm{GSh}(\mathcal{M}))$$

is called geometric if for every point $x \in U$ there is an open set V such that $x \in V \subseteq U$, and a section $m' \in \Gamma(V, \mathcal{M})$, such that

$$m|_V = m' \in \Gamma(V, \mathrm{GSh}(\mathcal{M})).$$

Specializing to our case it says that following:

Proposition 2.14. *Let* $(X, \mathcal{O}_X) := \big(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)}\big)$. *Then* $\mathcal{O}_X$ *is the subsheaf of* $\mathrm{GSh}(\mathcal{O}^{\mathrm{pre}})$ *consisting of the geometric sections. Specifically, let* $U \subseteq X$ *be an open set, and let*

$$f \in \Gamma(U, \mathrm{GSh}(\mathcal{O}^{\mathrm{pre}})) = \prod_{\mathfrak{p} \in U} A_{\mathfrak{p}}.$$

The section f *belongs to* $\Gamma(U, \mathcal{O}_X)$ *iff for every point* $x = \mathfrak{p} \in U$ *there is an open set* V *such that* $x \in V \subseteq U$*, and elements* $a \in A$ *and* $s \in \mathrm{S}(V)$*, such that*

$$f|_V = a/s \in \Gamma(V, \mathrm{GSh}(\mathcal{O}^{\mathrm{pre}})) = \prod_{\mathfrak{q} \in V} A_{\mathfrak{q}}.$$

See Figure 4.

Here are some example of affine schemes.

Example 2.15. Consider the ring $A = \mathbb{Z}$. The affine scheme $X := \mathrm{Spec}(\mathbb{Z})$ has these points: for every (positive) prime number p there is a maximal ideal $\mathfrak{m} := (p)$. These are closed points of X, since

$$\mathrm{Z}(p) = \{\mathfrak{m}\}.$$

The local ring is

$$\mathbb{Z}_{(p)} = \{a/s \mid s \notin (p)\} \subseteq \mathbb{Q}.$$

The residue field is $\mathbb{F}_p$.

The zero ideal $\mathfrak{p} := (0)$ is also prime. Is is the *generic point* of X; namely its topological closure is X. The local ring and the residue field at $\mathfrak{p}$ are $\mathbb{Q}$.

Exercise 2.16. Analyze the affine scheme $\mathrm{Spec}(A)$ for the ring $A := \mathbb{K}[t]$, the polynomial ring in one variable over a field $\mathbb{K}$.

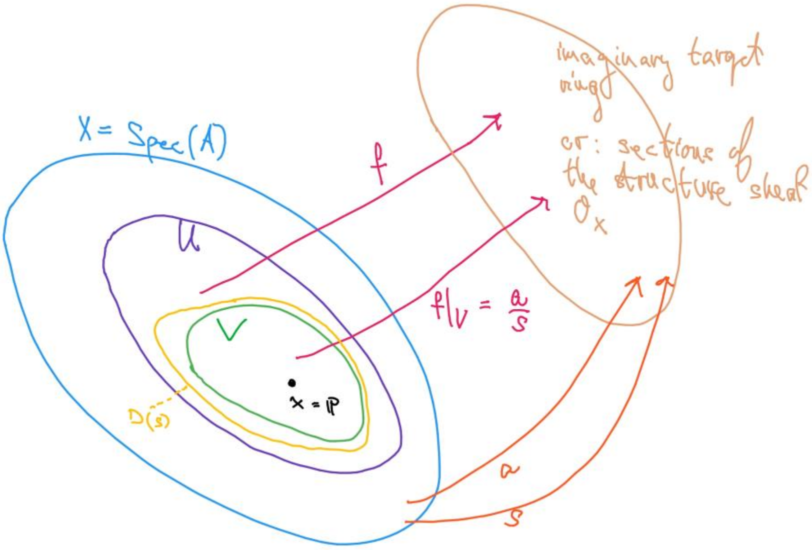

Figure 4. Picture for Prop 2.14

Example 2.17. Suppose $\mathbb{K}$ is a nonzero ring. For every $n \geq 0$ there is an affine scheme

$$\mathbf{A}^n_{\mathbb{K}} := \operatorname{Spec}(\mathbb{K}[t_1, \ldots, t_n])$$

called the *n dimensional affine space over* $\mathbb{K}$.

Recall that for an ideal $\mathfrak{a} \subseteq A$, its zero locus is

$$\mathrm{Z}(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid \mathfrak{a} \subseteq \mathfrak{p}\}.$$

The *radical* of $\mathfrak{a}$ is the ideal

$$\sqrt{\mathfrak{a}} := \{a \in A \mid a^i \in \mathfrak{a} \text{ for some } i > 0\} \subseteq A.$$

Lemma 2.18. *Let $\mathfrak{a} \subseteq A$ be an ideal. Then*

$$\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{p} \in \mathrm{Z}(\mathfrak{a})} \mathfrak{p}$$

and

$$\mathrm{Z}(\mathfrak{a}) = \mathrm{Z}(\sqrt{\mathfrak{a}}).$$

Exercise 2.19. Prove the lemma. (Hint: do not use the Nullstellensatz.)

Lemma 2.20. *For ideals $\mathfrak{a}, \mathfrak{b} \subseteq A$ the following are equivalent:*

(i) $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$.
(ii) $\mathrm{Z}(\mathfrak{a}) = \mathrm{Z}(\mathfrak{b})$.

Proof. Clear from Lemma 2.18. □

Definition 2.21. A topological space X is called *quasi-compact* if every open covering of X has a finite subcovering.

Remark 2.22. The term “compact” is usually reserved for spaces that are Hausdorff and quasi-compact. Schemes are almost never Hausdorff. There is an analogous notion of separation, that we will study later. Cf. Exercise 1.27 and Rem 1.29.

Proposition 2.23. *Let A be a ring. The topological space $X := \operatorname{Spec}(A)$ is quasi-compact.*

Proof. Let $X = \bigcup_{i \in I} U_i$ be an open covering. For each i there is an ideal $\mathfrak{a}_i$ such that $U_i = X - \mathrm{Z}(\mathfrak{a}_i)$. Write $\mathfrak{a} := \sum_i \mathfrak{a}_i$. Then there is equality

$$\mathrm{Z}(A) = \varnothing = \bigcap_{i \in I} \mathrm{Z}(\mathfrak{a}_i) = \mathrm{Z}(\mathfrak{a})$$

of subsets of X. By Lemma 2.20 we know that $\sqrt{A} = \sqrt{\mathfrak{a}}$. Since $1 \in A$, we see that $1 \in \sqrt{\mathfrak{a}}$, and hence $1 \in \mathfrak{a}$. This says that we can express 1 as a finite sum: $1 = \sum_{i \in I'} a_i$ with $I' \subseteq I$ a finite subset and $a_i \in \mathfrak{a}_i$. We see that $A = \sum_{i \in I'} \mathfrak{a}_i$, and therefore

$$\varnothing = \mathrm{Z}(A) = \bigcap_{i \in I'} \mathrm{Z}(\mathfrak{a}_i)$$

and $X = \bigcup_{i \in I'} U_i$. □

Lemma 2.24. *Let $s_1, \ldots, s_m \in A$. TFAE:*

(i) $\operatorname{Spec}(A) = \bigcup_i \mathrm{D}(s_i)$.
(ii) *There exists $a_1, \ldots, a_m \in A$ s.t. $1_A = \sum_i a_i \cdot s_i$.*

Proof.

(i) ⇒ (ii): As in the proof of the proposition, $A = \sum_i \mathfrak{a}_i$, where $\mathfrak{a}_i := (s_i)$. So the element 1_A is a linear combination $1_A = \sum_i a_i \cdot s_i$ for some $a_i \in A$.

(ii) ⇒ (i): Here $1_A \in \sum_i \mathfrak{a}_i$, so $A = \sum_i \mathfrak{a}_i$, and, as in the proof of the proposition, $\operatorname{Spec}(A) = \bigcup_i \mathrm{D}(s_i)$. □

By construction, for every $s \in A$ the element s is invertible in the rings $\Gamma(\mathrm{D}(s), \mathcal{O}^{\mathrm{pre}})$ and $\Gamma(\mathrm{D}(s), \mathcal{O}_X)$.

Lemma 2.25. *Let $(X, \mathcal{O}_X) := \big(\operatorname{Spec}(A), \mathcal{O}_{\operatorname{Spec}(A)}\big)$. Suppose $U \subseteq X$ is an open set, $f \in \Gamma(U, \mathcal{O}_X)$, and $x = \mathfrak{p} \in U$. Then there are elements $a, s \in A$ such that $x \in \mathrm{D}(s) \subseteq U$ and*

$$f|_{\mathrm{D}(s)} = a/s \in \Gamma(\mathrm{D}(s), \mathcal{O}_X).$$

Proof. By Proposition 2.14 the point $x = \mathfrak{p} \in U$ has an open neighborhood $V \subseteq U$, and elements $b, t \in A$, such that $t \in \mathrm{S}(V)$ and $f|_V = b/t \in \Gamma(V, \mathcal{O}_X)$.

According to Proposition 2.5 we can replace V with a smaller open neighborhood $\mathrm{D}(r)$ for some $r \in A$, i.e. $x \in \mathrm{D}(r) \subseteq V$.

Define the elements $a := b \cdot r \in A$ and $s := t \cdot r \in A$, and the open set $W := \mathrm{D}(s)$. Since $\mathrm{D}(r) \subseteq V \subseteq \mathrm{D}(t)$, it follows that

$$\mathrm{D}(r) = \mathrm{D}(t \cdot r) = \mathrm{D}(s) = W.$$

Also

$$a/s = (b \cdot r)/(t \cdot r) = f|_W \in \Gamma(W, \mathcal{O}_X).$$

□

Lemma 2.26. *Let $(X, \mathcal{O}_X) := \big(\operatorname{Spec}(A), \mathcal{O}_{\operatorname{Spec}(A)}\big)$. Suppose $V \subseteq X$ is a quasi-compact open set, and $f \in \Gamma(V, \mathcal{O}_X)$. Then there are finitely many elements $a_1, \ldots, a_m, s_1, \ldots, s_m \in A$ such that $V = \bigcup_{i=1}^m \mathrm{D}(s_i)$, and*

$$f|_{\mathrm{D}(s_i)} = a_i/s_i \in \Gamma(\mathrm{D}(s_i), \mathcal{O}_X)$$

for every i.

Proof. By Lemma 2.25, for every point $x = \mathfrak{p} \in V$ there are elements $a_x, s_x \in A$ such that $x \in \mathrm{D}(s_x) \subseteq V$ and

$$f|_{\mathrm{D}(s_x)} = a_x / s_x \in \Gamma(\mathrm{D}(s_x), \mathcal{O}_X).$$

We have an open covering $V = \bigcup_{x \in V} \mathrm{D}(s_x)$. Because of quasi-compactness we can pass to a finite subcovering, that is to a finite subset $\{x_1, \ldots, x_m\}$ of V. Finally, by letting $a_i := a_{x_i}$ and $s_i := s_{x_i}$ we are done. □

A ring homomorphism $\psi : A \to B$ induces a map of sets

$$\operatorname{Spec}(\psi) : \operatorname{Spec}(B) \to \operatorname{Spec}(A) \tag{2.27}$$

with formula

$$\operatorname{Spec}(\psi)(\mathfrak{q}) := \psi^{-1}(\mathfrak{q}).$$

Lemma 2.28. *Let $\psi : A \to B$ be a ring homomorphism.*

1. *The map $\operatorname{Spec}(\psi)$ is continuous.*
2. *Suppose $B = A_s$ for some $s \in A$. Then the image of $\operatorname{Spec}(\psi)$ is $\mathrm{D}(s)$, and*

$$\operatorname{Spec}(\psi) : \operatorname{Spec}(A_s) \to \mathrm{D}(s)$$

is a homeomorphism.

Exercise 2.29. Prove the last lemma.

Lecture 4, 27 Mar 2019

Theorem 2.30. *Let A be a ring and $s \in A$ an element. Consider the affine scheme $(X, \mathcal{O}_X) := \big(\operatorname{Spec}(A), \mathcal{O}_{\operatorname{Spec}(A)}\big)$. There is a unique A-ring isomorphism*

$$A_s \cong \Gamma(\mathrm{D}(s), \mathcal{O}_X).$$

Proof.

Step 1. Since the element s is invertible in the ring $\Gamma(\mathrm{D}(s), \mathcal{O}_X)$, there is a unique A-ring homomorphism

$$\phi : A_s \to \Gamma(\mathrm{D}(s), \mathcal{O}_X). \tag{2.31}$$

We need to prove that ϕ is bijective.

Step 2. In this step we we will prove that ϕ is injective. Let's write $U := \mathrm{D}(s)$. Since $\mathcal{O}_X$ is a subsheaf of the Godement sheaf $\operatorname{GSh}(\mathcal{O}_X)$, there is an embedding of A-rings

$$\Gamma(U, \mathcal{O}_X) \rightarrowtail \Gamma\big(U, \operatorname{GSh}(\mathcal{O}_X)\big) = \prod_{\mathfrak{p} \in U} A_{\mathfrak{p}}.$$

It thus suffices to prove that the homomorphism

$$\phi' : A_s \to \prod_{\mathfrak{p} \in U} A_{\mathfrak{p}} \tag{2.32}$$

is injective.

Let's write $B := A_s$. By Lemma 2.28 we know that $\operatorname{Spec}(B) = U$ as topological subspaces of X. For every $\mathfrak{p} \in U$ the element s is invertible in $A_{\mathfrak{p}}$, and hence the homomorphism $A_{\mathfrak{p}} \to B_{\mathfrak{p}}$ is bijective. So we can rewrite (2.32) as $\phi' : B \to \prod_{\mathfrak{p} \in U} B_{\mathfrak{p}}$, and we need to prove it is injective.

Suppose $b \in B$ is such that $\phi'(b) = 0$. Then $\phi'_{\mathfrak{p}}(b) = 0$ in $B_{\mathfrak{p}}$ for all $\mathfrak{p} \in U$. This means that there is some element $t_{\mathfrak{p}} \in B$ such that $\mathfrak{p} \in \mathrm{D}(t_{\mathfrak{p}})$ and $t_{\mathfrak{p}} \cdot b = 0$ in B. Now $U = \operatorname{Spec}(B) = \bigcup_{\mathfrak{p} \in U} \mathrm{D}(t_{\mathfrak{p}})$.

By quasi-compactness of U we can pass to a finite subcovering, indexed by $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}$. Let $t_i := t_{\mathfrak{p}_i} \in B$. Then $t_i \cdot b = 0$ in B, and $\operatorname{Spec}(B) = \bigcup_{i=1}^m \mathrm{D}(t_i)$. According to Lemma 2.24 as in the proof of Prop 2.23, there are elements $c_1, \ldots, c_m \in B$ such that $1_B = \sum_i c_i \cdot t_i$. Then $b = 1_B \cdot b = \sum_i c_i \cdot t_i \cdot b = 0$.

Step 3. We now prove that the homomorphism ϕ from (2.31) is surjective. As before let $U := \mathrm{D}(s)$. Take an element $f \in \Gamma(U, \mathcal{O}_X)$. We know that $U = \operatorname{Spec}(B)$ is a quasi-compact topological space. By Lem 2.26 there are finitely many elements $a_i, s_i \in A$, $1 \leq i \leq m$, such that $U = \bigcup_{i=1}^m \mathrm{D}(s_i)$, and $f|_{\mathrm{D}(s_i)} = a_i / s_i \in \Gamma(\mathrm{D}(s_i), \mathcal{O}_X)$ for every i.

Step 2, when applied to the element $s_i \cdot s_j \cdot s \in A$ instead of to s, shows that for every i, j the A-ring homomorphism

$$\phi_{i,j} : A_{s_i \cdot s_j \cdot s} \to \Gamma\big(\mathrm{D}(s_i \cdot s_j \cdot s), \mathcal{O}_X\big)$$

is injective. Recall that $B = A_s$, so that $A_{s_i \cdot s_j \cdot s} = B_{s_i \cdot s_j}$. We know that

$$f|_{\mathrm{D}(s_i \cdot s_j \cdot s)} = a_i \cdot s_i^{-1} = a_j \cdot s_j^{-1} \in \Gamma\big(\mathrm{D}(s_i \cdot s_j \cdot s), \mathcal{O}_X\big).$$

Therefore $a_i \cdot s_i^{-1} = a_j \cdot s_j^{-1}$ in $B_{s_i \cdot s_j}$. The kernel of the localization homomorphism $B \to B_{s_i \cdot s_j}$ is known: there is a positive integer $l_{i,j}$ such that

$$(s_i \cdot s_j)^{l_{i,j}} \cdot (a_i \cdot s_j - a_j \cdot s_i) = 0$$

in B. Taking $l := \max(\{l_{i,j}\})$ we obtain

$$(s_i \cdot s_j)^{l} \cdot (a_i \cdot s_j - a_j \cdot s_i) = 0 \tag{2.33}$$

in B.

Define $b_i := a_i \cdot s_i^l$ and $t_i := s_i^{l+1}$. Then $\mathrm{D}(t_i) = \mathrm{D}(s_i)$ and

$$f|_{\mathrm{D}(t_i)} = a_i \cdot s_i^{-1} = b_i \cdot t_i^{-1} \in \Gamma(\mathrm{D}(t_i), \mathcal{O}_X). \tag{2.34}$$

Also, from (2.33) we have

$$t_i \cdot b_j = t_j \cdot b_i \tag{2.35}$$

in B.

Since

$$\operatorname{Spec}(B) = \bigcup_{i=1}^m \mathrm{D}(t_i), \tag{2.36}$$

by Lem 2.24 we can find elements $c_1, \ldots, c_m \in B$ such that $1_B = \sum_i c_i \cdot t_i$. Let

$$b := \sum_i c_i \cdot b_i \in B.$$

For every i we have – using (2.35) and (2.34) –

$$\phi(b)|_{\mathrm{D}(t_i)} = \sum_j c_j \cdot b_j = t_i^{-1} \cdot \Big(\sum_j c_j \cdot t_i \cdot b_j\Big) = t_i^{-1} \cdot \Big(\sum_j c_j \cdot t_j\Big) \cdot b_i = t_i^{-1} \cdot b_i = f|_{\mathrm{D}(t_i)}$$

in $\Gamma(\mathrm{D}(t_i), \mathcal{O}_X)$. But by (2.36) and the sheaf axioms this implies that $\phi(b) = f$. □

Lecture 5, 3 Apr 2019

Convention 2.37. From now on the expression $\operatorname{Spec}(A)$ refers to the locally ringed space $\big(\operatorname{Spec}(A), \mathcal{O}_{\operatorname{Spec}(A)}\big)$.

Corollary 2.38. *For a ring A, with* $(X, \mathcal{O}_X) := \operatorname{Spec}(A)$, *the canonical ring homomorphism* $A \to \Gamma(X, \mathcal{O}_X)$ *is bijective.*

Proof. Take $s = 1$ in Thm 2.30. □

Definition 2.39. The category of *affine* $\mathbb{K}$-*schemes*, denoted by $\mathsf{Sch}_{\mathrm{aff}}/\mathbb{K}$, is the full subcategory of $\mathsf{LRSp}/\mathbb{K}$ on the affine schemes (see Definition 2.13).

Recall that $\mathsf{Rng}/\mathbb{K}$ is the category of (commutative) $\mathbb{K}$-rings.

Proposition 2.40. *The assignment that sends a locally ringed space* $(X, \mathcal{O}_X)$ *to the ring* $\Gamma(X, \mathcal{O}_X)$*, and a map of locally ringed spaces* $(f, \tilde{\phi}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ *to the ring homomorphism*

$$\Gamma(X, \phi) : \Gamma(Y, \mathcal{O}_Y) \to \Gamma(X, \mathcal{O}_X),$$

is a functor

$$\Gamma : (\mathsf{LRSp}/\mathbb{K})^{\mathrm{op}} \to \mathsf{Rng}/\mathbb{K}.$$

Exercise 2.41. Prove proposition 2.40.

Before continuing, here is a general useful notion.

Definition 2.42. A morphism $f : C \to D$ in a category C is called an *epimorphism* if it has this cancellation property: for every morphisms $g_0, g_1 : D \to E$ in C, if $g_0 \circ f = g_1 \circ f$ then $g_0 = g_1$.

Proposition 2.43. *Let* $f : A \to B$ *be a morphism in the category* Rng *of commutative rings. If* f *is either surjective or a localization, then* f *is an epimorphism in* Rng.

Exercise 2.44. Prove proposition 2.43.

Let $(Y, \mathcal{O}_Y)$ be a LRSp and $s \in \Gamma(Y, \mathcal{O}_Y)$. For a point $y \in Y$ we denote by $s(y)$ the image of the element s in the residue field $\boldsymbol{k}(y)$. And we write

$$\mathrm{D}(s) := \{y \in Y \mid s(y) \neq 0\}.$$

Lemma 2.45. *Let* $(Y, \mathcal{O}_Y)$ *be a LRSp and* $s \in \Gamma(Y, \mathcal{O}_Y)$*. Then:*

1. *The set* $\mathrm{D}(s)$ *is open in* Y.
2. *The element* s *is invertible in the ring* $\Gamma(\mathrm{D}(s), \mathcal{O}_Y)$.

Proof.

(1) Take a point $y \in \mathrm{D}(s)$. Since $s(y) \neq 0$, it is an invertible element of the field $\boldsymbol{k}(y)$. Because the stalk $\mathcal{O}_{Y,y}$ is a local ring, it follows that s is an invertible element of $\mathcal{O}_{Y,y}$. Let $t \in \mathcal{O}_{Y,y}$ be the inverse of s, so $s \cdot t = 1$ in $\mathcal{O}_{Y,y}$. There is an open neighborhood V of y such that $t \in \Gamma(V, \mathcal{O}_Y)$. There is a smaller open neighborhood V' of y s.t. $s \cdot t = 1$ in $\Gamma(V', \mathcal{O}_Y)$. Then $s(y') \neq 0$ for all $y' \in V'$, so $V' \subseteq \mathrm{D}(s)$.

(2) Let $V := \mathrm{D}(s)$. As we saw above, every point $y \in V$ has an open neighborhood $V_y \subseteq V$ and an element $t_y \in \Gamma(V_y, \mathcal{O}_Y)$ such that $s \cdot t_y = 1$. This means that $t_y = s^{-1}$ in $\Gamma(V_y, \mathcal{O}_Y)$, a fact that makes it unique. We conclude that $t_y = t_{y'}$ in $\Gamma(V_y \cap V_{y'}, \mathcal{O}_Y)$. By the sheaf property we get $t \in \Gamma(V, \mathcal{O}_Y)$, and it satisfies $s \cdot t = 1$. □

Theorem 2.46. *Let* $A \in \mathsf{Rng}/\mathbb{K}$ *and let* $(Y, \mathcal{O}_Y) \in \mathsf{LRSp}/\mathbb{K}$*. Write* $B := \Gamma(Y, \mathcal{O}_Y)$ *and* $(X, \mathcal{O}_X) := \mathrm{Spec}(A)$*. Given a* $\mathbb{K}$*-ring homomorphism* $\phi : A \to B$*, there is a unique map*

$$(f, \tilde{\phi}) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$$

in $\mathsf{LRSp}/\mathbb{K}$*, such that* $\Gamma(X, \tilde{\phi}) = \phi : A \to B$.

Proof.

Step 1. We prove uniqueness of f. Take a point $y \in Y$, and let $x = \mathfrak{p} := f(y) \in X$. Because $\tilde{\phi}_y : \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is a local homomorphism, and $\Gamma(X, \tilde{\phi}) = \phi$, we have a commutative diagram of rings

(2.47)
$$\begin{tikzcd}
A \arrow[r, "\phi"] \arrow[d] & B \arrow[d] \\
\mathcal{O}_{X,x} \arrow[r, "\tilde{\phi}_y"] \arrow[d] & \mathcal{O}_{Y,y} \arrow[d] \\
\boldsymbol{k}(x) \arrow[r, tail] & \boldsymbol{k}(y)
\end{tikzcd}$$

The homomorphism $\boldsymbol{k}(x) \to \boldsymbol{k}(y)$ is injective. Comparing the two paths in this diagram we see that

$$\operatorname{Ker}\big(A \xrightarrow{\phi} B \to \boldsymbol{k}(y)\big) = \operatorname{Ker}\big(A \to \boldsymbol{k}(\mathfrak{p})\big) = \mathfrak{p}.$$

This formula determines the value $f(y) = \mathfrak{p}$.

Step 2. Now we prove that the homomorphism of sheaves of rings

$$\tilde{\phi} : \mathcal{O}_X \to f_*(\mathcal{O}_Y)$$

is unique. Since the principal open sets $U = \mathrm{D}(s) \subseteq X$, for $s \in A$, are a basis for the topology, it is enough to prove the uniqueness of the ring homomorphism

$$\Gamma(U, \tilde{\phi}) : \Gamma(U, \mathcal{O}_X) \to \Gamma(V, \mathcal{O}_Y),$$

for $U := \mathrm{D}(s)$ and $V := f^{-1}(U) \subseteq Y$. Let's examine this commutative diagram of rings

(2.48)
$$\begin{tikzcd}
A \arrow[r, "\phi = \Gamma(X, \tilde{\phi})"] \arrow[d] & B \arrow[d] \\
A_s = \Gamma(U, \mathcal{O}_X) \arrow[r, "\Gamma(U, \tilde{\phi})"] & \Gamma(V, \mathcal{O}_Y)
\end{tikzcd}$$

Because the left vertical arrow is an epimorphism (see Prop 2.43), it follows that the ring homomorphism $\Gamma(U, \tilde{\phi})$ is unique.

Step 3. Here we start with the existence. We define the function $f : Y \to X$ by the formula from step 1, namely a point $y \in Y$ is sent to the prime ideal

(2.49)
$$\mathfrak{p} := \operatorname{Ker}\big(A \xrightarrow{\phi} B \to \boldsymbol{k}(y)\big) \in X.$$

We need to prove that f is continuous. It suffices to show that for a principal open set $U = \mathrm{D}(s) \subseteq X$, its preimage $f^{-1}(U)$ is open. A little calculation using formula (2.49) shows that $f^{-1}(U) = \mathrm{D}(\phi(s))$, and this is open in Y by Lemma 2.45.

Step 4. Now we construct the homomorphism of sheaves of rings

(2.50)
$$\tilde{\phi} : \mathcal{O}_X \to f_*(\mathcal{O}_Y)$$

on X. Let $U \subseteq X$ be an open set. We need to specify the ring homomorphism

(2.51)
$$\tilde{\phi}_U : \Gamma(U, \mathcal{O}_X) \to \Gamma(V, \mathcal{O}_Y)$$

where $V := f^{-1}(U) \subseteq Y$.

First we consider $U = \mathrm{D}(s)$ for $s \in A$. Then $V = \mathrm{D}(\phi(s))$. We know from Thm 2.30 that $\Gamma(U, \mathcal{O}_X) = A_s$. The element $\phi(s)$ is invertible in $\Gamma(V, \mathcal{O}_Y)$ by Lemma 2.45. So there is a unique A-ring homomorphism

$$\tilde{\phi}_s : A_s \to \Gamma(V, \mathcal{O}_Y). \tag{2.52}$$

In particular, for $s = 1$, we get $\Gamma(X, \tilde{\phi}) = \phi$.

By the uniqueness, if $t \in A$ is another element, then the diagram of A-rings

$$\begin{tikzcd}
A_s \arrow[r] \arrow[d, "\tilde{\phi}_s"'] & A_{s \cdot t} \arrow[d, "\tilde{\phi}_{s \cdot t}"] \\
\Gamma(\mathrm{D}(\phi(s)), \mathcal{O}_Y) \arrow[r] & \Gamma(\mathrm{D}(\phi(s \cdot t)), \mathcal{O}_Y)
\end{tikzcd} \tag{2.53}$$

is commutative.

Now we consider an arbitrary open set $U \subseteq X$. We can cover U by principal open sets: $U = \bigcup_{i \in I} \mathrm{D}(s_i)$. Then $V = \bigcup_{i \in I} \mathrm{D}(\phi(s_i))$. There are exact sequences

$$0 \to \Gamma(U, \mathcal{O}_X) \to \prod_i \Gamma(\mathrm{D}(s_i), \mathcal{O}_X) \to \prod_{i,j} \Gamma(\mathrm{D}(s_i \cdot s_j), \mathcal{O}_X) \tag{2.54}$$

and

$$0 \to \Gamma(V, \mathcal{O}_Y) \to \prod_i \Gamma(\mathrm{D}(\phi(s_i)), \mathcal{O}_Y) \to \prod_{i,j} \Gamma(\mathrm{D}(\phi(s_i \cdot s_j)), \mathcal{O}_Y). \tag{2.55}$$

By the previous paragraph there are A-ring homomorphisms

$$\tilde{\phi}_{s_i} : \Gamma(\mathrm{D}(s_i), \mathcal{O}_X) \to \Gamma(\mathrm{D}(\phi(s_i)), \mathcal{O}_Y)$$

and these agree on double intersections by diagram (2.53). So we obtain the ring homomorphism $\tilde{\phi}_U$ in (2.51). As the open set $U \subseteq X$ changes, these become a homomorphism of sheaves of rings $\tilde{\phi} : \mathcal{O}_X \to f_*(\mathcal{O}_Y)$.

Step 5. It remains to prove that $(f, \tilde{\phi})$ is local, i.e. $\tilde{\phi}_y : \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is a local homomorphism for every $y \in Y$ and $x := f(y)$.

Let's return to the construction of the map f. For a point $y \in Y$ with image $x = \mathfrak{p} = f(y) \in X$ we have this solid commutative diagram of rings:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\phi"'] & A/\mathfrak{p} \arrow[r, tail] \arrow[dr, hook] & \boldsymbol{k}(\mathfrak{p}) \arrow[d, dashed] \\
\Gamma(Y, \mathcal{O}_Y) \arrow[rr] & & \boldsymbol{k}(y)
\end{tikzcd} \tag{2.56}$$

Because the slanted arrow is an injection, it extends to the field of fractions $\boldsymbol{k}(\mathfrak{p})$, i.e. the dashed arrow exists. On the other hand, all our ring constructions are over A, so we have the next commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\phi"'] & A_{\mathfrak{p}} = \mathcal{O}_{X,x} \arrow[d, "\tilde{\phi}_y"'] \\
\Gamma(Y, \mathcal{O}_Y) \arrow[r] & \mathcal{O}_{Y,y}
\end{tikzcd} \tag{2.57}$$

Merging (2.56) and (2.57) we get this diagram:

(2.58)
$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\phi"'] & A_{\mathfrak{p}} = \mathcal{O}_{X,x} \arrow[r] \arrow[d, "\tilde{\phi}_y"'] & \boldsymbol{k}(\mathfrak{p}) \arrow[d] \\
\Gamma(Y, \mathcal{O}_Y) \arrow[r] & \mathcal{O}_{Y,y} \arrow[r] & \boldsymbol{k}(y)
\end{tikzcd}$$

The left square is the commutative diagram (2.57), and the outer rectangle is the commutative diagram (2.56). Because $A \to A_{\mathfrak{p}}$ is an epimorphism, it follows that the right square is commutative. But this implies that $\tilde{\phi}_y$ is a local homomorphism. □

Corollary 2.59. *The assignment that sends a ring A to the affine scheme* $\operatorname{Spec}(A)$ *is a functor*

$$\operatorname{Spec} : (\mathsf{Rng}/\mathbb{K})^{\mathrm{op}} \to \mathsf{Sch}_{\mathrm{aff}}/\mathbb{K}$$

Proof. Say $\phi : A \to B$ is a homomorphism in $\mathsf{Rng}/\mathbb{K}$. Let $(X, \mathcal{O}_X) := \operatorname{Spec}(A)$ and $(Y, \mathcal{O}_Y) := \operatorname{Spec}(B)$. By Thm 2.46 there is a map

$$\operatorname{Spec}(\phi) := (f, \tilde{\phi}) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$$

in $\mathsf{LRSp}/\mathbb{K}$. If $\psi : B \to C$ is another homomorphism in $\mathsf{Rng}/\mathbb{K}$, with $(Z, \mathcal{O}_Z) := \operatorname{Spec}(C)$, then there is a map

$$\operatorname{Spec}(\psi) := (g, \tilde{\psi}) : (Z, \mathcal{O}_Z) \to (Y, \mathcal{O}_Y)$$

in $\mathsf{LRSp}/\mathbb{K}$. Now

$$\Gamma(Y, \tilde{\psi}) \circ \Gamma(X, \tilde{\phi}) = \psi \circ \phi,$$

so the uniqueness clause in Thm 2.46 says that

$$\operatorname{Spec}(\phi) \circ \operatorname{Spec}(\psi) = \operatorname{Spec}(\psi \circ \phi).$$

Likewise $\operatorname{Spec}(\mathrm{id}_A) = \mathrm{id}_{\operatorname{Spec}(A)}$. □

Corollary 2.60. *The functor*

$$\operatorname{Spec} : (\mathsf{Rng}/\mathbb{K})^{\mathrm{op}} \to \mathsf{LRS}/\mathbb{K}$$

is adjoint to the functor

$$\Gamma : (\mathsf{LRS}/\mathbb{K})^{\mathrm{op}} \to \mathsf{Rng}/\mathbb{K}.$$

Proof. Thm 2.46 produces a bijection

$$\operatorname{Hom}_{\mathsf{Rng}/\mathbb{K}}\big(A, \Gamma(Y, \mathcal{O}_Y)\big) \xrightarrow{\simeq} \operatorname{Hom}_{\mathsf{LRSp}/\mathbb{K}}\big((Y, \mathcal{O}_Y), \operatorname{Spec}(A)\big). \tag{2.61}$$

We need to prove that this is bifunctorial. This is an exercise. □

Exercise 2.62. Prove that the bijection (2.61) is functorial in A and in $(Y, \mathcal{O}_Y)$.

Corollary 2.63. *The functor*

$$\operatorname{Spec} : (\mathsf{Rng}/\mathbb{K})^{\mathrm{op}} \to \mathsf{Sch}_{\mathrm{aff}}/\mathbb{K}$$

is an equivalence of categories, with quasi-inverse Γ.

Proof. By definition, $\mathsf{Sch}_{\mathrm{aff}}/\mathbb{K}$ is the essential image in $\mathsf{LRSp}/\mathbb{K}$ of the functor Spec.

We need to prove that Spec is fully faithful. Since there is adjunction (Cor 2.60), it is enough to prove that the unit of the adjunction

$$\eta_A : A \to \Gamma(X, \mathcal{O}_X),$$

where $(X, \mathcal{O}_X) := \operatorname{Spec}(A)$, is an isomorphism in $\mathsf{Rng}/\mathbb{K}$ for every A. But this is Cor 2.38. □

Next week we will talk about schemes and fiber products.

Lecture 6, 10 Apr 2019

Definition 2.64. Let $(X, \mathcal{O}_X)$ be a LRSp and let $U \subseteq X$ be an open subset. The LRSp $(U, \mathcal{O}_X|_U)$ is called an *open subspace* of $(X, \mathcal{O}_X)$.

It is clear that $(U, \mathcal{O}_X|_U)$ is a LRSp, and that the inclusion $(U, \mathcal{O}_X|_U) \to (X, \mathcal{O}_X)$ is a map in $\mathsf{LRSp}/\mathbb{K}$.

Proposition 2.65. *Let A be a ring and $s \in A$. Let $(X, \mathcal{O}_X) := \operatorname{Spec}(A)$. Consider the principal open set $U := \mathrm{D}(s) \subseteq X$ and the open subspace $(U, \mathcal{O}_X|_U)$ of $(X, \mathcal{O}_X)$. Let $\phi : A \to A_s$ be the canonical ring homomorphism. Then the map*

$$\operatorname{Spec}(\phi) : \operatorname{Spec}(A_s) \to \operatorname{Spec}(A)$$

induces an isomorphism

$$\operatorname{Spec}(A_s) \xrightarrow{\simeq} (U, \mathcal{O}_X|_U)$$

in $\mathsf{LRSp}/\mathbb{K}$.

Proof. Let's write $(Y, \mathcal{O}_Y) := \operatorname{Spec}(A_s)$. We know that the image of the set Y is the subset $U \subseteq X$, and moreover that $Y \to U$ is a homeomorphism of topological spaces. It remains to prove that the sheaf homomorphism $\tilde{\phi} : \mathcal{O}_X|_U \to \mathcal{O}_Y$ on the topological space $Y = U$ is an isomorphism. It is enough to check that it induces bijections on all stalks. But the stalks are $\mathcal{O}_{X,y} = \mathcal{O}_{Y,y} = A_{\mathfrak{p}}$ for $y = \mathfrak{p} \in U$, and the homomorphism induced by ϕ is the identity. □

3. Schemes

Recall that an affine $\mathbb{K}$-scheme is a locally ringed space $(U, \mathcal{O}_U) \in \mathsf{LRSp}/\mathbb{K}$ which is isomorphic to $\operatorname{Spec}(A)$ for some $\mathbb{K}$-ring A.

Definition 3.1. A $\mathbb{K}$-*scheme* is a locally ringed $\mathbb{K}$-space $(X, \mathcal{O}_X)$ that admits an open covering by affine $\mathbb{K}$-schemes. Namely there is an open covering $X = \bigcup_{i \in I} U_i$, such that each open subspace $(U_i, \mathcal{O}_X|_{U_i})$ is an affine $\mathbb{K}$-scheme.

Definition 3.2. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be $\mathbb{K}$-schemes. A map of $\mathbb{K}$-schemes

$$(f, \psi) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$$

is, by definition, a maps of LR $\mathbb{K}$-spaces. Thus the category $\mathsf{Sch}/\mathbb{K}$ of $\mathbb{K}$-schemes is the full subcategory of $\mathsf{LRSp}/\mathbb{K}$ on the $\mathbb{K}$-schemes.

Proposition 3.3. *Let $(X, \mathcal{O}_X)$ be a $\mathbb{K}$-scheme and let $U \subseteq X$ be an open set. Then the open subspace $(U, \mathcal{O}_X|_U)$ is a $\mathbb{K}$-scheme.*

Proof. Take a point $x \in U$. We need to find an open neighborhood V of x in U such that the open subspace $(V, (\mathcal{O}_X|_U)|_V)$ is an affine $\mathbb{K}$-scheme. We know (by Definition 3.1) that there is an open neighborhood W of x in X such that the open subspace $(W, \mathcal{O}_X|_W)$ is an affine $\mathbb{K}$-scheme; say $(W, \mathcal{O}_X|_W) = \operatorname{Spec}(A)$. Because $U \cap W$ is open in W, we can find an element $s \in A$ such that $x \in \mathrm{D}(s) \subseteq U \cap W$. Write $V := \mathrm{D}(s)$. Now $(V, (\mathcal{O}_X|_U)|_V) = (V, \mathcal{O}_X|_V)$, and by Prop 2.65 we know that $(V, \mathcal{O}_X|_V) \cong \operatorname{Spec}(A_s)$ as LRSp's. □

Here are two examples of schemes which are not affine.

Example 3.4. Let $\mathbb{K}$ be a nonzero ring. (If you prefer, you can assume that $\mathbb{K}$ is a field, or even an algebraically closed field.) We define 1-*dimensional projective space* to be the following scheme $\mathbf{P}^1_{\mathbb{K}}$. Let $U_0 := \operatorname{Spec}(\mathbb{K}[t_1])$ and $U_1 := \operatorname{Spec}(\mathbb{K}[t_0])$. So $U_0 \cong U_1 \cong \mathbf{A}^1_{\mathbb{K}}$, the 1-dimensional affine space over $\mathbb{K}$, see Exa 2.17. Inside U_0 we have the affine open subscheme $U_{0,1} := \operatorname{Spec}(\mathbb{K}[t_1, t_1^{-1}])$, and inside U_1 we have the affine open subscheme $U_{1,0} := \operatorname{Spec}(\mathbb{K}[t_0, t_0^{-1}])$. The ring isomorphism $\mathbb{K}[t_0, t_0^{-1}] \xrightarrow{\simeq} \mathbb{K}[t_1, t_1^{-1}]$, $t_0 \mapsto t_1^{-1}$, induces an isomorphism of schemes $\phi_{0,1} : U_{0,1} \xrightarrow{\simeq} U_{1,0}$. By Thm 1.24 we can glue U_0 and U_1 along $\phi_{0,1}$. The resulting LRSp is the scheme $\mathbf{P}^1_{\mathbb{K}}$.

Exercise 3.5. Show that $\mathbf{P}^1_{\mathbb{K}}$ is not an affine scheme. (Hint: write $(X, \mathcal{O}_X) := \mathbf{P}^1_{\mathbb{K}}$. By direct calculation show that the ring $\Gamma(X, \mathcal{O}_X) = \mathbb{K}$. Then use Cor 2.63.)

Example 3.6. Let $\mathbb{K}$ be a nonzero ring. (If you prefer, you can assume that $\mathbb{K}$ is a field, or even an algebraically closed field.) Consider the affine plane $\mathbf{A}^2_{\mathbb{K}} = \operatorname{Spec}(\mathbb{K}[t_1, t_2])$. The "origin" is the closed subset $Z := \mathrm{Z}(t_1, t_2) \subseteq \mathbf{A}^2_{\mathbb{K}}$. If $\mathbb{K}$ is a field, then Z contains only one point; but in general it is the topological space $\operatorname{Spec}(\mathbb{K})$, a closed subset of $\mathbf{A}^2_{\mathbb{K}}$.

Define the "punctured plane" $U := \mathbf{A}^2_{\mathbb{K}} - Z$. This is an open subscheme $(U, \mathcal{O}_U)$ of $\mathbf{A}^2_{\mathbb{K}}$.

Exercise 3.7. Show that the punctured plane $(U, \mathcal{O}_U)$ is not an affine scheme. (Hint: define

$$U_i := \operatorname{Spec}(\mathbb{K}[t_1, t_2, t_i^{-1}] \subseteq \mathbf{A}^2_{\mathbb{K}},$$

an affine open subscheme. Show that $U = U_1 \cup U_2$. Use this covering to calculate the ring $\Gamma(U, \mathcal{O}_U)$. Show that $\Gamma(U, \mathcal{O}_U) = \mathbb{K}[t_1, t_2]$. Then use Cor 2.63.)

Exercise 3.8. Let U be the punctured plane from Example 3.6. Show that there is a "canonical" surjective map of schemes $g : U \to \mathbf{P}^1_{\mathbb{K}}$. Show that when $\mathbb{K}$ is a field, then set-theoretically this is the usual way we get the projective line.

4. Fibered Products of Schemes

Fibered products are very important in the theory of schemes.

To be clear what we mean, here's a definition

Definition 4.1. Let $f : Y \to X$ and $g : Z \to X$ be morphisms in a category C. A *fibered product of Y and Z over X* is an object $W \in \mathsf{C}$, with morphisms $p : W \to Y$ and $q : W \to Z$, satisfying these two conditions:

(i) $f \circ p = g \circ q$.

(ii) Given an object W', and morphisms $p' : W' \to Y$ and $q' : W' \to Z$, such that $f \circ p' = g \circ q'$, there exists a unique morphism $r : W' \to W$ such that $p' = p \circ r$ and $q' = q \circ r$.

This is illustrated in the next commutative diagram.

$$\begin{tikzcd}
W' \arrow[rd, dashed, "r"] \arrow[rrd, dashed, bend left, "q'"] \arrow[rdd, dashed, bend right, "p'"'] & & \\
& W \arrow[r, "q"] \arrow[d, "p"] & Z \arrow[d, "g"] \\
& Y \arrow[r, "f"] & X
\end{tikzcd}$$

It is clear that the fibered product (W, p, q) is unique, up to a unique isomorphism. The notation for the fibered product is $Y \times_X Z$, leaving the morphisms implicit.

See Figure 5.

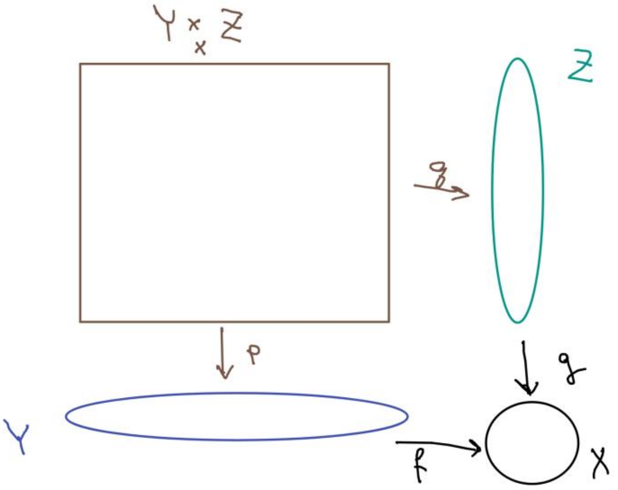

Figure 5. Fibered product

Example 4.2. In the category Set the fiber product is

$$Y \times_X Z = \{(y,z) \mid f(y) = g(z)\} \subseteq Y \times Z,$$

with the obvious projections p, q.

Here is another name for the same categorical construct.

Definition 4.3. A diagram

$$\begin{tikzcd} W \arrow[r, "q"] \arrow[d, "p"'] & Z \arrow[d, "g"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

in the category C is called *cartesian* if $W = Y \times_X Z$.

We can also describe the fibered product in terms of representable functors. Given $f : Y \to X$ and $g : Z \to X$ as in Definition 4.1, let $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ be this functor:

$$F(U) := \mathrm{Hom}_{\mathsf{C}}(U,Y) \times_{\mathrm{Hom}_{\mathsf{C}}(U,X)} \mathrm{Hom}_{\mathsf{C}}(U,Z) \in \mathsf{Set}\,.$$

I.e. the elements of $F(U)$ are pairs (p', q') of morphisms $p' : U \to Y$ and $q' : U \to Z$ such that $f \circ p' = g \circ q'$. The fibered product $W = Y \times_X Z$ is an object of C that represents F:

$$F \cong \mathrm{Hom}_{\mathsf{C}}(-, W)$$

as functors $\mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$. The pair of projections $(p, q) \in F(W)$ is the universal pair.

The next theorem gives a sufficient condition for the existence of the fibered product in $\mathsf{LRSp}/\mathbb{K}$.

Theorem 4.4. *Let $f : Y \to X$ and $g : Z \to X$ be maps in $\mathsf{LRSp}/\mathbb{K}$. Assume that there is a collection of open subspaces $\{X_i\}_{i \in I}$ of X, a collection of open subspaces $\{Y_i\}_{i \in I}$ of Y, and a collection of open subspaces $\{Z_i\}_{i \in I}$ of Z, such that these three conditions hold:*

(a) *For every $i \in I$ there are inclusions $f(Y_i) \subseteq X_i$ and $g(Z_i) \subseteq X_i$.*

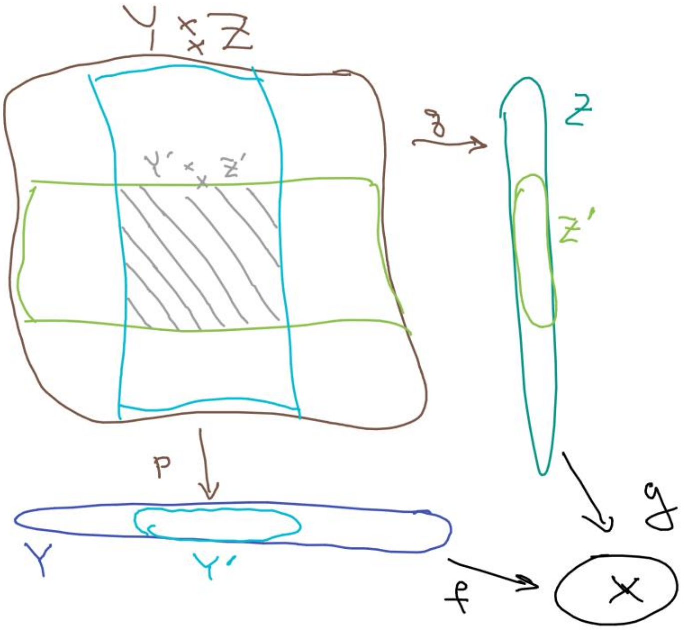

FIGURE 6. For Lem 4.6

(b) *For every $i \in I$ the fibered product $Y_i \times_{X_i} Z_i$ exists.*
(c) *For every pair of points $(y, z) \in Y \times Z$ such that $f(y) = g(z)$, there is an index $i \in I$ such that $y \in Y_i$ and $z \in Z_i$.*

Then the fibered product $Y \times_X Z$ exists. Moreover, for every i the canonical map

$$Y_i \times_{X_i} Z_i \to Y \times_X Z$$

is an open embedding, and

$$Y \times_X Z = \bigcup_{i \in I} Y_i \times_{X_i} Z_i .$$

The proof requires two lemmas.

Lemma 4.5. *Let $f : X' \to X$ be an open embedding in $\mathsf{LRSp}/\mathbb{K}$. Then f is a monomorphism in the category $\mathsf{LRSp}/\mathbb{K}$.*

Lemma 4.6. *Let $f : Y \to X$ and $g : Z \to X$ be maps in $\mathsf{LRSp}/\mathbb{K}$. Let $X' \subseteq X$, $Y' \subseteq Y$ and $Z' \subseteq Z$ be open subspaces, such that $f(Y') \subseteq X'$ and $g(Z') \subseteq X'$. Assume the fiber product $Y \times_X Z$ exists, with projections p, q. Let W' be the open subspace*

$$W' := p^{-1}(Y') \cap q^{-1}(Z') \subseteq Y \times_X Z,$$

with induced projections $p' : W \to Y'$ and $q' : W \to Z'$. Then $W' = Y' \times_{X'} Z'$.

See Figure 6 for an illustration of this lemma.

Exercise 4.7. Prove Lemmas 4.5 and 4.6.

Proof of Thm 4.4. Let's write $W_i := Y_i \times_{X_i} Z_i$. The projections are $p_i : W_i \to Y_i$ and $q_i : W_i \to Z_i$. See Figure 7 for an illustration.

For a pair of indices i, j define $X_{i,j} := X_i \cap X_j$, $Y_{i,j} := Y_i \cap Y_j$ and $Z_{i,j} := Z_i \cap Z_j$. Let

$$W_{i,j} := p_i^{-1}(Y_{i,j}) \cap q_i^{-1}(Z_{i,j}) \subseteq W_i .$$

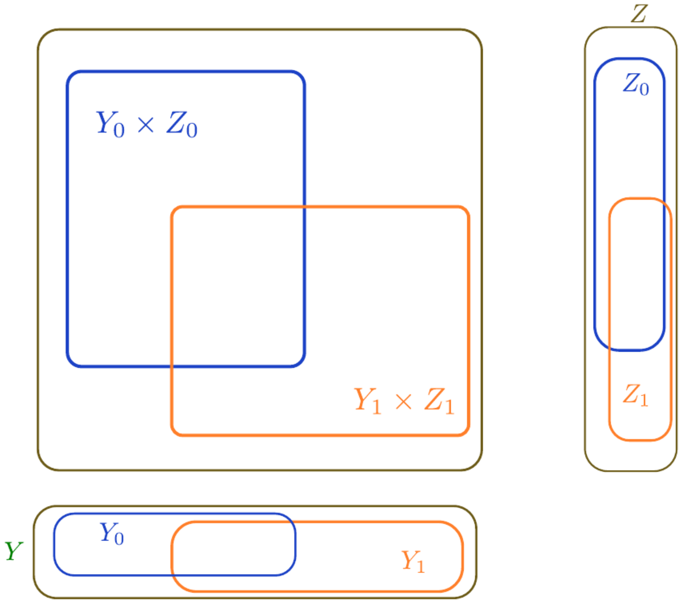

Figure 7.

By Lemma 4.6 we know that

$$W_{i,j} \cong Y_{i,j} \times_{X_{i,j}} Z_{i,j}.$$

But $Y_{i,j} = Y_{j,i}$, etc., so there is an isomorphism $h_{i,j} : W_{i,j} \xrightarrow{\simeq} W_{j,i}$, and this isomorphism is determined by the properties of fibered products, namely by its compatibility with the projections to Y and to Z. This implies that the conditions of Thm 1.24 are satisfied: $h_{j,k} \circ h_{i,j} = h_{i,k}$. In other words, these three isomorphism between three incarnations of the fibered product $Y_{i,j,k} \times_{X_{i,j,k}} Z_{i,j,k}$ are compatible. According to Thm 1.24 we get a locally ringed space W, with a covering $W = \bigcup_{i \in I} W_i$ by open subspaces.

The projections $p_i : W_i \to Y$ and $p_j : W_j \to Y$ agree on $W_{i,j} = W_{j,i}$, now seen as open subspaces of W. Therefore they glue to a map $p : W \to Y$. Likewise the projections $q_i : W_i \to Z$ glue to a map $q : W \to Z$. By Lemma 4.6 we know that $W_i = p^{-1}(Y_i) \cap q^{-1}(Z_i)$ as subspaces of W.

It remains to prove that (W, p, q) is the fibered product $Y \times_X Z$. Take a space U and maps $p' : U \to Y$ and $q' : U \to Z$ such that $f \circ p' = g \circ q'$. Define

$$U_i := p'^{-1}(Y_i) \cap q'^{-1}(Z_i) \subseteq U.$$

Then $U = \bigcup_{i \in I} U_i$ is an open covering. For every i there is a unique map $r_i : U_i \to W_i$ that's compatible with the projections, since $W_i = Y_i \times_{X_i} Z_i$. By the arguments used above, the maps r_i and r_j agree on $U_{i,j} := U_i \cap U_j$. So they glue to a map $r : U \to W$. This r is unique because all its local pieces r_i are unique. So $W = Y \times_X Z$ as claimed. □

This is where lecture 6 ended. From here on it is self-reading during the vacation.

Before proving the existence of the fibered product of schemes, a little reminder on rings.

The *fibered coproduct* is the dual (or opposite) notion of fibered product. To be a bit more precise, the fibered coproduct in a category C is the fibered product in the category C^{op}.

Lemma 4.8. *Let $f^* : A \to B$ and $g^* : A \to C$ be homomorphisms in $\mathsf{Rng}/\mathbb{K}$. Then the ring $D := B \otimes_A C$, with the canonical homomorphisms $p^* : B \to D$ and $q^* : C \to D$, is the fibered coproduct of f^* and g^* in $\mathsf{Rng}/\mathbb{K}$.*

Proof. We know from commutative algebra that for a ring E, the ring homomorphisms $r^* : D \to E$ are in bijection with the pairs of A-ring homomorphisms $p'^* : B \to E$ and $q'^* : C \to E$ such that $f^* \circ p'^* = g^* \circ q'^*$. □

The solid diagram

$$\begin{tikzcd}
A \arrow[r, "g^*"] \arrow[d, "f^*"'] & C \arrow[d, "q^*"] \arrow[ddr, dashed, "q'^*"] & \\
B \arrow[r, "p^*"] \arrow[drr, dashed, "p'^*"'] & D \arrow[dr, dashed, "r^*"] & \\
& & E
\end{tikzcd}$$

in the category $\mathsf{Rng}/\mathbb{K}$, as in the lemma, is called *cocartesian*.

Theorem 4.9. *Let $f : Y \to X$ and $g : Z \to X$ be maps in $\mathsf{LRSp}/\mathbb{K}$, and assume these three spaces are schemes. Then the fibered product $Y \times_X Z$ in $\mathsf{LRSp}/\mathbb{K}$ exists, and it is a scheme. Moreover, if X, Y, Z are all affine schemes, then the fibered product $Y \times_X Z$ is an affine scheme.*

Proof.

Step 1. Suppose $X = \operatorname{Spec}(A)$, $Y = \operatorname{Spec}(B)$ and $Z = \operatorname{Spec}(C)$. There are ring homomorphisms $f^* : A \to B$ and $g^* : A \to C$. Define $D := B \otimes_A C \in \mathsf{Rng}/\mathbb{K}$ and $W := \operatorname{Spec}(D)$. The canonical ring homomorphisms $B \to D$ and $C \to D$ give rise to maps $p : W \to Y$ and $q : W \to Z$. We claim that $(W, p, q) = Y \times_X Z$. This is a consequence of Thm 2.46 and Lem 4.8. Indeed, given a space $(U, \mathcal{O}_U) \in \mathsf{LRSp}/\mathbb{K}$ with ring of functions $E := (U, \mathcal{O}_U)$, we know from Thm 2.46 that morphisms from U to affine schemes are in bijection with ring homomorphisms to E. By Lem 4.8, with arrows reversed, we see that maps $U \to W$ are in bijection with pairs of maps $U \to Y$ and $U \to Z$ whose composed maps to X aree equal. This is the property of the fibered product.

Step 2. Take a pair of points $(y, z) \in Y \times Z$ such that $f(y) = g(z)$, and let $x := f(y)$. Choose some affine open neighborhood X_x of x in X. Then choose an affine open neighborhood Y_y of y in $f^{-1}(X_x)$, and an affine open neighborhood Z_z of z in $g^{-1}(X_x)$. We get a collection of open subspaces as in conditions (a) and (c) of Thm 4.4, with indexing set I consisting of these pairs (y, z), which is the fibered product of the underlying sets.

Step 3. We have collections of affine open subschemes $\{X_i\}_{i \in I}$ of X, $\{Y_i\}_{i \in I}$ of Y and $\{Z_i\}_{i \in I}$ of Z, satisfying conditions (a) and (c) of Thm 4.4. These collections could be from the construction in step 2 above, but not necessarily so. According to step 1, for every index $i \in I$ the fibered product $W_i := Y_i \times_{X_i} Z_i$ exists, and it is an affine scheme. Now Thm 4.4 applies. We get an object $W = Y \times_X Z$ in $\mathsf{LRSp}/\mathbb{K}$, and its has an open covering by the affine schemes W_i. Hence W is a scheme. □

Fiber products will be used a lot. One operation they provide is "base change". Here are a few examples of this feature.

But first a definition.

Definition 4.10. Fix a scheme X (say in $\mathsf{Sch}/\mathbb{Z}$). By an *X-scheme* we mean a scheme Y together with a map $\pi_Y : Y \to X$. The X-schemes form a category in the obvioous way, and we denote it by Sch/X.

If $X = \operatorname{Spec}(A)$ is affine, then the expressions Sch/X and Sch/A have the same meaning.

Example 4.11. The projective line is "defined over $\mathbb{Z}$", or is "gotten by base change from $\mathbf{P}^1_{\mathbb{Z}}$". Here is what this means. In Example 3.4, taking $\mathbb{K} := \mathbb{Z}$, we saw how to build the $\mathbb{Z}$-scheme $\mathbf{P}^1_{\mathbb{Z}}$. But for any ring A we can also build $\mathbf{P}^1_A$. It is not hard (and maybe useful to try) to show that

$$\mathbf{P}^1_A \cong \mathbf{P}^1_{\mathbb{Z}} \times_{\operatorname{Spec}(\mathbb{Z})} \operatorname{Spec}(A)$$

in Sch/A. (By abuse of notation we sometimes write $\mathbf{P}^1_{\mathbb{Z}} \times_{\mathbb{Z}} A$ instead of the longer expression above.)

We can also take an arbitrary scheme X (not affine), and form the scheme

$$\mathbf{P}^1_X := \mathbf{P}^1_{\mathbb{Z}} \times_{\operatorname{Spec}(\mathbb{Z})} X.$$

Example 4.12. Let X be a scheme. A *group scheme over X* is a scheme $G \in \mathsf{Sch}/X$ equipped with maps $\operatorname{mult} : G \times_X G \to G$, $\operatorname{inv} : G \to G$ and $\operatorname{unit} : X \to G$ in Sch/X , that satisfy the group axioms.

Example 4.13. To be concrete, for $n \geq 1$ we have the group scheme $\mathrm{GL}_{n,\mathbb{Z}}$. It is an affine group scheme: as a scheme over $\mathbb{Z}$ it is $\operatorname{Spec}(A)$, where A is the polynomial ring over $\mathbb{Z}$ in the n^2 variables $t_{i,j}$, localized w.r.t. the determinant. You can try to write the formulas for the ring homomorphisms that yield the group operations.

For an arbitrary scheme X we have the group scheme

$$\mathrm{GL}_{n,X} := \mathrm{GL}_{n,\mathbb{Z}} \times_{\operatorname{Spec}(\mathbb{Z})} X.$$

Example 4.14. We already know (from the last example) the $\mathbb{R}$-group scheme $G := \mathrm{GL}_{1,\mathbb{R}}$. As an $\mathbb{R}$-scheme it is $G = \operatorname{Spec}(A)$, where $A := \mathbb{R}[t, t^{-1}]$. Now let $B := \mathbb{R}[s, t]/(s^2 + t^2 - 1)$. The affine scheme $H := \operatorname{Spec}(B)$ is also a group scheme over $\mathbb{R}$. The multiplication is that of the circle. Try to guess the formulas.

One can show that G and H are not isomorphic as group schemes over $\mathbb{R}$. (In fact, they are not isomorphic as $\mathbb{R}$-schemes.)

However, after base change to $\mathbb{C}$ they become isomorphic:

$$G \times_{\operatorname{Spec}(\mathbb{R})} \operatorname{Spec}(\mathbb{C}) \cong H \times_{\operatorname{Spec}(\mathbb{R})} \operatorname{Spec}(\mathbb{C}) \cong \mathrm{GL}_{1,\mathbb{C}}$$

as group schemes over $\mathbb{C}$.

Here is a quick discussion of "points of a scheme", that could help understanding the examples on group schemes above. Let X be a $\mathbb{K}$-scheme. Given a $\mathbb{K}$-ring A, we define the set

$$X(A) := \operatorname{Hom}_{\mathsf{Sch}/\mathbb{K}}\big(\operatorname{Spec}(A), X\big).$$

The elements of $X(A)$ are called the *A-points of X*.

Key example: for the scheme

$$X = \mathbf{A}^n_{\mathbb{K}} = \operatorname{Spec}(\mathbb{K}[t_1, \ldots, t_n])$$

we have, by virtue of Cor 2.63,

$$\mathbf{A}^n_{\mathbb{K}}(A) = \operatorname{Hom}_{\mathsf{Rng}/\mathbb{K}}\big(\mathbb{K}[t_1, \ldots, t_n], A\big) = \operatorname{Hom}_{\mathsf{Set}}\big(\{t_1, \ldots, t_n\}, A\big) = A^n,$$

the set of n-tuples of elements of the ring A.

If G is a group scheme over $\mathbb{K}$, then the set $G(A)$ is a group, in the ordinary sense of the word. For $G = \mathrm{GL}_{n,\mathbb{K}}$, the group $G(A) = \mathrm{GL}_{n,\mathbb{K}}(A)$ is nothing but the group of invertible $n \times n$ matrices with entries in A.

The notions of "points" and "base change" play the following game with each other. Suppose we have a homomorphism $A \to B$ in $\mathsf{Rng}/\mathbb{K}$, and a scheme $X \in \mathsf{Sch}/\mathbb{K}$. Define the A-scheme

$$X_A := X \times_{\operatorname{Spec}(\mathbb{K})} \operatorname{Spec}(A).$$

Since B is both in $\mathsf{Rng}/\mathbb{K}$ and in Rng/A, we can consider the sets

$$X(B) = \operatorname{Hom}_{\mathsf{Sch}/\mathbb{K}}\big(\operatorname{Spec}(B), X\big)$$

and

$$X_A(B) = \operatorname{Hom}_{\mathsf{Sch}/A}\big(\operatorname{Spec}(B), X_A\big).$$

I leave it as an exercise to prove that these sets are equal; or more precisely, the canonical function $X_A(B) \to X(B)$ is bijective. For this reason we can write $\mathbf{A}^n(B)$ unambiguously – it can mean either $\mathbf{A}^n_{\mathbb{Z}}(B)$ or $\mathbf{A}^n_A(B)$, for $B \in \mathsf{Rng}/A$, but these sets are equal (to the set B^n). Likewise for the group $\mathrm{GL}_n(A)$.

Example 4.15. Continuing Exa 4.14, the group scheme $G = \mathrm{GL}_{1,\mathbb{R}}$ has the group $G(\mathbb{R}) = \mathrm{GL}_1(\mathbb{R}) = \mathbb{R}^\times$ as its group of $\mathbb{R}$-points. For the group scheme H, the group of $\mathbb{R}$-points $H(\mathbb{R})$ is the unit circle in the plane. This should help figure out the formulas for the operations of H: think of $H(\mathbb{R})$ as a subgroup of $\mathbb{C}^\times$, with s as the real part and t as the imaginary part.

The two group schemes G and H are not isomorphic. One way to see this is to compare their groups of $\mathbb{R}$-points. In the group $G(\mathbb{R}) = R^\times$ the only finite subgroup is $\{\pm 1\}$, whereas in $H(\mathbb{R})$ we have finite cyclic groups of all sizes.

However, there is an isomorphism $G_{\mathbb{C}} \cong H_{\mathbb{C}}$ of group schemes over $\mathbb{C}$, and hence the groups of $\mathbb{C}$-points are:

$$H(\mathbb{C}) = H_{\mathbb{C}}(\mathbb{C}) \cong G_{\mathbb{C}}(\mathbb{C}) = G(\mathbb{C}) = \mathbb{C}^\times.$$

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