Course Notes:

Algebraic Geometry – Schemes 2

Spring 2018-19

Amnon Yekutieli

Contents

1. Review of Prior Material

Lecture 1, 27 Feb 2019

We fix a nonzero commutative base ring $\mathbb{K}$. All rings will be commutative $\mathbb{K}$-rings by default.

Let X be a topological space. Recall that a *presheaf* of $\mathbb{K}$-modules on X is a functor

$$\mathcal{M} : \mathsf{Open}(X)^{\mathrm{op}} \to \mathsf{Mod}\,\mathbb{K},$$

where $\mathsf{Open}(X)$ is the category of open sets of X, and $\mathsf{Mod}\,\mathbb{K}$ is the category of $\mathbb{K}$-modules.

More concretely, the presheaf $\mathcal{M}$ is the data of a $\mathbb{K}$-module $\Gamma(U, \mathcal{M})$ for every open set $U \subseteq X$, and a module homomorphism

$$\mathrm{rest}_{V/U} : \Gamma(U, \mathcal{M}) \to \Gamma(V, \mathcal{M})$$

for every inclusion $V \subseteq U$. The conditions are that

$$\mathrm{rest}_{W/U} = \mathrm{rest}_{W/V} \circ \mathrm{rest}_{V/U}$$

for every double inclusion $W \subseteq V \subseteq U$, and that $\mathrm{rest}_{U/U} = \mathrm{id}$ for every U. We often use the abbreviation

$$m|_V := \mathrm{rest}_{V/U}(m) \in \Gamma(V, \mathcal{M})$$

for a section $m \in \Gamma(U, \mathcal{M})$.

The presehaves of $\mathbb{K}$-modules on X form a category. The morphisms are the obvious ones. We denote it by $\mathsf{Mod}^{\mathrm{pre}}\,\mathbb{K}_X$. Here $\mathbb{K}_X$ is the constant sheaf on X with values in $\mathbb{K}$ (but we didn't define sheaves yet...)

Given a presheaf $\mathcal{M}$ and a point $x \in X$, we have the *stalk* $\mathcal{M}_x$ of $\mathcal{M}$ at x. This is a $\mathbb{K}$-module. Recall the formula:

$$\mathcal{M}_x = \lim_{U \to} \Gamma(U, \mathcal{M}),$$

where the direct limit is on the open neighborhoods U of x. Taking the stalk at x is a functor

$$\mathsf{Mod}^{\mathrm{pre}}\,\mathbb{K}_X \to \mathsf{Mod}\,\mathbb{K}.$$

A presheaf $\mathcal{M}$ is a *sheaf* if it satisfies the two sheaf axioms. These can be encoded as follows: for every open set $U \subseteq X$ and every open covering $U = \bigcup_{i \in I} U_i$, the sequence of $\mathbb{K}$-modules

$$0 \to \Gamma(U, \mathcal{M}) \xrightarrow{\epsilon} \prod_{i_0 \in I} \Gamma(U_{i_0}, \mathcal{M}) \xrightarrow{\delta^0 - \delta^1} \prod_{i_0, i_1 \in I} \Gamma(U_{i_0} \cap U_{i_1}, \mathcal{M})$$

is exact. Here the homomorphisms ϵ, δ^i are induced by the restrictions. For more details see [Ye4, Sec 3 and Prop 7.10].

The sheaves of $\mathbb{K}$-modules on X, also called $\mathbb{K}_X$*-modules*, form a full subcategory of $\mathsf{Mod}^{\mathrm{pre}}\,\mathbb{K}_X$, that we denote by $\mathsf{Mod}\,\mathbb{K}_X$.

The *sheafification functor* assigns to each presheaf $\mathcal{M}$ a sheaf $\mathrm{Sh}(\mathcal{M})$, and a homomorphism of presheaves

$$\tau_{\mathcal{M}} : \mathcal{M} \to \mathrm{Sh}(\mathcal{M}),$$

which is universal for homomorphisms into sheaves. Namely: if $\mathcal{N}$ is a sheaf and $\phi : \mathcal{M} \to \mathcal{N}$ is a homomorphism of presehaves, then there is a unique homomorphism of sheaves $\phi' : \mathrm{Sh}(\mathcal{M}) \to \mathcal{N}$ such that $\phi = \phi' \circ \tau_{\mathcal{M}}$.

Exercise 1.1. Read the proof of the sheafication, including an understanding of the Godement sheaf $\mathrm{GSh}(\mathcal{M})$. This is [Ye4, Thm 6.1]. We will need this next week when we define the structure sheaf of an affine scheme.

Exercise 1.2. State the categorical property of the functor Sh, as an adjoint (from which side?) to the inclusion functor $\mathsf{Mod}\,\mathbb{K}_X \to \mathsf{Mod}^{\mathrm{pre}}\,\mathbb{K}_X$.

The sheafication does not change the stalks: for every $x \in X$ the homomorphism:

$$\tau_{\mathcal{M},x} : \mathcal{M}_x \xrightarrow{\simeq} \mathrm{Sh}(\mathcal{M})_x$$

is bijective.

A *ringed space* over $\mathbb{K}$ is a pair $(X, \mathcal{O}_X)$, consisting of a topological space X, and a sheaf of $\mathbb{K}$-rings $\mathcal{O}_X$ on X.

Let $(X, \mathcal{O}_X)$ be such a ringed space. A sheaf of $\mathcal{O}_X$-modules, also called an $\mathcal{O}_X$*-module*, is a sheaf of $\mathbb{K}$-modules on X, together with a structure of a $\Gamma(U, \mathcal{O}_X)$-module for every open set $U \subseteq X$, which respects restrictions to open subsets.

The category of $\mathcal{O}_X$-modules is denoted by $\mathsf{Mod}\,\mathcal{O}_X$. The morphisms are the obvious ones.

The sheafifcation functor respects the $\mathcal{O}_X$-module structure: if $\mathcal{M} \in \mathsf{Mod}^{\mathrm{pre}}\,\mathcal{O}_X$ then $\mathrm{Sh}(\mathcal{M}) \in \mathsf{Mod}\,\mathcal{O}_X$, and $\tau_{\mathcal{M}}$ is $\mathcal{O}_X$-linear.

Let $\phi : \mathcal{M} \to \mathcal{N}$ be a homomorphism in $\mathsf{Mod}\,\mathcal{O}_X$. Its *kernel* is the $\mathcal{O}_X$-module $\mathrm{Ker}(\phi)$ such that

$$\Gamma(U, \mathrm{Ker}(\phi)) = \mathrm{Ker}\Big(\Gamma(U, \phi) : \Gamma(U, \mathcal{M}) \to \Gamma(U, \mathcal{N})\Big).$$

The *image* of ϕ is the $\mathcal{O}_X$-module

$$\mathrm{Im}(\phi) := \mathrm{Sh}(\mathrm{Im}^{\mathrm{pre}}(\phi)),$$

where $\mathrm{Im}^{\mathrm{pre}}(\phi)$ is the presheaf defined by

$$\Gamma(U, \mathrm{Im}^{\mathrm{pre}}(\phi)) = \mathrm{Im}\Big(\Gamma(U, \phi) : \Gamma(U, \mathcal{M}) \to \Gamma(U, \mathcal{N})\Big).$$

Note that $\mathrm{Ker}(\phi)$ is a subsheaf of $\mathcal{M}$ and $\mathrm{Im}(\phi)$ is a subsheaf of $\mathcal{N}$.

A sequence of homomorphisms

$$\boldsymbol{S} = \Big(\cdots \mathcal{M}^i \xrightarrow{\phi^i} \mathcal{M}^{i+1} \xrightarrow{\phi^{i+1}} \mathcal{M}^{i+1} \cdots\Big)$$

in $\mathsf{Mod}\,\mathcal{O}_X$ is called *exact* if for every point $x \in X$ the sequence of homomorphisms

$$\cdots \mathcal{M}^i_x \xrightarrow{\phi^i_x} \mathcal{M}^{i+1}_x \xrightarrow{\phi^{i+1}_x} \mathcal{M}^{i+1}_x \cdots$$

in $\mathsf{Mod}\,\mathcal{O}_{X,x}$ is exact. We know that $\boldsymbol{S}$ is exact iff for every i there is equality

$$\mathrm{Im}(\phi^{i-1}) = \mathrm{Ker}(\phi^i)$$

of these subsheaves of $\mathcal{M}^i$.

Given an open set $U \subseteq X$ we write $\mathcal{O}_U := \mathcal{O}_X|_U$. Thus $(U, \mathcal{O}_U)$ is also a ringed space. There is a restriction functor

$$\mathsf{Mod}\,\mathcal{O}_X \to \mathsf{Mod}\,\mathcal{O}_U,\ \mathcal{M} \mapsto \mathcal{M}|_U.$$

Sheaves, and homomorphisms between sheaves, can be glued.

Notation: given an open covering $X = \bigcup_{i \in I} U_i$, and indices $i, j, \ldots \in I$, we often write

$$U_{i,j,\ldots} := U_i \cap U_j \cap \cdots . \tag{1.3}$$

Theorem 1.4 (Gluing Sheaf Homomorphisms)**.** *Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{O}_X$-modules, let $X = \bigcup_{i \in I} U_i$ be an open covering, and let*

$$\phi_i : \mathcal{M}|_{U_i} \to \mathcal{N}|_{U_i}$$

be homomorphisms of $\mathcal{O}_{U_i}$-modules satisfying the condition

$$\phi_i|_{U_{i,j}} = \phi_j|_{U_{i,j}} : \mathcal{M}|_{U_{i,j}} \to \mathcal{N}|_{U_{i,j}}.$$

(This is the 0*-cocycle condition.)*

Then there is a unique homomorphism of $\mathcal{O}_X$-modules

$$\phi : \mathcal{M} \to \mathcal{N}$$

such that

$$\phi|_{U_i} = \phi_i : \mathcal{M}|_{U_i} \to \mathcal{N}|_{U_i}$$

for all i.

Theorem 1.5 (Gluing Sheaves)**.** *Suppose $X = \bigcup_{i \in I} U_i$ is an open covering. For every i let $\mathcal{M}_i$ be an $\mathcal{O}_{U_i}$-module, and for every i, j let*

$$\phi_{i,j} : \mathcal{M}_i|_{U_{i,j}} \xrightarrow{\simeq} \mathcal{M}_j|_{U_{i,j}}$$

be an isomorphism of $\mathcal{O}_{U_{i,j}}$-modules. The condition is that

$$\phi_{j,k}|_{U_{i,j,k}} \circ \phi_{i,j}|_{U_{i,j,k}} = \phi_{i,k}|_{U_{i,j,k}}$$

as isomorphisms

$$\mathcal{M}_i|_{U_{i,j,k}} \xrightarrow{\simeq} \mathcal{M}_k|_{U_{i,j,k}}$$

for all i, j, k. (This is the 1-cocycle condition.)

Then there is an $\mathcal{O}_X$-module $\mathcal{M}$, together with isomorphisms

$$\phi_i : \mathcal{M}|_{U_i} \xrightarrow{\simeq} \mathcal{M}_i$$

of $\mathcal{O}_X|_{U_i}$-modules, such that

$$\phi_{i,j} \circ \phi_i|_{U_{i,j}} = \phi_j|_{U_{i,j}} : \mathcal{M}|_{U_{i,j}} \xrightarrow{\simeq} \mathcal{M}_j|_{U_{i,j}}.$$

Moreover, the $\mathcal{O}_X$-module $\mathcal{M}$, with the collection of isomorphisms $\{\phi_i\}$, is unique up to a unique isomorphism.

Exercise 1.6. Read and make sure you understand these last two thms. Proofs can be found here: [Ye4, Thm 7.3] and [Ye4, Thm 7.4].

Let $\mathcal{M}, \mathcal{N} \in \mathsf{Mod}\, \mathcal{O}_X$. The $\mathcal{O}_X$-module $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ is defined by

$$\Gamma(U, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})) := \mathrm{Hom}_{\mathsf{Mod}\, \mathcal{O}_X|_U}(\mathcal{M}|_U, \mathcal{N}|_U).$$

The $\mathcal{O}_X$-module $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$ is the sheaf associated to the presheaf

$$U \mapsto \Gamma(U, \mathcal{M}) \otimes_{\Gamma(U, \mathcal{O}_X)} \Gamma(U, \mathcal{N}). \tag{1.7}$$

Exercise 1.8. Find an example of $\mathcal{M}$ and $\mathcal{N}$ such that the presheaf tensor product (1.7) is not a sheaf. (Hint: line bundles on $\mathbf{P}^1$)

Let $f : Y \to X$ be a map of topological spaces. For a $\mathbb{K}_Y$-module $\mathcal{N}$, its pushforward, or *direct image*, is the $\mathbb{K}_X$-module $f_*(\mathcal{N})$ defined by

$$\Gamma(U, f_*(\mathcal{N})) := \Gamma(f^{-1}(U), \mathcal{N})$$

for open sets $U \subseteq X$. We get a functor

$$f_* : \mathsf{Mod}\, \mathbb{K}_Y \to \mathsf{Mod}\, \mathbb{K}_X. \tag{1.9}$$

For a $\mathbb{K}_X$-module $\mathcal{M}$, the pullback, or *inverse image*, is the $\mathbb{K}_Y$-module $f^{-1}(\mathcal{M})$ defined by

$$\Gamma(V, f^{-1}(\mathcal{M})) := \lim_{U \to} \Gamma(U, \mathcal{M}),$$

where $V \subseteq Y$ is open, and U runs over the open sets in X that contain $f(V)$. We get a functor

$$f^{-1} : \mathsf{Mod}\, \mathbb{K}_X \to \mathsf{Mod}\, \mathbb{K}_Y. \tag{1.10}$$

There is adjunction: an isomorphism of $\mathbb{K}$-modules

$$\mathrm{Hom}_{\mathsf{Mod}\, \mathbb{K}_X}(\mathcal{M}, f_*(\mathcal{N})) \cong \mathrm{Hom}_{\mathsf{Mod}\, \mathbb{K}_Y}(f^{-1}(\mathcal{M}), \mathcal{N}) \tag{1.11}$$

which is functorial in $\mathcal{M}$ and $\mathcal{N}$.

Exercise 1.12. Prove the adjunction formula (1.11).

Exercise 1.13. Prove that the functor f^{-1} in (1.10) is exact.

Exercise 1.14. Prove that the functor f_* in (1.9) is left exact. Find and example showing that it is not exact. (Hint: line bundles on $\mathbf{P}^1$)

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be ringed spaces. A map of ringed spaces

$$(f, \psi) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X) \tag{1.15}$$

consists of a map of topological spaces $f : Y \to X$, together with a homomorphism of $\mathbb{K}_X$-rings

$$\psi : \mathcal{O}_X \to f_*(\mathcal{O}_Y). \tag{1.16}$$

We shall often use the notation (f, f^*) instead of (f, ψ). The notation (f, ψ) is common in most textbooks, but it is a bit redundant, so we will only use it when it is need to clarify matters. Note that in many cases (see examples below) the homomorphism f^* is literally pullback of functions; and this makes the notation (f, f^*) good heuristically. On the other hand, we sometimes omit all mention of the structure sheaves, and just talk about a map $f : Y \to X$ of locally ringed spaces.

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Given a maps (1.15) of ringed spaces, the direct image is a functor

$$f_* : \mathsf{Mod}\, \mathcal{O}_Y \to \mathsf{Mod}\, \mathcal{O}_X .$$

There is another kind of inverse image here: for $\mathcal{M} \in \mathsf{Mod}\, \mathcal{O}_X$ we define $f^*(\mathcal{M}) \in \mathsf{Mod}\, \mathcal{O}_Y$ by

$$f^*(\mathcal{M}) := \mathcal{O}_Y \otimes_{f^{-1}(\mathcal{O}_X)} f^{-1}(\mathcal{M}).$$

Again there is adjunction:

$$\mathrm{Hom}_{\mathsf{Mod}\, \mathcal{O}_X}(\mathcal{M}, f_*(\mathcal{N})) \cong \mathrm{Hom}_{\mathsf{Mod}\, \mathcal{O}_Y}(f^*(\mathcal{M}), \mathcal{N}).$$

Definition 1.17. A *locally ringed* $\mathbb{K}$*-space* is a ringed $\mathbb{K}$-space $(X, \mathcal{O}_X)$, such that for every point $x \in X$ the stalk $\mathcal{O}_{X,x}$ is a local ring. The maximal ideal of $\mathcal{O}_{X,x}$ is denoted by $\mathfrak{m}_x$, and the residue field is denoted by $\boldsymbol{k}(x)$.

Definition 1.18. If $(Y, \mathcal{O}_Y)$ is another locally ringed space, then a map of locally ringed spaces

$$(f, \psi) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$$

is a map of ringed spaces, such that for every point $y \in Y$, with image $x := f(y) \in X$, the induced $\mathbb{K}$-ring homomorphism

$$\psi_x : \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$$

is a local homomorphism, namely $\psi_x(\mathfrak{m}_x) \subseteq \mathfrak{m}_y$.

The category of locally ringed $\mathbb{K}$-spaces is denoted by $\mathsf{LRSp}/\mathbb{K}$.

Here are three types of locally ringed spaces.

Example 1.19. The category Top of topological spaces. The base ring is $\mathbb{K} = \mathbb{R}$. A space $X \in \mathsf{Top}$ is made into a ringed space by putting on it the sheaf $\mathcal{O}_X$ of continous $\mathbb{R}$-valued functions (for the metric topology of $\mathbb{R}$). Then $(X, \mathcal{O}_X)$ belongs to $\mathsf{LRSp}/\mathbb{R}$. Moreover, the functor $\mathsf{Top} \to \mathsf{LRSp}/\mathbb{R}$ is fully faithful.

Example 1.20. The category Mfld of differentiable (of type C^∞) real manifolds. The base ring is $\mathbb{K} = \mathbb{R}$. A manifold $X \in \mathsf{Mfld}$ is made into a ringed space by putting on it the sheaf $\mathcal{O}_X$ of differentiable $\mathbb{R}$-valued functions. Then $(X, \mathcal{O}_X)$ belongs to $\mathsf{LRSp}/\mathbb{R}$. Moreover, the functor $\mathsf{Mfld} \to \mathsf{LRSp}/\mathbb{R}$ is fully faithful.

Example 1.21. Let $\mathbb{K}$ be an algebraically closed field, and let Var be the category of algebraic varieties over $\mathbb{K}$. Here $\mathcal{O}_X$ is the sheaf of algebraic $\mathbb{K}$-valued functions. Then $(X, \mathcal{O}_X)$ belongs to $\mathsf{LRSp}/\mathbb{K}$. Moreover, the functor $\mathsf{Var} \to \mathsf{LRSp}/\mathbb{K}$ is fully faithful.

Given a locally ringed space $(X, \mathcal{O}_X)$ and an open subset $U \subseteq X$, the pair $(U, \mathcal{O}_X|_U)$ is a locally ringed space. We call it an *open subspace* of X. The inclusion map

$$(U, \mathcal{O}_X|_U) \to (X, \mathcal{O}_X)$$

is called an *open embedding*.

Just like sheaves, locally ringed spaces and maps between them can be glued.

Theorem 1.22 (Gluing Maps of LR Spaces)**.** *Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be objects of $\mathsf{LRSp}/\mathbb{K}$, let $Y = \bigcup_{i \in I} V_i$ be an open covering, and for every i let*

$$(f_i, \psi_i) : (V_i, \mathcal{O}_Y|_{V_i}) \to (X, \mathcal{O}_X)$$

be a map in $\mathsf{LRSp}/\mathbb{K}$.

We assume that this condition holds: for every $i, j \in I$ there is equality

$$(f_i, \psi_i)|_{V_{i,j}} = (f_j, \psi_j)|_{V_{i,j}}$$

of maps

$$(V_{i,j}, \mathcal{O}_Y|_{V_{i,j}}) \to (X, \mathcal{O}_X)$$

in $\mathsf{LRSp}/\mathbb{K}$.

Then there is a unique map

$$(f, \psi) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$$

in $\mathsf{LRSp}/\mathbb{K}$ such that

$$(f, \psi)|_{V_i} = (f_i, \psi_i)$$

for every i.

See picture in figure 1.

Proof. The existence and uniqueness of a map of topological spaces $f : Y \to X$ satisfying $f|_{V_i} = f_i$ is clear.

We need to produce the homomorphism of sheaves of rings

$$\psi : \mathcal{O}_X \to f_*(\mathcal{O}_Y)$$

on X. By the adjunction formula (1.11), this amounts to producing a homomorphism of sheaves of rings

$$\psi : f^{-1}(\mathcal{O}_X) \to \mathcal{O}_Y \tag{1.23}$$

on Y. Now we are given homomorphisms

$$\psi_i : f^{-1}(\mathcal{O}_X)|_{V_i} \to \mathcal{O}_Y|_{V_i}$$

that agree on double intersections. According to Theorem 1.4 these can be glued uniquely to a homomorphism of sheaves of $\mathbb{K}$-modules ψ as in (1.23), such that $\psi|_{V_i} = \psi_i$. This is a homomorphism of sheaves of rings, because this propeorty can be checked locally. Also on stalks it is a local homomorphism. So (f, ψ) is a map of locally ringed spaces. □

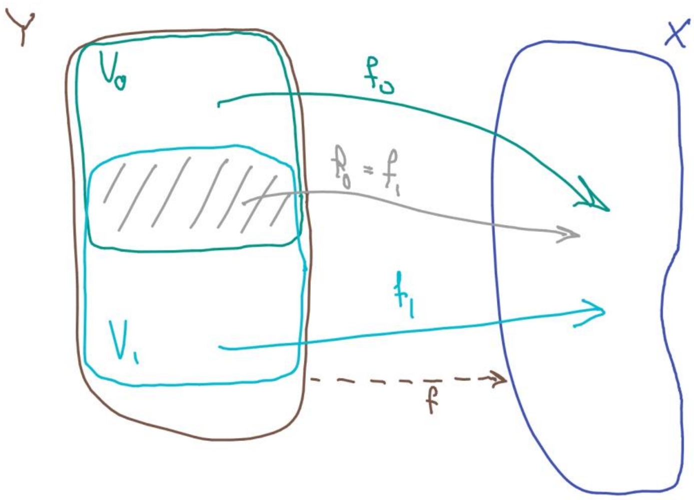


FIGURE 1. Gluing maps of spaces

Theorem 1.24 (Gluing LR Spaces). *Let $\{(U_i, \mathcal{O}_{U_i})\}_{i \in I}$ be a collection of objects of* $\mathsf{LRSp}/\mathbb{K}$*. For every $i, j \in I$ there is an open subset $U_{i,j} \subseteq U_i$, and an isomorphism*

$$(f_{i,j}, \psi_{i,j}) : (U_{i,j}, \mathcal{O}_{U_i}|_{U_{i,j}}) \xrightarrow{\simeq} (U_{j,i}, \mathcal{O}_{U_j}|_{U_{j,i}})$$

in $\mathsf{LRSp}/\mathbb{K}$.

These conditions hold:

(a) *For every i there are equalities $U_{i,i} = U_i$ and $(f_{i,i}, \psi_{i,i}) = \mathrm{id}$.*

(b) *For every i, j, k there are equalities*

$$f_{i,j}(U_{i,j} \cap U_{i,k}) = U_{j,i} \cap U_{j,k}$$

of subsets of U_i, and

$$(f_{j,k}, \psi_{j,k}) \circ (f_{i,j}, \psi_{i,j}) = (f_{i,k}, \psi_{i,k})$$

of isomorphisms

$$(U_{i,j} \cap U_{i,k}, \mathcal{O}_{U_i}|_{U_{i,j} \cap U_{i,k}}) \to (U_{k,i} \cap U_{k,j}, \mathcal{O}_{U_k}|_{U_{k,i} \cap U_{k,j}})$$

in $\mathsf{LRSp}/\mathbb{K}$.

Then there is an object $(X, \mathcal{O}_X)$ in $\mathsf{LRSp}/\mathbb{K}$*, with open embeddings*

$$(f_i, \psi_i) : (U_i, \mathcal{O}_{U_i}) \to (X, \mathcal{O}_X),$$

such that

$$(f_j, \psi_j) \circ (f_{i,j}, \psi_{i,j}) = (f_i, \psi_i)$$

as morphisms

$$(U_{i,j}, \mathcal{O}_{U_i}|_{U_{i,j}}) \to (X, \mathcal{O}_X),$$

and such that

$$X = \bigcup_{i \in I} f_i(U_i).$$

Moreover, the space $(X, \mathcal{O}_X)$, with the collection of morphisms $\{(f_i, \psi_i)\}_{i \in I}$, are unique up to a unique isomorphism.

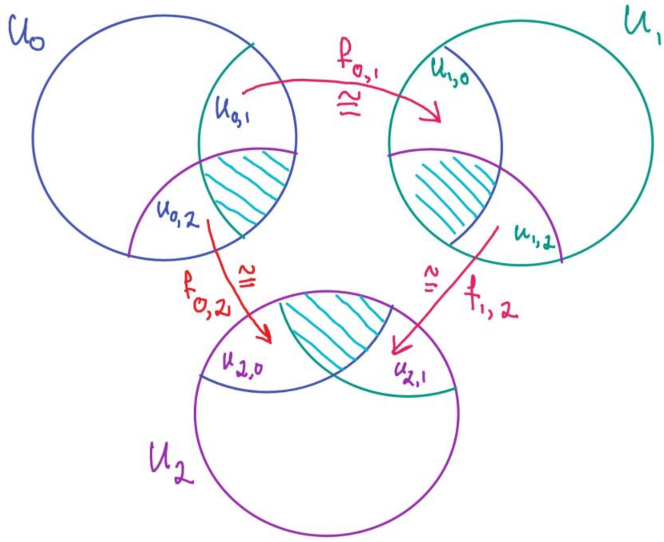


Figure 2. Gluing spaces: the input

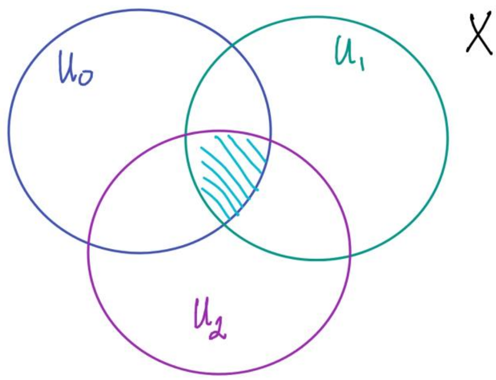


Figure 3. Gluing spaces: the output

See figures 2 and 3 for an illustration.

The data

$$\Big(\big\{ (U_i, \mathcal{O}_{U_i}) \big\}_{i \in I}, \ \big\{ (f_{i,j}, \psi_{i,j}) \big\}_{i,j \in I} \Big) \tag{1.25}$$

is called *gluing data* or *descent data*..

Proof. This is done in a few steps.

Step 1. We define the set X. Consider the disjoint union $U := \coprod_{i \in I} U_i$. We define a relation $\sim$ on this set as follows: two points $x, y \in U$ are in the relation $x \sim y$ if there are $i, j \in I$

such that $x \in U_{i,j} \subseteq U_i$, $y \in U_{j,i} \subseteq U_j$, and $f_{i,j}(x) = y$. This is an equivalence relation (exercise). We let $X := U/{\sim}$, the quotient set. The canonical surjection is $\pi : U \to X$.

For each i there a map $f_i : U_i \to X$, gotten by composing the inclusion $U_i \subseteq U$ with π. The map f_i is injective (exercise). We identify U_i with its image $f_i(U_i) \subseteq X$. With this identification, there is equality $U_{i,j} = U_{j,i}$ of subsets of X.

Step 2. We put a topology on X. The disjoint union U gets the disjoint union topology. Then we put on X the quotient topology relative to the surjection $\pi : U \to X$.

Each U_i is an open set of X, and so $X = \bigcup_i U_i$ is an open covering (exercise).

Step 3. Now we construct the sheaf of rings $\mathcal{O}_X$. On each open set $U_i \subseteq X$ we have a sheaf of rings $\mathcal{O}_{U_i}$. On double intersections we have isomorphisms of sheaves of rings

$$\psi_{i,j} : \mathcal{O}_{U_i}|_{U_{i,j}} \xrightarrow{\simeq} \mathcal{O}_{U_j}|_{U_{i,j}},$$

and these agree on triple intersections. By Theorem 1.5 we get a sheaf $\mathcal{O}_X$ on X, with isomorphisms of sheaves

$$\psi_i : \mathcal{O}_{U_i} \xrightarrow{\simeq} \mathcal{O}_X|_{U_i}$$

such that $\psi_j \circ \psi_{i,j} = \psi_i$ on $U_{i,j}$. We see that $\mathcal{O}_X$ is a sheaf of rings, and the stalks $\mathcal{O}_{X,x}$ are local rings. So $(X, \mathcal{O}_X)$ is a locally ringed $\mathbb{K}$-space.

By construction there are open embeddings

$$(f_i, \psi_i) : (U_i, \mathcal{O}_{U_i}) \to (X, \mathcal{O}_X)$$

satisfying the required compatibility.

Step 4. The uniqueness is due to Theorem 1.22. □

Exercise 1.26. Finish the proof of the theorem.

Exercise 1.27. Find an example of a collection of spaces as in the theorem, such that all the spaces U_i are separated topological spaces (i.e. Hausdorff), yet X is not separetd.

Exercise 1.28. Suppose that the spaces and the gluing data are in Mfld (Example 1.20), the indexing set I is countable, and the topological space X is Hausdorff. Prove that $(X, \mathcal{O}_X)$ is an object of Mfld.

Remark 1.29. The gluing in Thm 1.24 will be used to construct fiber products of schemes. Then the fiber products will allow us to define the notion of a *separated map of schemes*. This is the relative and generalized variant of the Hausdorff condition.

2. Affine Schemes

Let A be a $\mathbb{K}$-ring. The *prime spectrum* of A is the set $\operatorname{Spec}(A)$ of prime ideals of A. As we all know, this is a topological space with the *Zariski topology*. By definition the closed sets of $\operatorname{Spec}(A)$ are the sets

$$\mathrm{Z}(\mathfrak{a}) := \{\mathfrak{p} \in \operatorname{Spec}(A) \mid \mathfrak{a} \subseteq \mathfrak{p}\},$$

where $\mathfrak{a}$ is some ideal in A.

We sometimes refer to $\mathrm{Z}(\mathfrak{a})$ as the *zero locus* of the ideal $\mathfrak{a}$. Here is the reason: to a prime ideal $\mathfrak{p}$ we associate the local ring $A_{\mathfrak{p}}$ and the residue field

$$\boldsymbol{k}(\mathfrak{p}) := A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}.$$

An element $a \in A$ has a class $a(\mathfrak{p}) \in \boldsymbol{k}(\mathfrak{p})$, coming from the canonical ring homomorphism $A \to \boldsymbol{k}(\mathfrak{p})$. Then

$$\mathrm{Z}(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid a(\mathfrak{p}) = 0 \text{ for all } a \in \mathfrak{a}\}. \tag{2.1}$$

Exercise 2.2. Prove the formula above.

For an element $s \in A$ we define

$$D(s) = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid s \notin \mathfrak{p}\}. \tag{2.3}$$

This is an open set: it it the complement of the closed set $Z(\mathfrak{a})$, where $\mathfrak{a} := (s)$, the principal ideal generated by s. We call such an open set a *principal open set*. Analogously to (2.1) we have

$$D(s) = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid s(\mathfrak{p}) \neq 0\}. \tag{2.4}$$

Proposition 2.5. *The principal open sets are a basis of the topology of* $\operatorname{Spec}(A)$. *Namely every open set set U is a union $U = \bigcup_i D(s_i)$ for a suitable collection $\{s_i\}$ of elements of A.*

Exercise 2.6. Prove the proposition above.

Definition 2.7. Let A be a ring, and write $X := \operatorname{Spec}(A)$ for this topological space. For an open set $U \subseteq X$ we let $S(U) \subseteq A$ be the multiplicatively closed set

$$S(U) := \{s \in A \mid s(\mathfrak{p}) \neq 0 \text{ for all } \mathfrak{p} \in U\}.$$

Let $A_{S(U)}$ be the localization of A w.r.t. $S(U)$.

Lemma 2.8. *The assignment $U \mapsto S(U)$ is a presheaf of rings on $X = \operatorname{Spec}(A)$, that we denote by $\mathcal{O}^{\mathrm{pre}}$.*

Proof. This is easy: if $V \subseteq U$ is a smaller open set, then $S(U) \subseteq S(V)$, so by the universal property of localization there is a unique A-ring homomorphism $A_{S(U)} \to A_{S(V)}$. □

Definition 2.9. Let A be a ring. The *structure sheaf* of $\operatorname{Spec}(A)$ is the sheaf of rings

$$\mathcal{O}_{\operatorname{Spec}(A)} := \operatorname{Sh}(\mathcal{O}^{\mathrm{pre}}).$$

Lecture 3, 13 Mar 2019

By construction, $\mathcal{O}^{\mathrm{pre}}$ is a presheafs of A-rings on $X := \operatorname{Spec}(A)$, and $\mathcal{O}_X$ is a sheaf of A-rings on X.

Proposition 2.10. *Let A be a ring and write $X := \operatorname{Spec}(A)$. For every point $x = \mathfrak{p} \in X$ the stalk of $\mathcal{O}_X$ at x is $\mathcal{O}_{X,x} = A_{\mathfrak{p}}$, the local ring at $\mathfrak{p}$. More precisely, there is a unique A-ring isomorphism $\mathcal{O}_{X,x} \cong A_{\mathfrak{p}}$.*

Proof. Let $S(x) := A - \mathfrak{p}$, the complement of $\mathfrak{p}$. By definition we have $A_{\mathfrak{p}} = A_{S(x)}$. The universal property of localization says that there is at most one A-ring isomorphism $A_{\mathfrak{p}} \xrightarrow{\simeq} \mathcal{O}_{X,x}$. We will produce it.

Let us denote by $\boldsymbol{U}_x$ the set of principal open neighborhoods of x. These are the open sets $U = D(s)$ for some $s \in S(x)$. By Proposition 2.5, $\boldsymbol{U}_x$ is a basis of open neighborhoods of x. Because the stalks of the presheaf $\mathcal{O}^{\mathrm{pre}}$ and its associated sheaf $\mathcal{O}$ are the same, we have

$$\mathcal{O}_{X,x} = \lim_{U \to} \Gamma(U, \mathcal{O}^{\mathrm{pre}}) = \lim_{U \to} A_{S(U)} \tag{2.11}$$

where U runs over $\boldsymbol{U}_x$.

For each $U = D(s) \in \boldsymbol{U}_x$ we have $S(U) \subseteq S(x)$, so there is a unique A-ring homomorphism $A_{S(U)} \to A_{S(x)} = A_{\mathfrak{p}}$. Going to the limit in U we get an A-ring homomorphism

$$\psi : \mathcal{O}_{X,x} \to A_{\mathfrak{p}}.$$

In the other direction, every element $s \in S(x)$ belongs to $S(U)$ for $U := D(s) \in \boldsymbol{U}_x$, and hence s is invertible in $A_{S(U)}$. This means that every $s \in S(x)$ is invertible in the A-ring $\mathcal{O}_{X,x}$. Hence there is a unique A-ring homomorphism

$$\phi : A_{\mathfrak{p}} = A_{S(x)} \to \mathcal{O}_{X,x}.$$

The homomorphism ϕ is surjective, because every $f \in \mathcal{O}_{X,x}$ is the image of some fraction $a/s \in A_{\mathrm{S}(U)}$, see (2.11). But $s \in \mathrm{S}(x)$, so $a/s \in A_{\mathfrak{p}}$ and $f = \phi(a/s)$.

Finally consider the A-ring homomorphism $\psi \circ \phi$ from $A_{\mathfrak{p}}$ to itself. By uniqueness there is equality $\psi \circ \phi = \mathrm{id}_{A_{\mathfrak{p}}}$. This shows that ϕ is injective. In conclusion, ϕ is an isomorphism of A-rings. □

Corollary 2.12. $\big(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)}\big)$ *is a locally ringed space.*

Definition 2.13. An *affine* $\mathbb{K}$*-scheme* is a locally ringed space $(X, \mathcal{O}_X) \in \mathsf{LRSp}/\mathbb{K}$ which is isomorphic to $\big(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)}\big)$ for some $\mathbb{K}$-ring A.

We now provide a more explicit description of the structur sheaf $\mathcal{O}_{\mathrm{Spec}(A)}$ in terms of its *Godement sheaf* from [Ye4, Sec 6]. Recall that for a presheaf of $\mathbb{K}$-modules $\mathcal{M}$ on a space X, its Godement sheaf $\mathrm{GSh}(\mathcal{M})$ is defined by

$$\Gamma(U, \mathrm{GSh}(\mathcal{M})) := \prod_{x \in U} \mathcal{M}_x$$

for an open set $U \subseteq X$. Then $\mathrm{Sh}(\mathcal{M}) \subseteq \mathrm{GSh}(\mathcal{M})$ is the subsheaf of *geometric sections*. Here is what this means. There is a canonical homomorphism of presehaves $\mathcal{M} \to \mathrm{GSh}(\mathcal{M})$. For an open set $U \subseteq X$, a section

$$m \in \Gamma(U, \mathrm{GSh}(\mathcal{M}))$$

is called geometric if for every point $x \in U$ there is an open set V such that $x \in V \subseteq U$, and a section $m' \in \Gamma(V, \mathcal{M})$, such that

$$m|_V = m' \in \Gamma(V, \mathrm{GSh}(\mathcal{M})).$$

Specializing to our case it says that following:

Proposition 2.14. *Let* $(X, \mathcal{O}_X) := \big(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)}\big)$. *Then* $\mathcal{O}_X$ *is the subsheaf of* $\mathrm{GSh}(\mathcal{O}^{\mathrm{pre}})$ *consisting of the geometric sections. Specifically, let* $U \subseteq X$ *be an open set, and let*

$$f \in \Gamma(U, \mathrm{GSh}(\mathcal{O}^{\mathrm{pre}})) = \prod_{\mathfrak{p} \in U} A_{\mathfrak{p}}.$$

The section f *belongs to* $\Gamma(U, \mathcal{O}_X)$ *iff for every point* $x = \mathfrak{p} \in U$ *there is an open set* V *such that* $x \in V \subseteq U$*, and elements* $a \in A$ *and* $s \in \mathrm{S}(V)$*, such that*

$$f|_V = a/s \in \Gamma(V, \mathrm{GSh}(\mathcal{O}^{\mathrm{pre}})) = \prod_{\mathfrak{q} \in V} A_{\mathfrak{q}}.$$

See Figure 4.

Here are some example of affine schemes.

Example 2.15. Consider the ring $A = \mathbb{Z}$. The affine scheme $X := \mathrm{Spec}(\mathbb{Z})$ has these points: for every (positive) prime number p there is a maximal ideal $\mathfrak{m} := (p)$. These are closed points of X, since

$$\mathrm{Z}(p) = \{\mathfrak{m}\}.$$

The local ring is

$$\mathbb{Z}_{(p)} = \{a/s \mid s \notin (p)\} \subseteq \mathbb{Q}.$$

The residue field is $\mathbb{F}_p$.

The zero ideal $\mathfrak{p} := (0)$ is also prime. Is is the *generic point* of X; namely its topological closure is X. The local ring and the residue field at $\mathfrak{p}$ are $\mathbb{Q}$.

Exercise 2.16. Analyze the affine scheme $\mathrm{Spec}(A)$ for the ring $A := \mathbb{K}[t]$, the polynomial ring in one variable over a field $\mathbb{K}$.

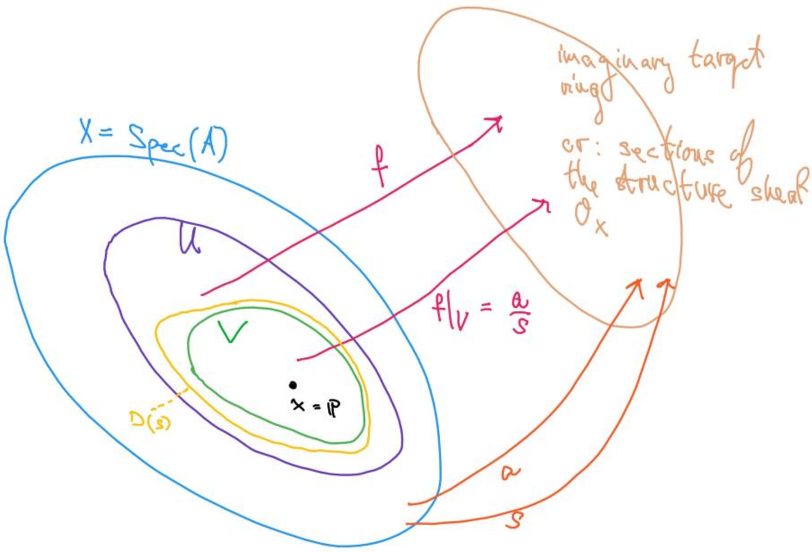


FIGURE 4. Picture for Prop 2.14

Example 2.17. Suppose $\mathbb{K}$ is a nonzero ring. For every $n \geq 0$ there is an affine scheme

$$\mathbf{A}^n_{\mathbb{K}} := \operatorname{Spec}(\mathbb{K}[t_1, \ldots, t_n])$$

called the *n dimensional affine space over* $\mathbb{K}$.

Recall that for an ideal $\mathfrak{a} \subseteq A$, its zero locus is

$$\mathrm{Z}(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid \mathfrak{a} \subseteq \mathfrak{p}\}.$$

The *radical* of $\mathfrak{a}$ is the ideal

$$\sqrt{\mathfrak{a}} := \{a \in A \mid a^i \in \mathfrak{a} \text{ for some } i > 0\} \subseteq A.$$

Lemma 2.18. *Let $\mathfrak{a} \subseteq A$ be an ideal. Then*

$$\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{p} \in \mathrm{Z}(\mathfrak{a})} \mathfrak{p}$$

and

$$\mathrm{Z}(\mathfrak{a}) = \mathrm{Z}(\sqrt{\mathfrak{a}}).$$

Exercise 2.19. Prove the lemma. (Hint: do not use the Nullstellensatz.)

Lemma 2.20. *For ideals $\mathfrak{a}, \mathfrak{b} \subseteq A$ the following are equivalent:*

(i) $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$.
(ii) $\mathrm{Z}(\mathfrak{a}) = \mathrm{Z}(\mathfrak{b})$.

Proof. Clear from Lemma 2.18. □

Definition 2.21. A topological space X is called *quasi-compact* if every open covering of X has a finite subcovering.

Remark 2.22. The term "compact" is usually reserved for spaces that are Hausdorff and quasi-compact. Schemes are almost never Hausdorff. There is an analogous notion of separation, that we will study later. Cf. Exercise 1.27 and Rem 1.29.

Proposition 2.23. *Let A be a ring. The topological space $X := \mathrm{Spec}(A)$ is quasi-compact.*

Proof. Let $X = \bigcup_{i \in I} U_i$ be an open covering. For each i there is an ideal $\mathfrak{a}_i$ such that $U_i = X - \mathrm{Z}(\mathfrak{a}_i)$. Write $\mathfrak{a} := \sum_i \mathfrak{a}_i$. Then there is equality

$$\mathrm{Z}(A) = \varnothing = \bigcap_{i \in I} \mathrm{Z}(\mathfrak{a}_i) = \mathrm{Z}(\mathfrak{a})$$

of subsets of X. By Lemma 2.20 we know that $\sqrt{A} = \sqrt{\mathfrak{a}}$. Since $1 \in A$, we see that $1 \in \sqrt{\mathfrak{a}}$, and hence $1 \in \mathfrak{a}$. This says that we can express 1 as a finite sum: $1 = \sum_{i \in I'} a_i$ with $I' \subseteq I$ a finite subset and $a_i \in \mathfrak{a}_i$. We see that $A = \sum_{i \in I'} \mathfrak{a}_i$, and therefore

$$\varnothing = \mathrm{Z}(A) = \bigcap_{i \in I'} \mathrm{Z}(\mathfrak{a}_i)$$

and $X = \bigcup_{i \in I'} U_i$. □

Lemma 2.24. *Let $s_1, \ldots, s_m \in A$. TFAE:*

(i) $\mathrm{Spec}(A) = \bigcup_i \mathrm{D}(s_i)$.
(ii) *There exists $a_1, \ldots, a_m \in A$ s.t. $1_A = \sum_i a_i \cdot s_i$.*

Proof.

(i) ⇒ (ii): As in the proof of the proposition, $A = \sum_i \mathfrak{a}_i$, where $\mathfrak{a}_i := (s_i)$. So the element 1_A is a linear combination $1_A = \sum_i a_i \cdot s_i$ for some $a_i \in A$.

(ii) ⇒ (i): Here $1_A \in \sum_i \mathfrak{a}_i$, so $A = \sum_i \mathfrak{a}_i$, and, as in the proof of the proposition, $\mathrm{Spec}(A) = \bigcup_i \mathrm{D}(s_i)$. □

By construction, for every $s \in A$ the element s is invertible in the rings $\Gamma(\mathrm{D}(s), \mathcal{O}^{\mathrm{pre}})$ and $\Gamma(\mathrm{D}(s), \mathcal{O}_X)$.

Lemma 2.25. *Let $(X, \mathcal{O}_X) := \big(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)}\big)$. Suppose $U \subseteq X$ is an open set, $f \in \Gamma(U, \mathcal{O}_X)$, and $x = \mathfrak{p} \in U$. Then there are elements $a, s \in A$ such that $x \in \mathrm{D}(s) \subseteq U$ and*

$$f|_{\mathrm{D}(s)} = a/s \in \Gamma(\mathrm{D}(s), \mathcal{O}_X).$$

Proof. By Proposition 2.14 the point $x = \mathfrak{p} \in U$ has an open neighborhood $V \subseteq U$, and elements $b, t \in A$, such that $t \in \mathrm{S}(V)$ and $f|_V = b/t \in \Gamma(V, \mathcal{O}_X)$.

According to Proposition 2.5 we can replace V with a smaller open neighborhood $\mathrm{D}(r)$ for some $r \in A$, i.e. $x \in \mathrm{D}(r) \subseteq V$.

Define the elements $a := b \cdot r \in A$ and $s := t \cdot r \in A$, and the open set $W := \mathrm{D}(s)$. Since $\mathrm{D}(r) \subseteq V \subseteq \mathrm{D}(t)$, it follows that

$$\mathrm{D}(r) = \mathrm{D}(t \cdot r) = \mathrm{D}(s) = W.$$

Also

$$a/s = (b \cdot r)/(t \cdot r) = f|_W \in \Gamma(W, \mathcal{O}_X).$$

□

Lemma 2.26. *Let $(X, \mathcal{O}_X) := \big(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)}\big)$. Suppose $V \subseteq X$ is a quasi-compact open set, and $f \in \Gamma(V, \mathcal{O}_X)$. Then there are finitely many elements $a_1, \ldots, a_m, s_1, \ldots, s_m \in A$ such that $V = \bigcup_{i=1}^{m} \mathrm{D}(s_i)$, and*

$$f|_{\mathrm{D}(s_i)} = a_i/s_i \in \Gamma(\mathrm{D}(s_i), \mathcal{O}_X)$$

for every i.

Proof. By Lemma 2.25, for every point $x = \mathfrak{p} \in V$ there are elements $a_x, s_x \in A$ such that $x \in \mathrm{D}(s_x) \subseteq V$ and

$$f|_{\mathrm{D}(s_x)} = a_x / s_x \in \Gamma(\mathrm{D}(s_x), \mathcal{O}_X).$$

We have an open covering $V = \bigcup_{x \in V} \mathrm{D}(s_x)$. Because of quasi-compactness we can pass to a finite subcovering, that is to a finite subset $\{x_1, \ldots, x_m\}$ of V. Finally, by letting $a_i := a_{x_i}$ and $s_i := s_{x_i}$ we are done. □

A ring homomorphism $\psi : A \to B$ induces a map of sets

$$\mathrm{Spec}(\psi) : \mathrm{Spec}(B) \to \mathrm{Spec}(A) \tag{2.27}$$

with formula

$$\mathrm{Spec}(\psi)(\mathfrak{q}) := \psi^{-1}(\mathfrak{q}).$$

Lemma 2.28. *Let $\psi : A \to B$ be a ring homomorphism.*

(1) *The map* $\mathrm{Spec}(\psi)$ *is continuous.*
(2) *Suppose $B = A_s$ for some $s \in A$. Then the image of* $\mathrm{Spec}(\psi)$ *is* $\mathrm{D}(s)$*, and*

$$\mathrm{Spec}(\psi) : \mathrm{Spec}(A_s) \to \mathrm{D}(s)$$

is a homeomorphism.

Exercise 2.29. Prove the last lemma.

Lecture 4, 27 Mar 2019

Theorem 2.30. *Let A be a ring and $s \in A$ an element. Consider the affine scheme $(X, \mathcal{O}_X) := \big(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)}\big)$. There is a unique A-ring isomorphism*

$$A_s \cong \Gamma(\mathrm{D}(s), \mathcal{O}_X).$$

Proof.

Step 1. Since the element s is invertible in the ring $\Gamma(\mathrm{D}(s), \mathcal{O}_X)$, there is a unique A-ring homomorphism

$$\phi : A_s \to \Gamma(\mathrm{D}(s), \mathcal{O}_X). \tag{2.31}$$

We need to prove that ϕ is bijective.

Step 2. In this step we we will prove that ϕ is injective. Let's write $U := \mathrm{D}(s)$. Since $\mathcal{O}_X$ is a subsheaf of the Godement sheaf $\mathrm{GSh}(\mathcal{O}_X)$, there is an embedding of A-rings

$$\Gamma(U, \mathcal{O}_X) \rightarrowtail \Gamma\big(U, \mathrm{GSh}(\mathcal{O}_X)\big) = \prod_{\mathfrak{p} \in U} A_{\mathfrak{p}}.$$

It thus suffices to prove that the homomorphism

$$\phi' : A_s \to \prod_{\mathfrak{p} \in U} A_{\mathfrak{p}} \tag{2.32}$$

is injective.

Let's write $B := A_s$. By Lemma 2.28 we know that $\mathrm{Spec}(B) = U$ as topological subspaces of X. For every $\mathfrak{p} \in U$ the element s is invertible in $A_{\mathfrak{p}}$, and hence the homomorphism $A_{\mathfrak{p}} \to B_{\mathfrak{p}}$ is bijective. So we can rewrite (2.32) as $\phi' : B \to \prod_{\mathfrak{p} \in U} B_{\mathfrak{p}}$, and we need to prove it is injective.

Suppose $b \in B$ is such that $\phi'(b) = 0$. Then $\phi'_{\mathfrak{p}}(b) = 0$ in $B_{\mathfrak{p}}$ for all $\mathfrak{p} \in U$. This means that there is some element $t_{\mathfrak{p}} \in B$ such that $\mathfrak{p} \in \mathrm{D}(t_{\mathfrak{p}})$ and $t_{\mathfrak{p}} \cdot b = 0$ in B. Now $U = \mathrm{Spec}(B) = \bigcup_{\mathfrak{p} \in U} \mathrm{D}(t_{\mathfrak{p}})$.

By quasi-compactness of U we can pass to a finite subcovering, indexed by $\{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$. Let $t_i := t_{\mathfrak{p}_i} \in B$. Then $t_i \cdot b = 0$ in B, and $\mathrm{Spec}(B) = \bigcup_{i=1}^m \mathrm{D}(t_i)$. According to Lemma 2.24 as in the proof of Prop 2.23, there are elements $c_1, \dots, c_m \in B$ such that $1_B = \sum_i c_i \cdot t_i$. Then $b = 1_B \cdot b = \sum_i c_i \cdot t_i \cdot b = 0$.

Step 3. We now prove that the homomorphism ϕ from (2.31) is surjective. As before let $U := \mathrm{D}(s)$. Take an element $f \in \Gamma(U, \mathcal{O}_X)$. We know that $U = \mathrm{Spec}(B)$ is a quasi-compact topological space. By Lem 2.26 there are finitely many elements $a_i, s_i \in A$, $1 \leq i \leq m$, such that $U = \bigcup_{i=1}^m \mathrm{D}(s_i)$, and $f|_{\mathrm{D}(s_i)} = a_i / s_i \in \Gamma(\mathrm{D}(s_i), \mathcal{O}_X)$ for every i.

Step 2, when applied to the element $s_i \cdot s_j \cdot s \in A$ instead of to s, shows that for every i, j the A-ring homomorphism

$$\phi_{i,j} : A_{s_i \cdot s_j \cdot s} \to \Gamma\big(\mathrm{D}(s_i \cdot s_j \cdot s), \mathcal{O}_X\big)$$

is injective. Recall that $B = A_s$, so that $A_{s_i \cdot s_j \cdot s} = B_{s_i \cdot s_j}$. We know that

$$f|_{\mathrm{D}(s_i \cdot s_j \cdot s)} = a_i \cdot s_i^{-1} = a_j \cdot s_j^{-1} \in \Gamma\big(\mathrm{D}(s_i \cdot s_j \cdot s), \mathcal{O}_X\big).$$

Therefore $a_i \cdot s_i^{-1} = a_j \cdot s_j^{-1}$ in $B_{s_i \cdot s_j}$. The kernel of the localization homomorphism $B \to B_{s_i \cdot s_j}$ is known: there is a positive integer $l_{i,j}$ such that

$$(s_i \cdot s_j)^{l_{i,j}} \cdot (a_i \cdot s_j - a_j \cdot s_i) = 0$$

in B. Taking $l := \max(\{l_{i,j}\})$ we obtain

$$(s_i \cdot s_j)^{l} \cdot (a_i \cdot s_j - a_j \cdot s_i) = 0 \tag{2.33}$$

in B.

Define $b_i := a_i \cdot s_i^l$ and $t_i := s_i^{l+1}$. Then $\mathrm{D}(t_i) = \mathrm{D}(s_i)$ and

$$f|_{\mathrm{D}(t_i)} = a_i \cdot s_i^{-1} = b_i \cdot t_i^{-1} \in \Gamma(\mathrm{D}(t_i), \mathcal{O}_X). \tag{2.34}$$

Also, from (2.33) we have

$$t_i \cdot b_j = t_j \cdot b_i \tag{2.35}$$

in B.

Since

$$\mathrm{Spec}(B) = \bigcup_{i=1}^m \mathrm{D}(t_i), \tag{2.36}$$

by Lem 2.24 we can find elements $c_1, \dots, c_m \in B$ such that $1_B = \sum_i c_i \cdot t_i$. Let

$$b := \sum_i c_i \cdot b_i \in B.$$

For every i we have – using (2.35) and (2.34) –

$$\phi(b)|_{\mathrm{D}(t_i)} = \sum_j c_j \cdot b_j = t_i^{-1} \cdot \Big(\sum_j c_j \cdot t_i \cdot b_j\Big) = t_i^{-1} \cdot \Big(\sum_j c_j \cdot t_j\Big) \cdot b_i = t_i^{-1} \cdot b_i = f|_{\mathrm{D}(t_i)}$$

in $\Gamma(\mathrm{D}(t_i), \mathcal{O}_X)$. But by (2.36) and the sheaf axioms this implies that $\phi(b) = f$. □

Lecture 5, 3 Apr 2019

Convention 2.37. From now on the expression $\mathrm{Spec}(A)$ refers to the locally ringed space $\big(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)}\big)$.

Corollary 2.38. *For a ring A, with $(X, \mathcal{O}_X) := \mathrm{Spec}(A)$, the canonical ring homomorphism $A \to \Gamma(X, \mathcal{O}_X)$ is bijective.*

Proof. Take $s = 1$ in Thm 2.30. □

Definition 2.39. The category of *affine* $\mathbb{K}$*-schemes*, denoted by $\mathsf{Sch}_{\mathrm{aff}}/\mathbb{K}$, is the full subcategory of $\mathsf{LRSp}/\mathbb{K}$ on the affine schemes (see Definition 2.13).

Recall that $\mathsf{Rng}/\mathbb{K}$ is the category of (commutative) $\mathbb{K}$-rings.

Proposition 2.40. *The assignment that sends a locally ringed space* $(X, \mathcal{O}_X)$ *to the ring* $\Gamma(X, \mathcal{O}_X)$*, and a map of locally ringed spaces* $(f, \tilde{\phi}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ *to the ring homomorphism*

$$\Gamma(X, \phi) : \Gamma(Y, \mathcal{O}_Y) \to \Gamma(X, \mathcal{O}_X),$$

is a functor

$$\Gamma : (\mathsf{LRSp}/\mathbb{K})^{\mathrm{op}} \to \mathsf{Rng}/\mathbb{K}.$$

Exercise 2.41. Prove proposition 2.40.

Before continuing, here is a general useful notion.

Definition 2.42. A morphism $f : C \to D$ in a category C is called an *epimorphism* if it has this cancellation property: for every morphisms $g_0, g_1 : D \to E$ in C, if $g_0 \circ f = g_1 \circ f$ then $g_0 = g_1$.

Proposition 2.43. *Let* $f : A \to B$ *be a morphism in the category* Rng *of commutative rings. If* f *is either surjective or a localization, then* f *is an epimorphism in* Rng.

Exercise 2.44. Prove proposition 2.43.

Let $(Y, \mathcal{O}_Y)$ be a LRSp and $s \in \Gamma(Y, \mathcal{O}_Y)$. For a point $y \in Y$ we denote by $s(y)$ the image of the element s in the residue field $\boldsymbol{k}(y)$. And we write

$$\mathrm{D}(s) := \{y \in Y \mid s(y) \neq 0\}.$$

Lemma 2.45. *Let* $(Y, \mathcal{O}_Y)$ *be a LRSp and* $s \in \Gamma(Y, \mathcal{O}_Y)$*. Then:*

(1) *The set* $\mathrm{D}(s)$ *is open in* Y.
(2) *The element* s *is invertible in the ring* $\Gamma(\mathrm{D}(s), \mathcal{O}_Y)$.

Proof.

(1) Take a point $y \in \mathrm{D}(s)$. Since $s(y) \neq 0$, it is an invertible element of the field $\boldsymbol{k}(y)$. Because the stalk $\mathcal{O}_{Y,y}$ is a local ring, it follows that s is an invertible element of $\mathcal{O}_{Y,y}$. Let $t \in \mathcal{O}_{Y,y}$ be the inverse of s, so $s \cdot t = 1$ in $\mathcal{O}_{Y,y}$. There is an open neighborhood V of y such that $t \in \Gamma(V, \mathcal{O}_Y)$. There is a smaller open neighborhood V' of y s.t. $s \cdot t = 1$ in $\Gamma(V', \mathcal{O}_Y)$. Then $s(y') \neq 0$ for all $y' \in V'$, so $V' \subseteq \mathrm{D}(s)$.

(2) Let $V := \mathrm{D}(s)$. As we saw above, every point $y \in V$ has an open neighborhood $V_y \subseteq V$ and an element $t_y \in \Gamma(V_y, \mathcal{O}_Y)$ such that $s \cdot t_y = 1$. This means that $t_y = s^{-1}$ in $\Gamma(V_y, \mathcal{O}_Y)$, a fact that makes it unique. We conclude that $t_y = t_{y'}$ in $\Gamma(V_y \cap V_{y'}, \mathcal{O}_Y)$. By the sheaf property we get $t \in \Gamma(V, \mathcal{O}_Y)$, and it satisfies $s \cdot t = 1$. □

Theorem 2.46. *Let* $A \in \mathsf{Rng}/\mathbb{K}$ *and let* $(Y, \mathcal{O}_Y) \in \mathsf{LRSp}/\mathbb{K}$*. Write* $B := \Gamma(Y, \mathcal{O}_Y)$ *and* $(X, \mathcal{O}_X) := \mathrm{Spec}(A)$*. Given a* $\mathbb{K}$*-ring homomorphism* $\phi : A \to B$*, there is a unique map*

$$(f, \tilde{\phi}) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$$

in $\mathsf{LRSp}/\mathbb{K}$*, such that* $\Gamma(X, \tilde{\phi}) = \phi : A \to B$.

Proof.

Step 1. We prove uniqueness of f. Take a point $y \in Y$, and let $x = \mathfrak{p} := f(y) \in X$. Because $\tilde{\phi}_y : \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is a local homomorphism, and $\Gamma(X, \tilde{\phi}) = \phi$, we have a commutative diagram of rings

(2.47)
$$\begin{array}{ccc} A & \xrightarrow{\ \phi\ } & B \\ \downarrow & & \downarrow \\ \mathcal{O}_{X,x} & \xrightarrow{\ \tilde{\phi}_y\ } & \mathcal{O}_{Y,y} \\ \downarrow & & \downarrow \\ \boldsymbol{k}(x) & \rightarrowtail & \boldsymbol{k}(y) \end{array}$$

The homomorphism $\boldsymbol{k}(x) \to \boldsymbol{k}(y)$ is injective. Comparing the two paths in this diagram we see that

$$\operatorname{Ker}\big(A \xrightarrow{\phi} B \to \boldsymbol{k}(y)\big) = \operatorname{Ker}\big(A \to \boldsymbol{k}(\mathfrak{p})\big) = \mathfrak{p}.$$

This formula determines the value $f(y) = \mathfrak{p}$.

Step 2. Now we prove that the homomorphism of sheaves of rings

$$\tilde{\phi} : \mathcal{O}_X \to f_*(\mathcal{O}_Y)$$

is unique. Since the principal open sets $U = \mathrm{D}(s) \subseteq X$, for $s \in A$, are a basis for the topology, it is enough to prove the uniqueness of the ring homomorphism

$$\Gamma(U, \tilde{\phi}) : \Gamma(U, \mathcal{O}_X) \to \Gamma(V, \mathcal{O}_Y),$$

for $U := \mathrm{D}(s)$ and $V := f^{-1}(U) \subseteq Y$. Let's examine this commutative diagram of rings

(2.48)
$$\begin{array}{ccc} A & \xrightarrow{\ \phi = \Gamma(X, \tilde{\phi})\ } & B \\ \downarrow & & \downarrow \\ A_s = \Gamma(U, \mathcal{O}_X) & \xrightarrow{\ \Gamma(U, \tilde{\phi})\ } & \Gamma(V, \mathcal{O}_Y) \end{array}$$

Because the left vertical arrow is an epimorphism (see Prop 2.43), it follows that the ring homomorphism $\Gamma(U, \tilde{\phi})$ is unique.

Step 3. Here we start with the existence. We define the function $f : Y \to X$ by the formula from step 1, namely a point $y \in Y$ is sent to the prime ideal

$$\mathfrak{p} := \operatorname{Ker}\big(A \xrightarrow{\phi} B \to \boldsymbol{k}(y)\big) \in X. \tag{2.49}$$

We need to prove that f is continuous. It suffices to show that for a principal open set $U = \mathrm{D}(s) \subseteq X$, its preimage $f^{-1}(U)$ is open. A little calculation using formula (2.49) shows that $f^{-1}(U) = \mathrm{D}(\phi(s))$, and this is open in Y by Lemma 2.45.

Step 4. Now we construct the homomorphism of sheaves of rings

$$\tilde{\phi} : \mathcal{O}_X \to f_*(\mathcal{O}_Y) \tag{2.50}$$

on X. Let $U \subseteq X$ be an open set. We need to specify the ring homomorphism

$$\tilde{\phi}_U : \Gamma(U, \mathcal{O}_X) \to \Gamma(V, \mathcal{O}_Y) \tag{2.51}$$

where $V := f^{-1}(U) \subseteq Y$.

First we consider $U = \mathrm{D}(s)$ for $s \in A$. Then $V = \mathrm{D}(\phi(s))$. We know from Thm 2.30 that $\Gamma(U, \mathcal{O}_X) = A_s$. The element $\phi(s)$ is invertible in $\Gamma(V, \mathcal{O}_Y)$ by Lemma 2.45. So there is a unique A-ring homomorphism

$$\tilde{\phi}_s : A_s \to \Gamma(V, \mathcal{O}_Y). \tag{2.52}$$

In particular, for $s = 1$, we get $\Gamma(X, \tilde{\phi}) = \phi$.

By the uniqueness, if $t \in A$ is another element, then the diagram of A-rings

$$\begin{array}{ccc} A_s & \longrightarrow & A_{s \cdot t} \\ {\scriptstyle \tilde{\phi}_s}\downarrow & & \downarrow{\scriptstyle \tilde{\phi}_{s \cdot t}} \\ \Gamma(\mathrm{D}(\phi(s)), \mathcal{O}_Y) & \longrightarrow & \Gamma(\mathrm{D}(\phi(s \cdot t)), \mathcal{O}_Y) \end{array} \tag{2.53}$$

is commutative.

Now we consider an arbitrary open set $U \subseteq X$. We can cover U by principal open sets: $U = \bigcup_{i \in I} \mathrm{D}(s_i)$. Then $V = \bigcup_{i \in I} \mathrm{D}(\phi(s_i))$. There are exact sequences

$$0 \to \Gamma(U, \mathcal{O}_X) \to \prod_i \Gamma(\mathrm{D}(s_i), \mathcal{O}_X) \to \prod_{i,j} \Gamma(\mathrm{D}(s_i \cdot s_j), \mathcal{O}_X) \tag{2.54}$$

and

$$0 \to \Gamma(V, \mathcal{O}_Y) \to \prod_i \Gamma(\mathrm{D}(\phi(s_i)), \mathcal{O}_Y) \to \prod_{i,j} \Gamma(\mathrm{D}(\phi(s_i \cdot s_j)), \mathcal{O}_Y). \tag{2.55}$$

By the previous paragraph there are A-ring homomorphisms

$$\tilde{\phi}_{s_i} : \Gamma(\mathrm{D}(s_i), \mathcal{O}_X) \to \Gamma(\mathrm{D}(\phi(s_i)), \mathcal{O}_Y)$$

and these agree on double intersections by diagram (2.53). So we obtain the ring homomorphism $\tilde{\phi}_U$ in (2.51). As the open set $U \subseteq X$ changes, these become a homomorphism of sheaves of rings $\tilde{\phi} : \mathcal{O}_X \to f_*(\mathcal{O}_Y)$.

Step 5. It remains to prove that $(f, \tilde{\phi})$ is local, i.e. $\tilde{\phi}_y : \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is a local homomorphism for every $y \in Y$ and $x := f(y)$.

Let's return to the construction of the map f. For a point $y \in Y$ with image $x = \mathfrak{p} = f(y) \in X$ we have this solid commutative diagram of rings:

$$\begin{array}{ccccc} A & \longrightarrow & A/\mathfrak{p} & \rightarrowtail & \boldsymbol{k}(\mathfrak{p}) \\ {\scriptstyle \phi}\downarrow & & & \searrow & \downarrow \\ \Gamma(Y, \mathcal{O}_Y) & & \longrightarrow & & \boldsymbol{k}(y) \end{array} \tag{2.56}$$

Because the slanted arrow is an injection, it extends to the field of fractions $\boldsymbol{k}(\mathfrak{p})$, i.e. the dashed arrow exists. On the other hand, all our ring constructions are over A, so we have the next commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & A_{\mathfrak{p}} = \mathcal{O}_{X,x} \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \tilde{\phi}_y} \\ \Gamma(Y, \mathcal{O}_Y) & \longrightarrow & \mathcal{O}_{Y,y} \end{array} \tag{2.57}$$

Merging (2.56) and (2.57) we get this diagram:

$$
\begin{array}{ccccc}
A & \longrightarrow & A_{\mathfrak{p}} = \mathcal{O}_{X,x} & \longrightarrow & \boldsymbol{k}(\mathfrak{p}) \\
{\scriptstyle \phi}\downarrow & & {\scriptstyle \tilde{\phi}_y}\downarrow & & \downarrow \\
\Gamma(Y, \mathcal{O}_Y) & \longrightarrow & \mathcal{O}_{Y,y} & \longrightarrow & \boldsymbol{k}(y)
\end{array}
\tag{2.58}
$$

The left square is the commutative diagram (2.57), and the outer rectangle is the commutative diagram (2.56). Because $A \to A_{\mathfrak{p}}$ is an epimorphism, it follows that the right square is commutative. But this implies that $\tilde{\phi}_y$ is a local homomorphism. □

Corollary 2.59. *The assignment that sends a ring A to the affine scheme* $\operatorname{Spec}(A)$ *is a functor*

$$\operatorname{Spec} : (\mathsf{Rng}/\mathbb{K})^{\mathrm{op}} \to \mathsf{Sch}_{\mathrm{aff}}/\mathbb{K}$$

Proof. Say $\phi : A \to B$ is a homomorphism in $\mathsf{Rng}/\mathbb{K}$. Let $(X, \mathcal{O}_X) := \operatorname{Spec}(A)$ and $(Y, \mathcal{O}_Y) := \operatorname{Spec}(B)$. By Thm 2.46 there is a map

$$\operatorname{Spec}(\phi) := (f, \tilde{\phi}) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$$

in $\mathsf{LRSp}/\mathbb{K}$. If $\psi : B \to C$ is another homomorphism in $\mathsf{Rng}/\mathbb{K}$, with $(Z, \mathcal{O}_Z) := \operatorname{Spec}(C)$, then there is a map

$$\operatorname{Spec}(\psi) := (g, \tilde{\psi}) : (Z, \mathcal{O}_Z) \to (Y, \mathcal{O}_Y)$$

in $\mathsf{LRSp}/\mathbb{K}$. Now

$$\Gamma(Y, \tilde{\psi}) \circ \Gamma(X, \tilde{\phi}) = \psi \circ \phi,$$

so the uniqueness clause in Thm 2.46 says that

$$\operatorname{Spec}(\phi) \circ \operatorname{Spec}(\psi) = \operatorname{Spec}(\psi \circ \phi).$$

Likewise $\operatorname{Spec}(\mathrm{id}_A) = \mathrm{id}_{\operatorname{Spec}(A)}$. □

Corollary 2.60. *The functor*

$$\operatorname{Spec} : (\mathsf{Rng}/\mathbb{K})^{\mathrm{op}} \to \mathsf{LRS}/\mathbb{K}$$

is adjoint to the functor

$$\Gamma : (\mathsf{LRS}/\mathbb{K})^{\mathrm{op}} \to \mathsf{Rng}/\mathbb{K}.$$

Proof. Thm 2.46 produces a bijection

$$\operatorname{Hom}_{\mathsf{Rng}/\mathbb{K}}\big(A, \Gamma(Y, \mathcal{O}_Y)\big) \xrightarrow{\simeq} \operatorname{Hom}_{\mathsf{LRSp}/\mathbb{K}}\big((Y, \mathcal{O}_Y), \operatorname{Spec}(A)\big). \tag{2.61}$$

We need to prove that this is bifunctorial. This is an exercise. □

Exercise 2.62. Prove that the bijection (2.61) is functorial in A and in $(Y, \mathcal{O}_Y)$.

Corollary 2.63. *The functor*

$$\operatorname{Spec} : (\mathsf{Rng}/\mathbb{K})^{\mathrm{op}} \to \mathsf{Sch}_{\mathrm{aff}}/\mathbb{K}$$

is an equivalence of categories, with quasi-inverse Γ.

Proof. By definition, $\mathsf{Sch}_{\mathrm{aff}}/\mathbb{K}$ is the essential image in $\mathsf{LRSp}/\mathbb{K}$ of the functor Spec.

We need to prove that Spec is fully faithful. Since there is adjunction (Cor 2.60), it is enough to prove that the unit of the adjunction

$$\eta_A : A \to \Gamma(X, \mathcal{O}_X),$$

where $(X, \mathcal{O}_X) := \operatorname{Spec}(A)$, is an isomorphism in $\mathsf{Rng}/\mathbb{K}$ for every A. But this is Cor 2.38. □

Next week we will talk about schemes and fiber products.

Lecture 6, 10 Apr 2019

Definition 2.64. Let $(X, \mathcal{O}_X)$ be a LRSp and let $U \subseteq X$ be an open subset. The LRSp $(U, \mathcal{O}_X|_U)$ is called an *open subspace* of $(X, \mathcal{O}_X)$.

It is clear that $(U, \mathcal{O}_X|_U)$ is a LRSp, and that the inclusion $(U, \mathcal{O}_X|_U) \to (X, \mathcal{O}_X)$ is a map in $\mathsf{LRSp}/\mathbb{K}$.

Proposition 2.65. *Let A be a ring and $s \in A$. Let $(X, \mathcal{O}_X) := \operatorname{Spec}(A)$. Consider the principal open set $U := \mathrm{D}(s) \subseteq X$ and the open subspace $(U, \mathcal{O}_X|_U)$ of $(X, \mathcal{O}_X)$. Let $\phi : A \to A_s$ be the canonical ring homomorphism. Then the map*

$$\operatorname{Spec}(\phi) : \operatorname{Spec}(A_s) \to \operatorname{Spec}(A)$$

induces an isomorphism

$$\operatorname{Spec}(A_s) \xrightarrow{\simeq} (U, \mathcal{O}_X|_U)$$

in $\mathsf{LRSp}/\mathbb{K}$.

Proof. Let's write $(Y, \mathcal{O}_Y) := \operatorname{Spec}(A_s)$. We know that the image of the set Y is the subset $U \subseteq X$, and moreover that $Y \to U$ is a homeomorphism of topological spaces. It remains to prove that the sheaf homomorphism $\tilde{\phi} : \mathcal{O}_X|_U \to \mathcal{O}_Y$ on the topological space $Y = U$ is an isomorphism. It is enough to check that it induces bijections on all stalks. But the stalks are $\mathcal{O}_{X,y} = \mathcal{O}_{Y,y} = A_{\mathfrak{p}}$ for $y = \mathfrak{p} \in U$, and the homomorphism induced by ϕ is the identity. □

3. Schemes

Recall that an affine $\mathbb{K}$-scheme is a locally ringed space $(U, \mathcal{O}_U) \in \mathsf{LRSp}/\mathbb{K}$ which is isomorphic to $\operatorname{Spec}(A)$ for some $\mathbb{K}$-ring A.

Definition 3.1. A $\mathbb{K}$-*scheme* is a locally ringed $\mathbb{K}$-space $(X, \mathcal{O}_X)$ that admits an open covering by affine $\mathbb{K}$-schemes. Namely there is an open covering $X = \bigcup_{i \in I} U_i$, such that each open subspace $(U_i, \mathcal{O}_X|_{U_i})$ is an affine $\mathbb{K}$-scheme.

Definition 3.2. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be $\mathbb{K}$-schemes. A map of $\mathbb{K}$-schemes

$$(f, \psi) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$$

is, by definition, a maps of LR $\mathbb{K}$-spaces. Thus the category $\mathsf{Sch}/\mathbb{K}$ of $\mathbb{K}$-schemes is the full subcategory of $\mathsf{LRSp}/\mathbb{K}$ on the $\mathbb{K}$-schemes.

Proposition 3.3. *Let $(X, \mathcal{O}_X)$ be a $\mathbb{K}$-scheme and let $U \subseteq X$ be an open set. Then the open subspace $(U, \mathcal{O}_X|_U)$ is a $\mathbb{K}$-scheme.*

Proof. Take a point $x \in U$. We need to find an open neighborhood V of x in U such that the open subspace $(V, (\mathcal{O}_X|_U)|_V)$ is an affine $\mathbb{K}$-scheme. We know (by Definition 3.1) that there is an open neighborhood W of x in X such that the open subspace $(W, \mathcal{O}_X|_W)$ is an affine $\mathbb{K}$-scheme; say $(W, \mathcal{O}_X|_W) = \operatorname{Spec}(A)$. Because $U \cap W$ is open in W, we can find an element $s \in A$ such that $x \in \mathrm{D}(s) \subseteq U \cap W$. Write $V := \mathrm{D}(s)$. Now $(V, (\mathcal{O}_X|_U)|_V) = (V, \mathcal{O}_X|_V)$, and by Prop 2.65 we know that $(V, \mathcal{O}_X|_V) \cong \operatorname{Spec}(A_s)$ as LRSp's. □

This proposition makes the next definition sensible.

comment: [(190425) next def new]

Definition 3.4. Let $(X, \mathcal{O}_X)$ be a scheme.

(1) An *open subscheme* of $(X, \mathcal{O}_X)$ is a scheme $(U, \mathcal{O}_U)$ such that $U \subseteq X$ is an open subset, and $\mathcal{O}_U = \mathcal{O}_X|_U$.
(2) An *affine open subscheme* of $(X, \mathcal{O}_X)$ is an open subscheme $(U, \mathcal{O}_U)$ which is an affine scheme, i.e. $(U, \mathcal{O}_U) \cong \operatorname{Spec}(A)$ for some ring A.
(3) An *affine open set* of $(X, \mathcal{O}_X)$ is an open subset $U \subseteq X$ such that the open subscheme $(U, \mathcal{O}_X|_U)$ is an affine scheme.

In item (2) above the ring A is $\Gamma(U, \mathcal{O}_U)$, by Corollary 2.38.
Here are two examples of schemes which are not affine.

Example 3.5. Let $\mathbb{K}$ be a nonzero ring. (If you prefer, you can assume that $\mathbb{K}$ is a field, or even an algebraically closed field.) We define 1*-dimensional projective space* to be the following scheme $\mathbf{P}^1_{\mathbb{K}}$. Let $U_0 := \operatorname{Spec}(\mathbb{K}[t_1])$ and $U_1 := \operatorname{Spec}(\mathbb{K}[t_0])$. So $U_0 \cong U_1 \cong \mathbf{A}^1_{\mathbb{K}}$, the 1-dimensional affine space over $\mathbb{K}$, see Exa 2.17. Inside U_0 we have the affine open subscheme $U_{0,1} := \operatorname{Spec}(\mathbb{K}[t_1, t_1^{-1}])$, and inside U_1 we have the affine open subscheme $U_{1,0} := \operatorname{Spec}(\mathbb{K}[t_0, t_0^{-1}])$. The ring isomorphism $\mathbb{K}[t_0, t_0^{-1}] \xrightarrow{\simeq} \mathbb{K}[t_1, t_1^{-1}]$, $t_0 \mapsto t_1^{-1}$, induces an isomorphism of schemes $\phi_{0,1} : U_{0,1} \xrightarrow{\simeq} U_{1,0}$. By Thm 1.24 we can glue U_0 and U_1 along $\phi_{0,1}$. The resulting LRSp is the scheme $\mathbf{P}^1_{\mathbb{K}}$.

Exercise 3.6. Show that $\mathbf{P}^1_{\mathbb{K}}$ is not an affine scheme. (Hint: write $(X, \mathcal{O}_X) := \mathbf{P}^1_{\mathbb{K}}$. By direct calculation show that the ring $\Gamma(X, \mathcal{O}_X) = \mathbb{K}$. Then use Cor 2.63.)

Example 3.7. Let $\mathbb{K}$ be a nonzero ring. (If you prefer, you can assume that $\mathbb{K}$ is a field, or even an algebraically closed field.) Consider the affine plane $\mathbf{A}^2_{\mathbb{K}} = \operatorname{Spec}(\mathbb{K}[t_1, t_2])$. The "origin" is the closed subset $Z := \mathrm{Z}(t_1, t_2) \subseteq \mathbf{A}^2_{\mathbb{K}}$. If $\mathbb{K}$ is a field, then Z contains only one point; but in general it is the topological space $\operatorname{Spec}(\mathbb{K})$, a closed subset of $\mathbf{A}^2_{\mathbb{K}}$.

Define the "punctured plane" $U := \mathbf{A}^2_{\mathbb{K}} - Z$. This is an open subscheme $(U, \mathcal{O}_U)$ of $\mathbf{A}^2_{\mathbb{K}}$.

Exercise 3.8. Show that the punctured plane $(U, \mathcal{O}_U)$ is not an affine scheme. (Hint: define

$$U_i := \operatorname{Spec}(\mathbb{K}[t_1, t_2, t_i^{-1}] \subseteq \mathbf{A}^2_{\mathbb{K}},$$

an affine open subscheme. Show that $U = U_1 \cup U_2$. Use this covering to calculate the ring $\Gamma(U, \mathcal{O}_U)$. Show that $\Gamma(U, \mathcal{O}_U) = \mathbb{K}[t_1, t_2]$. Then use Cor 2.63.)

Exercise 3.9. Let U be the punctured plane from Example 3.7. Show that there is a "canonical" surjective map of schemes $g : U \to \mathbf{P}^1_{\mathbb{K}}$. Show that when $\mathbb{K}$ is a field, then set-theoretically this is the usual way we get the projective line.

4. Fibered Products of Schemes

Fibered products are very important in the theory of schemes.
To be clear what we mean, here's a definition

Definition 4.1. Let $f : Y \to X$ and $g : Z \to X$ be morphisms in a category C. A *fibered product of* Y *and* Z *over* X is an object $W \in \mathsf{C}$, with morphisms $p : W \to Y$ and $q : W \to Z$, satisfying these two conditions:

(i) $f \circ p = g \circ q$.
(ii) Given an object W', and morphisms $p' : W' \to Y$ and $q' : W' \to Z$, such that $f \circ p' = g \circ q'$, there exists a unique morphism $r : W' \to W$ such that $p' = p \circ r$ and $q' = q \circ r$.

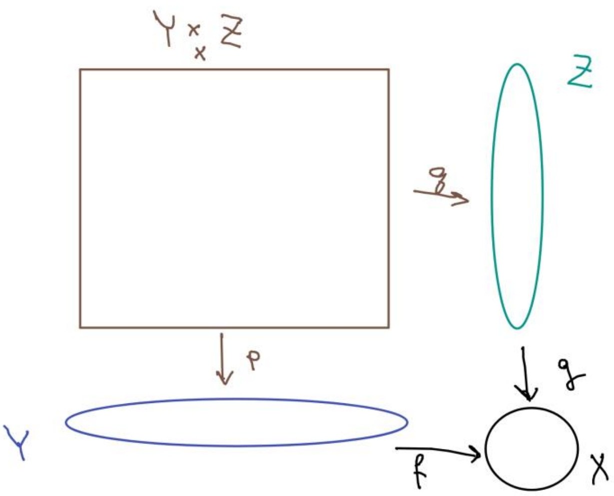


Figure 5. Fibered product

This is illustrated in the next commutative diagram.

$$\begin{tikzcd} W' \arrow[dr, dashed, "r"] \arrow[drr, dashed, bend left, "q'"] \arrow[ddr, dashed, bend right, "p'"'] & & \\ & W \arrow[r, "q"] \arrow[d, "p"'] & Z \arrow[d, "g"] \\ & Y \arrow[r, "f"] & X \end{tikzcd}$$

It is clear that the fibered product (W, p, q) is unique, up to a unique isomorphism. The notation for the fibered product is $Y \times_X Z$, leaving the morphisms implicit.

See Figure 5.

Example 4.2. In the category Set the fiber product is

$$Y \times_X Z = \{(y, z) \mid f(y) = g(z)\} \subseteq Y \times Z,$$

with the obvious projections p, q.

Here is another name for the same categorical construct.

Definition 4.3. A diagram

$$\begin{tikzcd} W \arrow[r, "q"] \arrow[d, "p"'] & Z \arrow[d, "g"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

in the category C is called *cartesian* if $W = Y \times_X Z$.

We can also describe the fibered product in terms of representable functors. Given $f : Y \to X$ and $g : Z \to X$ as in Definition 4.1, let $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ be this functor:

$$F(U) := \mathrm{Hom}_{\mathsf{C}}(U, Y) \times_{\mathrm{Hom}_{\mathsf{C}}(U,X)} \mathrm{Hom}_{\mathsf{C}}(U, Z) \in \mathsf{Set}\,.$$

I.e. the elements of $F(U)$ are pairs (p', q') of morphisms $p' : U \to Y$ and $q' : U \to Z$ such that $f \circ p' = g \circ q'$. The fibered product $W = Y \times_X Z$ is an object of C that represents F:

$$F \cong \mathrm{Hom}_{\mathsf{C}}(-, W)$$

as functors $\mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$. The pair of projections $(p, q) \in F(W)$ is the universal pair.

The next theorem gives a sufficient condition for the existence of the fibered product in $\mathsf{LRSp}/\mathbb{K}$.

Theorem 4.4. *Let $f : Y \to X$ and $g : Z \to X$ be maps in $\mathsf{LRSp}/\mathbb{K}$. Assume that there is a collection of open subspaces $\{X_i\}_{i \in I}$ of X, a collection of open subspaces $\{Y_i\}_{i \in I}$ of Y, and a collection of open subspaces $\{Z_i\}_{i \in I}$ of Z, such that these three conditions hold:*

(a) *For every $i \in I$ there are inclusions $f(Y_i) \subseteq X_i$ and $g(Z_i) \subseteq X_i$.*
(b) *For every $i \in I$ the fibered product $Y_i \times_{X_i} Z_i$ exists.*
(c) *For every pair of points $(y, z) \in Y \times Z$ such that $f(y) = g(z)$, there is an index $i \in I$ such that $y \in Y_i$ and $z \in Z_i$.*

Then the fibered product $Y \times_X Z$ exists. Moreover, for every i the canonical map

$$Y_i \times_{X_i} Z_i \to Y \times_X Z$$

is an open embedding, and

$$Y \times_X Z = \bigcup_{i \in I} Y_i \times_{X_i} Z_i .$$

The proof requires two lemmas.

Lemma 4.5. *Let $f : X' \to X$ be an open embedding in $\mathsf{LRSp}/\mathbb{K}$. Then f is a monomorphism in the category $\mathsf{LRSp}/\mathbb{K}$.*

Lemma 4.6. *Let $f : Y \to X$ and $g : Z \to X$ be maps in $\mathsf{LRSp}/\mathbb{K}$. Let $X' \subseteq X$, $Y' \subseteq Y$ and $Z' \subseteq Z$ be open subspaces, such that $f(Y') \subseteq X'$ and $g(Z') \subseteq X'$. Assume the fiber product $Y \times_X Z$ exists, with projections p, q. Let W' be the open subspace*

$$W' := p^{-1}(Y') \cap q^{-1}(Z') \subseteq Y \times_X Z,$$

with induced projections $p' : W \to Y'$ and $q' : W \to Z'$. Then $W' = Y' \times_{X'} Z'$.

See Figure 6 for an illustration of this lemma.

Exercise 4.7. Prove Lemmas 4.5 and 4.6.

Proof of Thm 4.4. Let's write $W_i := Y_i \times_{X_i} Z_i$. The projections are $p_i : W_i \to Y_i$ and $q_i : W_i \to Z_i$. See Figure 7 for an illustration.

For a pair of indices i, j define $X_{i,j} := X_i \cap X_j$, $Y_{i,j} := Y_i \cap Y_j$ and $Z_{i,j} := Z_i \cap Z_j$. Let

$$W_{i,j} := p_i^{-1}(Y_{i,j}) \cap q_i^{-1}(Z_{i,j}) \subseteq W_i .$$

By Lemma 4.6 we know that

$$W_{i,j} \cong Y_{i,j} \times_{X_{i,j}} Z_{i,j} .$$

But $Y_{i,j} = Y_{j,i}$, etc., so there is an isomorphism $h_{i,j} : W_{i,j} \xrightarrow{\simeq} W_{j,i}$, and this isomorphism is determined by the properties of fibered products, namely by its compatibility with the projections to Y and to Z. This implies that the conditions of Thm 1.24 are satisfied: $h_{j,k} \circ h_{i,j} = h_{i,k}$. In other words, these three isomorphism between three incarnations of the fibered product $Y_{i,j,k} \times_{X_{i,j,k}} Z_{i,j,k}$ are compatible. According to Thm 1.24 we get a locally ringed space W, with a covering $W = \bigcup_{i \in I} W_i$ by open subspaces.

The projections $p_i : W_i \to Y$ and $p_j : W_j \to Y$ agree on $W_{i,j} = W_{j,i}$, now seen as open subspaces of W. Therefore they glue to a map $p : W \to Y$. Likewise the projections $q_i : W_i \to Z$ glue to a map $q : W \to Z$. By Lemma 4.6 we know that $W_i = p^{-1}(Y_i) \cap q^{-1}(Z_i)$ as subspaces of W.

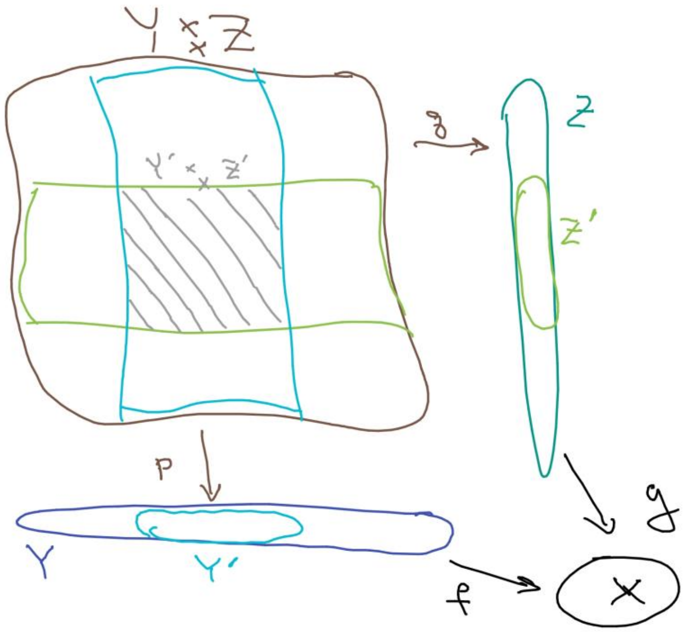


FIGURE 6. For Lem 4.6

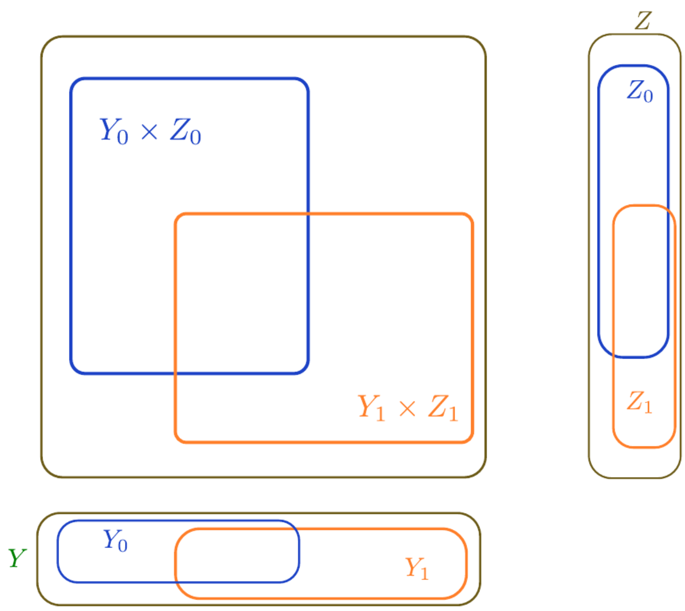


FIGURE 7.

It remains to prove that (W, p, q) is the fibered product $Y \times_X Z$. Take a space U and maps $p' : U \to Y$ and $q' : U \to Z$ such that $f \circ p' = g \circ q'$. Define

$$U_i := p'^{-1}(Y_i) \cap q'^{-1}(Z_i) \subseteq U.$$

Then $U = \bigcup_{i \in I} U_i$ is an open covering. For every i there is a unique map $r_i : U_i \to W_i$ that's compatible with the projections, since $W_i = Y_i \times_{X_i} Z_i$. By the arguments used above, the maps r_i and r_j agree on $U_{i,j} := U_i \cap U_j$. So they glue to a map $r : U \to W$. This r is unique because all its local pieces r_i are unique. So $W = Y \times_X Z$ as claimed. □

This is where lecture 6 ended. From here on it is self-reading during the vacation.

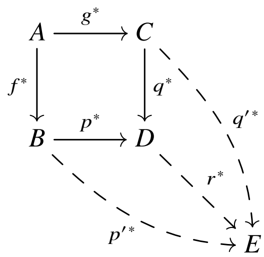
Before proving the existence of the fibered product of schemes, a little reminder on rings.

The *fibered coproduct* is the dual (or opposite) notion of fibered product. To be a bit more precise, the fibered coproduct in a category C is the fibered product in the category C^{op}.

Lemma 4.8. *Let $f^* : A \to B$ and $g^* : A \to C$ be homomorphisms in $\mathsf{Rng}/\mathbb{K}$. Then the ring $D := B \otimes_A C$, with the canonical homomorphisms $p^* : B \to D$ and $q^* : C \to D$, is the fibered coproduct of f^* and g^* in $\mathsf{Rng}/\mathbb{K}$.*

Proof. We know from commutative algebra that for a ring E, the ring homomorphisms $r^* : D \to E$ are in bijection with the pairs of A-ring homomorphisms $p'^* : B \to E$ and $q'^* : C \to E$ such that $f^* \circ p'^* = g^* \circ q'^*$. □

The solid diagram



in the category $\mathsf{Rng}/\mathbb{K}$, as in the lemma, is called *cocartesian*.

Theorem 4.9. *Let $f : Y \to X$ and $g : Z \to X$ be maps in $\mathsf{LRSp}/\mathbb{K}$, and assume these three spaces are schemes. Then the fibered product $Y \times_X Z$ in $\mathsf{LRSp}/\mathbb{K}$ exists, and it is a scheme. Moreover, if X, Y, Z are all affine schemes, then the fibered product $Y \times_X Z$ is an affine scheme.*

Proof.

Step 1. Suppose $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$ and $Z = \mathrm{Spec}(C)$. There are ring homomorphisms $f^* : A \to B$ and $g^* : A \to C$. Define $D := B \otimes_A C \in \mathsf{Rng}/\mathbb{K}$ and $W := \mathrm{Spec}(D)$. The canonical ring homomorphisms $B \to D$ and $C \to D$ give rise to maps $p : W \to Y$ and $q : W \to Z$. We claim that $(W, p, q) = Y \times_X Z$. This is a consequence of Thm 2.46 and Lem 4.8. Indeed, given a space $(U, \mathcal{O}_U) \in \mathsf{LRSp}/\mathbb{K}$ with ring of functions $E := (U, \mathcal{O}_U)$, we know from Thm 2.46 that morphisms from U to affine schemes are in bijection with ring homomorphisms to E. By Lem 4.8, with arrows reversed, we see that maps $U \to W$ are in bijection with pairs of maps $U \to Y$ and $U \to Z$ whose composed maps to X aree equal. This is the property of the fibered product.

Step 2. Take a pair of points $(y, z) \in Y \times Z$ such that $f(y) = g(z)$, and let $x := f(y)$. Choose some affine open neighborhood X_x of x in X. Then choose an affine open neighborhood Y_y of y in $f^{-1}(X_x)$, and an affine open neighborhood Z_z of z in $g^{-1}(X_x)$. We get a collection of open subspaces as in conditions (a) and (c) of Thm 4.4, with indexing set I consisting of these pairs (y, z), which is the fibered product of the underlying sets.

Step 3. We have collections of affine open subschemes $\{X_i\}_{i \in I}$ of X, $\{Y_i\}_{i \in I}$ of Y and $\{Z_i\}_{i \in I}$ of Z, satisfying conditions (a) and (c) of Thm 4.4. These collections could be from the construction in step 2 above, but not necessarily so. According to step 1, for every index $i \in I$ the fibered product $W_i := Y_i \times_{X_i} Z_i$ exists, and it is an affine scheme. Now Thm 4.4 applies. We get an object $W = Y \times_X Z$ in $\mathsf{LRSp}/\mathbb{K}$, and its has an open covering by the affine schemes W_i. Hence W is a scheme. □

Fiber products will be used a lot. One operation they provide is "base change". Here are a few examples of this feature.

But first a definition.

Definition 4.10. Fix a scheme X (say in $\mathsf{Sch}/\mathbb{Z}$). By an *X-scheme* we mean a scheme Y together with a map $\pi_Y : Y \to X$. The X-schemes form a category in the obvioous way, and we denote it by Sch/X.

If $X = \operatorname{Spec}(A)$ is affine, then the expressions Sch/X and Sch/A have the same meaning.

Example 4.11. The projective line is "defined over $\mathbb{Z}$", or is "gotten by base change from $\mathbf{P}^1_{\mathbb{Z}}$". Here is what this means. In Example 3.5, taking $\mathbb{K} := \mathbb{Z}$, we saw how to build the $\mathbb{Z}$-scheme $\mathbf{P}^1_{\mathbb{Z}}$. But for any ring A we can also build $\mathbf{P}^1_A$. It is not hard (and maybe useful to try) to show that

$$\mathbf{P}^1_A \cong \mathbf{P}^1_{\mathbb{Z}} \times_{\operatorname{Spec}(\mathbb{Z})} \operatorname{Spec}(A)$$

in Sch/A. (By abuse of notation we sometimes write $\mathbf{P}^1_{\mathbb{Z}} \times_{\mathbb{Z}} A$ instead of the longer expression above.)

We can also take an arbitrary scheme X (not affine), and form the scheme

$$\mathbf{P}^1_X := \mathbf{P}^1_{\mathbb{Z}} \times_{\operatorname{Spec}(\mathbb{Z})} X.$$

Example 4.12. Let X be a scheme. A *group scheme over* X is a scheme $G \in \mathsf{Sch}/X$ equipped with maps $\operatorname{mult} : G \times_X G \to G$, $\operatorname{inv} : G \to G$ and $\operatorname{unit} : X \to G$ in Sch/X, that satisfy the group axioms.

Example 4.13. To be concrete, for $n \geq 1$ we have the group scheme $\mathrm{GL}_{n,\mathbb{Z}}$. It is an affine group scheme: as a scheme over $\mathbb{Z}$ it is $\operatorname{Spec}(A)$, where A is the polynomial ring over $\mathbb{Z}$ in the n^2 variables $t_{i,j}$, localized w.r.t. the determinant. You can try to write the formulas for the ring homomorphisms that yield the group operations.

For an arbitrary scheme X we have the group scheme

$$\mathrm{GL}_{n,X} := \mathrm{GL}_{n,\mathbb{Z}} \times_{\operatorname{Spec}(\mathbb{Z})} X.$$

Example 4.14. We already know (from the last example) the $\mathbb{R}$-group scheme $G := \mathrm{GL}_{1,\mathbb{R}}$. As an $\mathbb{R}$-scheme it is $G = \operatorname{Spec}(A)$, where $A := \mathbb{R}[t, t^{-1}]$. Now let $B := \mathbb{R}[s, t]/(s^2 + t^2 - 1)$. The affine scheme $H := \operatorname{Spec}(B)$ is also a group scheme over $\mathbb{R}$. The multiplication is that of the circle. Try to guess the formulas.

One can show that G and H are not isomorphic as group schemes over $\mathbb{R}$. (In fact, they are not isomorphic as $\mathbb{R}$-schemes.)

However, after base change to $\mathbb{C}$ they become isomorphic:

$$G \times_{\operatorname{Spec}(\mathbb{R})} \operatorname{Spec}(\mathbb{C}) \cong H \times_{\operatorname{Spec}(\mathbb{R})} \operatorname{Spec}(\mathbb{C}) \cong \mathrm{GL}_{1,\mathbb{C}}$$

as group schemes over $\mathbb{C}$.

Here is a quick discussion of “points of a scheme”, that could help understanding the examples on group schemes above. Let X be a $\mathbb{K}$-scheme. Given a $\mathbb{K}$-ring A, we define the set

$$X(A) := \operatorname{Hom}_{\mathsf{Sch}/\mathbb{K}}\big(\operatorname{Spec}(A), X\big).$$

The elements of $X(A)$ are called the *A-points of X*.

Key example: for the scheme

$$X = \mathbf{A}^n_{\mathbb{K}} = \operatorname{Spec}(\mathbb{K}[t_1, \dots, t_n])$$

we have, by virtue of Cor 2.63,

$$\mathbf{A}^n_{\mathbb{K}}(A) = \operatorname{Hom}_{\mathsf{Rng}/\mathbb{K}}\big(\mathbb{K}[t_1, \dots, t_n], A\big) = \operatorname{Hom}_{\mathsf{Set}}\big(\{t_1, \dots, t_n\}, A\big) = A^n,$$

the set of n-tuples of elements of the ring A.

If G is a group scheme over $\mathbb{K}$, then the set $G(A)$ is a group, in the ordinary sense of the word. For $G = \mathrm{GL}_{n,\mathbb{K}}$, the group $G(A) = \mathrm{GL}_{n,\mathbb{K}}(A)$ is nothing but the group of invertible $n \times n$ matrices with entries in A.

The notions of "points" and "base change" play the following game with each other. Suppose we have a homomorphism $A \to B$ in $\mathsf{Rng}/\mathbb{K}$, and a scheme $X \in \mathsf{Sch}/\mathbb{K}$. Define the A-scheme

$$X_A := X \times_{\operatorname{Spec}(\mathbb{K})} \operatorname{Spec}(A).$$

Since B is both in $\mathsf{Rng}/\mathbb{K}$ and in Rng/A, we can consider the sets

$$X(B) = \operatorname{Hom}_{\mathsf{Sch}/\mathbb{K}}\big(\operatorname{Spec}(B), X\big)$$

and

$$X_A(B) = \operatorname{Hom}_{\mathsf{Sch}/A}\big(\operatorname{Spec}(B), X_A\big).$$

I leave it as an exercise to prove that these sets are equal; or more precisely, the canonical function $X_A(B) \to X(B)$ is bijective. For this reason we can write $\mathbf{A}^n(B)$ unambiguously – it can mean either $\mathbf{A}^n_{\mathbb{Z}}(B)$ or $\mathbf{A}^n_A(B)$, for $B \in \mathsf{Rng}/A$, but these sets are equal (to the set B^n). Likewise for the group $\mathrm{GL}_n(A)$.

Example 4.15. Continuing Exa 4.14, the group scheme $G = \mathrm{GL}_{1,\mathbb{R}}$ has the group $G(\mathbb{R}) = \mathrm{GL}_1(\mathbb{R}) = \mathbb{R}^\times$ as its group of $\mathbb{R}$-points. For the group scheme H, the group of $\mathbb{R}$-points $H(\mathbb{R})$ is the unit circle in the plane. This should help figure out the formulas for the operations of H: think of $H(\mathbb{R})$ as a subgroup of $\mathbb{C}^\times$, with s as the real part and t as the imaginary part.

The two group schemes G and H are not isomorphic. One way to see this is to compare their groups of $\mathbb{R}$-points. In the group $G(\mathbb{R}) = R^\times$ the torsion subgroup is $\{\pm 1\}$, whereas in the $H(\mathbb{R})$ the torsion subgroup is the group of roots of 1 in $\mathbb{C}$, that's isomorphic to $\mathbb{Q}/\mathbb{Z}$.

However, there is an isomorphism $G_{\mathbb{C}} \cong H_{\mathbb{C}}$ of group schemes over $\mathbb{C}$, and hence the groups of $\mathbb{C}$-points are:

$$H(\mathbb{C}) = H_{\mathbb{C}}(\mathbb{C}) \cong G_{\mathbb{C}}(\mathbb{C}) = G(\mathbb{C}) = \mathbb{C}^\times.$$

comment: [(190426) More on these group schemes]

The group scheme H is called the *1-dim nonsplit real torus*, and the group scheme $G = \mathrm{GL}_{1,\mathbb{R}}$ is called the *1-dim split real torus*, and it is also called the *mutiplicative group* $\mathbf{G}_{\mathrm{m},\mathbb{R}}$. The groups G and H are called the *real forms* of the group $\mathrm{GL}_{1,\mathbb{C}} = \mathbf{G}_{\mathrm{m},\mathbb{C}}$. Also H is called a *twisted form* of the group G.

There is a way to classify these forms, and to show there are only two isoclasses of them – this is done by 1-st *Galois cohomology*, which the 0-dimensional version of *étale cohomology*. Here is a sketch.

The classification is analogous to the classification of vector bundles in [Ye4, Thm 11.14] by 1-st nonabelian Čech cohomology, with values in the sheaf of automorphisms of the trivial vector bundle, i.e. in $\mathrm{GL}_n(\mathcal{O}_X)$. For vector bundles on a scheme $(X, \mathcal{O}_X)$ we do use the Zariski topology. However, for more complicated "bundles" the Zariski topology is not fine enough. (For the scheme $\operatorname{Spec}(\mathbb{R})$, that has a single point, the Zariski topology is useless!)

Instead of the Zariski topology, we must use the *étale topology* of X, that we might not have time to study in this course. An *étale covering* $f : Y \to X$ is the algebraic geometry version of a *covering space* in algebraic topology. (The conditions are these: f is finite and étale.) The covering $Y \to X$ is called *Galois* if the group $\mathrm{Aut}_{\mathsf{Sch}/X}(Y)$ has the same size as the fibers of f (assuming X is connected). The covering is *universal* if Y itself has no nontrivial étale coverings. This theory was discovered by Grothendieck. In the case of $X = \mathrm{Spec}(\mathbb{R})$, the map

$$f : \mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathbb{R}) \tag{4.16}$$

is a universal covering space, with Galois group $\Delta = \{\mathrm{id}, \sigma\}$, where σ is complex conjugation.

We want to classify twisted forms of the $\mathbb{R}$-group-scheme $\mathbf{G}_{\mathrm{m},\mathbb{R}}$. For that we need to knoe the group-sheaf of automorphisms of the group-scheme $\mathbf{G}_{\mathrm{m},\mathbb{R}}$ in the étale topology. (Think of the vector bundle analogy.) Let's do it slowly.

We start by identifying the group $N(\mathbb{R})$ of automorphisms of $\mathbf{G}_{\mathrm{m},\mathbb{R}}$ as an $\mathbb{R}$-group-scheme. Since $\mathbf{G}_{\mathrm{m},\mathbb{R}} = \mathrm{Spec}(\mathbb{R}[t, t^{-1}])$, and we know the group structure, the conclusion is that the only automorphisms are id and $\gamma : t \mapsto t^{-1}$. So the group is $N(\mathbb{R}) = \{\mathrm{id}, \gamma\}$. We now guess that these are the $\mathbb{R}$-points of the $\mathbb{R}$-group-scheme

$$N := \mathrm{Spec}(\mathbb{R}) \sqcup \mathrm{Spec}(\mathbb{R})$$

with two points, and with the obvious group structure.

It turns out that the sheaf of automorphisms of $\mathbf{G}_{\mathrm{m},\mathbb{R}}$ in the étale topology of $\mathrm{Spec}(\mathbb{R})$ is indeed represented by the group scheme N. Most importantly, on the universal covering (4.16) the group of automorphisms of $\mathbf{G}_{\mathrm{m},\mathbb{C}}$ as a $\mathbb{C}$-group-scheme is $N(\mathbb{C}) = \{\mathrm{id}, \gamma\}$.

The cohomology calculation we need now is of $\mathrm{H}^1_{\mathrm{et}}(\mathrm{Spec}(\mathbb{R}), N)$, which is the étale analogue of $\check{\mathrm{H}}^1(\boldsymbol{U}, \mathrm{GL}_n(\mathcal{O}_X))$ on a covering $\boldsymbol{U}$. Since (4.16) is a Galois covering, the Čech cohomology is the same as the group cohomology. The action of the Galois group Δ on the group $N(\mathbb{C})$ is trivial. Therefore the pointed set (by chance an abelian group) is

$$\mathrm{H}^1_{\mathrm{et}}(\mathrm{Spec}(\mathbb{R}), N) \cong \mathrm{H}^1(\Delta, N(\mathbb{C})) \cong \mathrm{Ext}^1_{\mathbb{Z}[\Delta]}(\mathbb{Z}, N(\mathbb{C})) \cong N(\mathbb{C}) = \{\mathrm{id}, \gamma\}.$$

So there are two twisted forms of $\mathbf{G}_{\mathrm{m},\mathbb{R}}$, and we already know them. Details can be found in found in [Mil, Section 3.k].

Lecture 7, 1 May 2019

Recall that for a locally ringed space $(X, \mathcal{O}_X)$, a point $x \in X$ and an element (a "function") $a \subseteq \Gamma(X, \mathcal{O}_X)$, we let $a(x) \in \boldsymbol{k}(x)$ be the image of a in the residue field $\boldsymbol{k}(x)$.

New notation, to replace $\mathrm{Z}(a)$ and $\mathrm{D}(a)$:

comment: [(190429) is this good notation?]

Definition 4.17. Let $(X, \mathcal{O}_X)$ be a locally ringed space.

(1) Given an element $a \subseteq \Gamma(X, \mathcal{O}_X)$, its *zero locus* is the set

$$\mathrm{Zer}_X(a) := \{x \in X \mid a(x) = 0\} \subseteq X.$$

(2) Given an ideal $\mathfrak{a} \subseteq \Gamma(X, \mathcal{O}_X)$, its *zero locus* is the set

$$\mathrm{Zer}_X(\mathfrak{a}) := \bigcap_{a \in \mathfrak{a}} \mathrm{Zer}_X(a) \subseteq X.$$

(3) Given an element $a \subseteq \Gamma(X, \mathcal{O}_X)$, its *non-zero locus* is the set

$$\mathrm{NZer}_X(a) := \{x \in X \mid a(x) \neq 0\} \subseteq X.$$

Proposition 4.18. *Let $(X, \mathcal{O}_X)$ be a locally ringed space. The set* $\mathrm{NZer}_X(a)$ *is open in* X*, and the sets* $\mathrm{Zer}_X(a)$ *and* $\mathrm{Zer}_X(\mathfrak{a})$ *are closed in* X*.*

Proof. According to Lemma 2.45 the set $\mathrm{NZer}_X(a)$ is open. Then its complement $\mathrm{Zer}_X(a)$ is closed. The claim about $\mathrm{Zer}_X(\mathfrak{a})$ is now clear. □

5. Closed Subspaces

comment: [(190425) new def 3.4 – open subscheme]

In Definition 2.64 we learned about open subspaces, and in Definition 3.4 we saw open subschemes. Closed subspaces and closed subschemes are much more complicated. Before talking about them we need to understand a few more basic things: *supports* and *quotients* of sheaves.

Definition 5.1. Suppose $(X, \mathcal{A})$ is a ringed space.

(1) Let $\mathcal{M}$ be a sheaf of $\mathcal{A}$-modules on X. The *support* of $\mathcal{M}$ is the set

$$\mathrm{Supp}(\mathcal{M}) := \{x \in X \mid \mathcal{M}_x \neq 0\} \subseteq X.$$

(2) Let $\mathcal{M} \in \mathsf{Mod}\,\mathcal{A}$ and let $Y \subseteq X$. We say that $\mathcal{M}$ is *supported on* Y if $\mathrm{Supp}(\mathcal{M}) \subseteq Y$.

(3) Given a subset $Y \subseteq X$, we denote by $\mathsf{Mod}_Y\,\mathcal{A}$ the full subcategory of $\mathsf{Mod}\,\mathcal{A}$ on the sheaves supported on Y.

Note that here we are looking at the stalks $\mathcal{M}_x$, and $(X, \mathcal{A})$ need not be a locally ringed space.

Exercise 5.2. Let $\mathcal{M} \twoheadrightarrow \mathcal{N}$ be a surjection in $\mathsf{Mod}\,\mathcal{A}$. Show that

$$\mathrm{Supp}(\mathcal{N}) \subseteq \mathrm{Supp}(\mathcal{M}).$$

Definition 5.3. Let $(X, \mathcal{A})$ be a ringed space and let $\mathcal{M}$ be an $\mathcal{A}$-module. We say that $\mathcal{M}$ is *locally finitely generated* over $\mathcal{A}$ if there is an open covering $\{U_i\}_{i \in I}$ of X, and for each index i there is a surjection of $\mathcal{A}|_{U_i}$-modules $(\mathcal{A}|_{U_i})^{\oplus e_i} \twoheadrightarrow \mathcal{M}|_{U_i}$ for some $e_i \in \mathbb{N}$.

Proposition 5.4. *Let $(X, \mathcal{A})$ be a ringed space and let $\mathcal{M}$ be an $\mathcal{A}$-module. Assume that $\mathcal{M}$ is locally finitely generated. Then* $\mathrm{Supp}(\mathcal{M})$ *is a closed subset of* X*.*

Note that, unlike Proposition 4.18, here this need not be a LR space.

Proof. Being closed is a local question: it is enough to prove that $Y \cap U_i \subseteq U_i$ is closed for all i. So, by replacing X with U_i, we may assume that there is a surjection $\mathcal{A}^{\oplus e} \twoheadrightarrow \mathcal{M}$ of $\mathcal{A}$-modules. Let $m_1, \ldots, m_e \in \Gamma(X, \mathcal{M})$ be the corresponding generators on $\mathcal{M}$ (the images of the e copies of $1_{\mathcal{A}}$). Then

$$\mathrm{Supp}(\mathcal{M}) = \bigcup_{i=1,\ldots,e} \mathrm{Supp}(m_i),$$

where for a global section $m \in \Gamma(X, \mathcal{M})$ we define

$$\mathrm{Supp}(m) := \{x \in X \mid m_x \neq 0 \text{ in } \mathcal{M}_x\} \subseteq X.$$

It now suffices to prove that $\mathrm{Supp}(m)$ is closed in X. We will prove that the complement $V := X - \mathrm{Supp}(m)$ is open in X. Take a point $x \in V$. Then $m_x = 0$ in the stalk $\mathcal{M}_x$. By the direct limit definition of the stalk, there is some open neighborhood W of x s.t. $m_{x'} = 0$ for all $x' \in W$. This means that $W \subseteq V$. So V is open in X. □

Exercise 5.5. Find an example of a scheme $(X, \mathcal{O}_X)$ and an $\mathcal{O}_X$-module $\mathcal{M}$ whose support is not closed.

Try to characterize the sets $Y \subseteq X$ that can occur as $\mathrm{Supp}(\mathcal{M})$ fore some $\mathcal{O}_X$-module $\mathcal{M}$.

Recall that for a map $f : Y \to X$ in Top there is adjunction between f_* and f^{-1}, i.e. an isomorphism of $\mathbb{K}$-modules

$$\mathrm{Hom}_{\mathsf{Mod}\,\mathbb{K}_X}(\mathcal{M}, f_*(\mathcal{N})) \cong \mathrm{Hom}_{\mathsf{Mod}\,\mathbb{K}_Y}(f^{-1}(\mathcal{M}), \mathcal{N}) \tag{5.6}$$

which is functorial in $\mathcal{M} \in \mathsf{Mod}\,\mathbb{K}_X$ and $\mathcal{N} \in \mathsf{Mod}\,\mathbb{K}_Y$.

Proposition 5.7. *Suppose X is a topological space, and $Y \subseteq X$ is a closed subset, with inclusion map $f : Y \to X$. Then:*

(1) *Let $\mathcal{N}$ be a sheaf of $\mathbb{K}$-modules on Y. Then the sheaf $f_*(\mathcal{N})$ is supported on Y, and the adjunction homomorphism $\mathcal{N} \to f^{-1}(f_*(\mathcal{N}))$ is an isomorphism in $\mathsf{Mod}\,\mathbb{K}_Y$.*
(2) *The functor*

$$f_* : \mathsf{Mod}\,\mathbb{K}_Y \to \mathsf{Mod}_Y\,\mathbb{K}_X$$

is an equivalence.

Exercise 5.8. Prove this proposition.

The proposition tells us that when $Y \subseteq X$ is a closed set, we can safely identify $\mathbb{K}_Y$-modules with $\mathbb{K}_X$-modules supported on Y. This is false for subsets that are not closed, as the next exercise shows.

Exercise 5.9. Find an example of a scheme $(X, \mathcal{O}_X)$, an open set $U \subseteq X$ with inclusion $g : U \to X$, and an $\mathcal{O}_U$-module $\mathcal{M}$, such that the set $\mathrm{Supp}(g_*(\mathcal{M}))$ is bigger than U.

Given a sheaf $\mathcal{M}$, we know what is a *subsheaf* $\mathcal{N} \subseteq \mathcal{M}$. But what's a *quotient sheaf* of $\mathcal{M}$?

Let X be a topological space, and let $\mathcal{M} \in \mathsf{Mod}\,\mathbb{K}_X$. A *quotient sheaf of* $\mathcal{M}$ is a $\mathbb{K}_X$-module $\mathcal{N}$ together with a *surjection* $\pi : \mathcal{M} \twoheadrightarrow \mathcal{N}$ in $\mathsf{Mod}\,\mathbb{K}_X$. Suppose $\pi' : \mathcal{M} \twoheadrightarrow \mathcal{N}'$ is another quotient sheaf of $\mathcal{M}$. A *homomorphism of quotients* $\phi : \mathcal{N} \to \mathcal{N}'$ is a homomorphism in $\mathsf{Mod}\,\mathbb{K}_X$ such that $\pi = \phi \circ \pi$. In this way the quotients of $\mathcal{M}$ form a category.

Note that if $\pi : \mathcal{M} \twoheadrightarrow \mathcal{N}$ and $\pi' : \mathcal{M} \twoheadrightarrow \mathcal{N}'$ are quotients of $\mathcal{M}$, then there is at most one homomorphism of quotients $\phi : \mathcal{N} \to \mathcal{N}'$, and it is a surjection too. In case ϕ is an isomorphism, then of course there is an isomorphism of quotients $\phi^{-1} : \mathcal{N}' \to \mathcal{N}$, and these are unique. This means that we can *identify* the isomorphic quotients $\mathcal{N}$ and $\mathcal{N}'$ via the unique isomorphism ϕ between them. (Formally this means passing to isomorphism classes, and then choosing one representative from each isomorphism class.) Having done so, the set of (isoclasses of) quotients of $\mathcal{M}$ becomes partially ordered, by the relation of existence of morphism of quotients.

Exercise 5.10. This is a continuation of Exercise 5.2. Fix $\mathcal{M} \in \mathsf{Mod}\,\mathbb{K}_X$. Show that the function Supp, from the set of quotients of $\mathcal{M}$ to the set of subsets of X, reverses the partial orders. (I.e. it is a contravariant functor.)

We now want to talk about quotients of sheaves of rings. Let $(X, \mathcal{A})$ be a ringed space. A *quotient ring of* $\mathcal{A}$ is an $\mathcal{A}$-ring $\mathcal{B}$ such that the structural homomorphism $\pi : \mathcal{A} \to \mathcal{B}$ is *surjective*. Suppose $\pi : \mathcal{A} \twoheadrightarrow \mathcal{B}$ and $\pi' : \mathcal{A} \twoheadrightarrow \mathcal{B}'$ are quotient rings of $\mathcal{A}$. A *homomorphism of quotient rings* $\phi : \mathcal{B} \to \mathcal{B}'$ is an $\mathcal{A}$-ring homomorphism. It is necessarily surjective, and there is at most one. It two quotient rings $\mathcal{B}$ and $\mathcal{B}'$ of $\mathcal{A}$ are isomorphic, then this isomorphism is unique. Hence, as in the case of $\mathbb{K}_X$-modules, we can identify isomorphic quotient rings of $\mathcal{A}$. Having done so, the quotient rings of $\mathcal{A}$ are partially ordered.

Definition 5.11. Let $(X, \mathcal{A})$ be a ringed space. An *ideal of* $\mathcal{A}$ is an $\mathcal{A}$-submodule $\mathcal{I} \subseteq \mathcal{A}$.

Lemma 5.12. *Let $(X, \mathcal{A})$ be a ringed space. The assignment*

$$\big(\pi : \mathcal{A} \twoheadrightarrow \mathcal{B}\big) \mapsto \big(\mathcal{I} := \operatorname{Ker}(\pi) \subseteq \mathcal{A}\big)$$

is an order preserving bijection from the set of (isoclasses of) quotient rings of $\mathcal{A}$ to the set if ideals of $\mathcal{A}$.

Exercise 5.13. Prove this lemma.

Definition 5.14. Let $(X, \mathcal{A})$ be a ringed space.

(1) A *closed subspace* of $(X, \mathcal{A})$ is a ringed space $(Y, \mathcal{B})$, such that $Y \subseteq X$ is a closed subset, and $\mathcal{B}$ a quotient ring of $\mathcal{A}$.
(2) Suppose $(Z, \mathcal{C})$ is another closed subspace of $(X, \mathcal{A})$. A map of closed subspaces $(Z, \mathcal{C}) \to (Y, \mathcal{B})$ is an inclusion $Z \subseteq Y$ of closed subsets of X, and a homomorphism $\mathcal{B} \to \mathcal{C}$ of quotient rings of $\mathcal{A}$.

Lemma 5.15. *Let $(X, \mathcal{A})$ be a ringed space, and let $(Y, \mathcal{B})$ and $(Z, \mathcal{C})$ be closed subspaces of $(X, \mathcal{A})$. There is at most one map of closed subspaces $(Y, \mathcal{B}) \to (Z, \mathcal{C})$.*

Exercise 5.16. Prove this lemma.

In view of this lemma, we can identify isomorphic closed subspaces of $(X, \mathcal{A})$.

comment: [(190506) Next lem was a prop. And it was wrong. Here's the corrected version.]

Lemma 5.17. *Let $(X, \mathcal{A})$ be a ringed space. The assignment*

$$\big(\mathcal{I} \subseteq \mathcal{A}\big) \mapsto \big((Y, \mathcal{B}) \subseteq (X, \mathcal{A})\big),$$

sending an ideal $\mathcal{I} \subseteq \mathcal{A}$ to the quotient ring $\mathcal{B} := \mathcal{A}/\mathcal{I}$ and the closed set $Y := \operatorname{Supp}(\mathcal{B}) \subseteq X$, is an order reversing bijection from the set of ideals $\mathcal{I} \subseteq \mathcal{A}$ to the set of (isoclasses of) closed subspaces $(Y, \mathcal{B}) \subseteq (X, \mathcal{A})$ such that $\operatorname{Supp}(\mathcal{B}) = Y$.

Exercise 5.18. Prove this lemma. (Hint: Use Lemma 5.12 and Proposition 5.4.)

Example 5.19. In the lemma above, suppose $(Y, \mathcal{B}) \subseteq (X, \mathcal{A})$ is such that $\operatorname{Supp}(\mathcal{B}) = Y$ and $Y \subsetneq X$, a closed strict subset. Then the pair $(X, \mathcal{B})$ is also a closed subspace of $(X, \mathcal{A})$, and it does not match an ideal $\mathcal{I} \subseteq \mathcal{A}$. This is why the earlier version of the result above was wrong.

Lecture 8, 6 May 2019

Last week we learned about closed subspaces of ringed spaces.

Lemma 5.20. *Let $(X, \mathcal{A})$ be a locally ringed space, and let $(Y, \mathcal{B})$ be a closed subspace of it, such that $\operatorname{Supp}(\mathcal{B}) = Y$. Then $(Y, \mathcal{B})$ is a locally ringed space, and the inclusion*

$$(f, \phi) : (Y, \mathcal{B}) \to (X, \mathcal{A})$$

is a map of LR spaces.

Exercise 5.21. Prove this lemma. (Hint: a nonzero quotient of a local ring is a local ring.)

Proposition 5.22. *Let $(X, \mathcal{A})$ be a locally ringed ringed space. The assignment*

$$\big(\mathcal{I} \subseteq \mathcal{A}\big) \mapsto \big((Y, \mathcal{B}) \subseteq (X, \mathcal{A})\big),$$

sending an ideal $\mathcal{I} \subseteq \mathcal{A}$ to the quotient ring $\mathcal{B} := \mathcal{A}/\mathcal{I}$ and the closed set $Y := \operatorname{Supp}(\mathcal{B}) \subseteq X$, is an order reversing bijection from the set of ideals $\mathcal{I} \subseteq \mathcal{A}$ to the set of (isoclasses of) closed locally ringed subspaces $(Y, \mathcal{B}) \subseteq (X, \mathcal{A})$.

Proof. Combine Lemma 5.17 and 5.20. □

Definition 5.23. A map of locally ringed spaces

$$(f, \phi) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$$

is called a *closed embedding* if it factors through an isomorphism $(Y, \mathcal{O}_Y) \xrightarrow{\simeq} (Y', \mathcal{O}_{Y'})$, where $(Y', \mathcal{O}_{Y'}) \subseteq (X, \mathcal{O}_X)$ is a closed subspace.

Definition 5.24. Let $(X, \mathcal{O}_X)$ be a scheme. A *closed subscheme* of $(X, \mathcal{O}_X)$ a closed subspace $(Y, \mathcal{O}_Y)$, as in Definition 5.14, such that $(Y, \mathcal{O}_Y)$ is a scheme.

The next proposition gives us the first example of a closed subscheme.

Proposition 5.25. *Take a ring A and an ideal $\mathfrak{a} \subseteq A$, and define the quotient ring $B := A/\mathfrak{a}$, with surjection $\pi : A \twoheadrightarrow B$. Consider the of affine schemes $(X, \mathcal{O}_X) :=$* Spec(A) *and $(Y, \mathcal{O}_Y) :=$* Spec(B). *Then the map of schemes*

$$(f, \tilde{\pi}) := \operatorname{Spec}(\pi) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$$

is a closed embedding, and $f(Y) = \operatorname{Zer}_X(\mathfrak{a})$.

Exercise 5.26. Prove this prop. (Hint: this is easy!)

Remark 5.27. Closed subschemes are delicate (much more complicated than open subschemes). Suppose $(X, \mathcal{O}_X)$ is a scheme. We will prove later that:

- A closed LR subspace $(Y, \mathcal{O}_Y)$ is a closed subscheme iff the corresponding ideal sheaf $\mathcal{I}_Y := \operatorname{Ker}(\mathcal{O}_X \twoheadrightarrow \mathcal{O}_Y)$ is a *quasi-coherent $\mathcal{O}_X$-module*.
- Every closed subset Y of a scheme $(X, \mathcal{O}_X)$ has at least one closed subscheme structure: the *reduced structure*. See Thm 5.38 below.
- If $(X, \mathcal{O}_X)$ is an affine scheme and $(Y, \mathcal{O}_Y)$ is a closed subscheme, then $(Y, \mathcal{O}_Y)$ is also affine. This is the converse of Prop 5.25. It will also require the study of quasi-coherent $\mathcal{O}_X$-modules.

The next two exercises will show how to get closed LR subspaces of schemes that are usually not schemes.

Exercise 5.28. Let $(X, \mathcal{O}_X)$ be a locally ringed space, let $Y \subseteq X$ be a closed subset, with inclusion map $f : Y \to X$, and let $\mathcal{O}_Y = \mathcal{O}_X|_Y := f^{-1}(\mathcal{O}_X)$. By Proposition 5.7 we can view $\mathcal{O}_Y$ as a sheaf of rings on X, supported on Y. Prove that:

(1) The pair $(Y, \mathcal{O}_Y)$ is a locally ringed space.
(2) Let $\psi : \mathcal{O}_X \to \mathcal{O}_Y$ be the homomorphism of $\mathbb{K}_X$-rings that corresponds to the identity of $\mathcal{O}_Y$ under the adjunction (5.6). Then ψ is surjective, and for every point $y \in Y$ the homomorphism on stalks $\psi_y : \mathcal{O}_{X,y} \to \mathcal{O}_{Y,y}$ is an isomorphism.
(3) The pair (f, ψ) is a closed embedding of locally ringed spaces $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$.
(4) The functor

$$f_* : \mathsf{Mod}\, \mathcal{O}_Y \to \mathsf{Mod}_Y\, \mathcal{O}_X$$

is an equivalence.

Exercise 5.29. Take $(X, \mathcal{O}_X) := \mathbf{A}^1_{\mathbb{K}} = \operatorname{Spec}(\mathbb{K}[t])$ for a field $\mathbb{K}$. Let $y \in X$ be the origin, and let $Y := \{y\}$, i.e. $Y = \operatorname{Zer}_X(t) \subseteq X$. By the previous exercise, $(Y, \mathcal{O}_Y) := (Y, \mathcal{O}_X|_Y)$ is a closed LR subspace of $(X, \mathcal{O}_X)$. Show that the locally ringed space $(Y, \mathcal{O}_Y)$ is not a scheme. (Hint: If $(Y, \mathcal{O}_Y)$ were a scheme, then it would have to be affine, say Spec(C) for a ring C, and then $C \cong \mathcal{O}_{Y,y}$.)

Try to identify the ideal $\mathcal{I} \subseteq \mathcal{O}_X$ that matches $\mathcal{O}_Y$ under the bijection in Prop 5.22. (Hint: $\mathcal{I} = g_!(\mathcal{O}_U)$, where $U := X - Y$ and g is the inclusion. Read about $g_!$, called *extension by* 0, in [Har, Sec III.6].)

We end this section with a general way of producing closed subschemes.

A ring A is called *reduced* if the only nilpotent element in it is 0. Any ring A has a maximal reduced quotient ring A_{red}. It is gotten as follows: let $\sqrt{0} \subseteq A$ be *radical* of A, i.e. the ideal consisting of all nilpotent elements. Then $A_{\mathrm{red}} = A/\sqrt{0}$. It is the universal quotient ring of A for homomorphisms into reduced rings. An ideal $\mathfrak{a} \subseteq A$ is called a *radical ideal* if $\mathfrak{a} = \sqrt{\mathfrak{a}}$; namely if the quotient ring $A/\mathfrak{a}$ is reduced.

Lemma 5.30. *If A is a reduced ring, then any localization A_S is also reduced.*

Exercise 5.31. Prove this lem.

Definition 5.32. A scheme $(X, \mathcal{O}_X)$ is called a *reduced scheme* if for every open set $U \subseteq X$, the ring $\Gamma(U, \mathcal{O}_X)$ is reduced.

Proposition 5.33. *Let $(X, \mathcal{O}_X)$ be a scheme. The following three conditions are equivalent.*

(i) *$(X, \mathcal{O}_X)$ is a reduced scheme.*
(ii) *X has an affine open covering $\{U_i\}_{i \in I}$ such that the rings $A_i := \Gamma(U_i, \mathcal{O}_X)$ are all reduced.*
(iii) *For every point $x \in X$ the local ring $\mathcal{O}_{X,x}$ is reduced.*

Exercise 5.34. Prove Proposition 5.33.

Lemma 5.35. *Let $(X, \mathcal{O}_X) = \mathrm{Spec}(A)$ be an affine scheme. Then ring A is reduced iff the scheme $(X, \mathcal{O}_X)$ is reduced.*

Exercise 5.36. Prove Lemma 5.35.

Example 5.37. Suppose $\mathbb{K}$ is an algebraically closed field and Y is an affine variety over $\mathbb{K}$. This means that Y is the zero locus inside $X := \mathbf{A}^n(\mathbb{K})$, for some $n \geq 0$, of an ideal $\mathfrak{a} \subseteq A := \mathbb{K}[t_1, \ldots, t_n]$. Because $\mathrm{Zer}_X(\mathfrak{a}) = \mathrm{Zer}_X(\sqrt{\mathfrak{a}})$, we can assume that $\mathfrak{a}$ is a radical ideal.

Let $B := A/\mathfrak{a}$, and let $(Y_{\mathrm{rsc}}, \mathcal{O}_{Y_{\mathrm{rsc}}}) := \mathrm{Spec}(B)$. By Lemma 5.35 and Proposition 5.25 we know that $(Y_{\mathrm{rsc}}, \mathcal{O}_{Y_{\mathrm{rsc}}})$ is a reduced closed subscheme of $\mathbf{A}^n_{\mathbb{K}} = \mathrm{Spec}(A)$. (The letters "rsc" stand for "reduced scheme".)

It turns out that $Y = Y_{\mathrm{rsc}}(\mathbb{K})$, i.e. the points of the variety Y are the $\mathbb{K}$-points of the scheme Y_{rsc}. The Hilbert NZS implies that the inclusion $f : Y \to Y_{\mathrm{rsc}}$ is a "homeomorphism", in the sense that these topological spaces have the same open sets. And the sheaf of rings $\mathcal{O}_Y := f^{-1}(\mathcal{O}_{Y_{\mathrm{rsc}}})$ is the sheaf of $\mathbb{K}$-valued functions on the variety $\mathbb{K}$.

The same holds for any quasi-projective variety Y over $\mathbb{K}$.

Theorem 5.38. *Let $(X, \mathcal{O}_X)$ be a scheme and let $Y \subseteq X$ be a closed subset. Then there is a unique (up to isomorphism) sheaf of $\mathcal{O}_X$-rings $\mathcal{O}_Y$, whose support is Y, and such that $(Y, \mathcal{O}_Y)$ is a reduced closed subscheme of $(X, \mathcal{O}_X)$.*

comment: [(190506) The proof was not done in class. Please read it and try to understand the details.]

Proof. Recall that for an open set $U \subseteq X$, an element $a \in \Gamma(U, \mathcal{O}_X)$ and a point $x \in U$, we denote by $a(x)$ the image of a in the residue field $\boldsymbol{k}(x)$. Let $\mathcal{I}_Y \subseteq \mathcal{O}_X$ be the subpresheaf defined by the formula

$$\Gamma(U, \mathcal{I}_Y) := \left\{ a \in \Gamma(U, \mathcal{O}_X) \;\middle|\; a(y) = 0 \text{ for all } y \in U \cap Y \right\} \tag{5.39}$$

for an open set $U \subseteq X$. Then $\mathcal{I}_Y$ is actually a sheaf, so it is an ideal sheaf of $\mathcal{O}_X$. Moreover, since $\boldsymbol{k}(y)$ is field, it follows that $\Gamma(U, \mathcal{I}_Y)$ is a radical ideal of $\Gamma(U, \mathcal{O}_X)$.

Define the $\mathcal{O}_X$-ring $\mathcal{O}_Y := \mathcal{O}_X/\mathcal{I}_Y$. By Proposition 5.22 we get a closed locally ringed subspace $(Y, \mathcal{O}_Y) \subseteq (X, \mathcal{O}_X)$.

Let us prove that $(Y, \mathcal{O}_Y)$ is a scheme, and that it is reduced. Both are local questions, so we may assume that $(X, \mathcal{O}_X) = \operatorname{Spec}(A)$ for a ring A. Let

$$I := \Gamma(X, \mathcal{I}_Y) = \{ a \in A \mid a(y) = 0 \text{ for all } y \in Y \} \subseteq A.$$

As noticed above, I is a radical ideal of A. Define $B := A/I$, which is a reduced ring. We know that $\operatorname{Zer}_X(I) = Y$. According to Proposition 5.25 the affine scheme $(Y, \mathcal{O}'_Y) := \operatorname{Spec}(B)$ is a closed subscheme of $(X, \mathcal{O}_X)$; and by Lemma 5.35 the scheme $(Y, \mathcal{O}'_Y)$ is reduced. We will prove that $(Y, \mathcal{O}_Y) = (Y, \mathcal{O}'_Y)$. This will be accomplished by proving that there is equality $\mathcal{I}_Y = \mathcal{I}'_Y$ of ideals of $\mathcal{O}_X$, where $\mathcal{I}'_Y := \operatorname{Ker}(\mathcal{O}_X \twoheadrightarrow \mathcal{O}'_Y)$.

Take a principal open set $U := \operatorname{NZer}_X(s) \subseteq X$ for some $s \in A$, so $\Gamma(U, \mathcal{O}_X) = A_s$. By formula (5.39) we have

$$\Gamma(U, \mathcal{I}_Y) = \{ a \in A_s \mid a(y) = 0 \text{ for all } y \in U \cap Y \} \subseteq A_s. \tag{5.40}$$

This is a radical ideal of A_s with zero locus $U \cap Y$.

According to Proposition 5.7(2), used for the closed embedding $U \cap Y \rightarrowtail U$, we have

$$\Gamma(U, \mathcal{O}_{Y'}) = \Gamma(U \cap Y, \mathcal{O}_{Y'}) = \Gamma(\operatorname{NZer}_Y(s), \mathcal{O}_{Y'}) = B_s.$$

By the left exactness of $\Gamma(U, -)$ and the flatness of localization we see that

$$\Gamma(U, \mathcal{I}'_Y) = \Gamma(U, \operatorname{Ker}(\mathcal{O}_X \to \mathcal{O}'_Y)) = \operatorname{Ker}(A_s \to B_s) = I_s \subseteq A_s. \tag{5.41}$$

Since I is a radical ideal, so is I_s. Thus $\Gamma(U, \mathcal{I}'_Y)$ is a radical ideal of A_s with zero locus $U \cap Y$. By Lemma 2.20 it follows that $\Gamma(U, \mathcal{I}_Y) = \Gamma(U, \mathcal{I}'_Y)$.

Finally, because the principal open sets are a basis of the topology of X, we see that $\mathcal{I}_Y = \mathcal{I}'_Y$. □

Definition 5.42. The reduced closed subscheme $(Y, \mathcal{O}_Y)$ from Thm 5.38 is called *the reduced closed subscheme of* $(X, \mathcal{O}_X)$ *supported on* Y, and the sheaf $\mathcal{O}_Y$ is called *the reduced subscheme structure on* Y.

Lecture 9, 15 May 2019

Definition 5.43. A topological space X is called *noetherian* if it has the descending chain condition on closed subsets; namely every strictly descending chain of closed subsets $X_0 \supsetneq X_1 \supsetneq X_2 \supsetneq \cdots$ in X is finite.

This sort of topological space almost never occurs in "classical topology". However it is very typical in algebraic geometry.

Example 5.44. If A is a noetherian ring, then $\operatorname{Spec}(A)$ is a noetherian topological space. Cf. Lemma 2.20.

Definition 5.45. A topological space X is called *irreducible* if it is nonempty, and it is not a union $X = X_1 \cup X_2$, where $X_i \subsetneq X$ are closed strict subsets.

Example 5.46. Let X be a topological space and $x \in X$ a point. The closure $Z := \overline{\{x\}}$ in X of the set $\{x\}$ is irreducible. The point x is called a *generic point* of Z.

Lemma 5.47. *If X is an irreducible topological space, then every nonempty open subset $U \subseteq X$ is irreducible, and it is is dense in X.*

Exercise 5.48. Prove the lemma.

Proposition 5.49. *Let X be a noetherian topological space. Then:*

(1) *X is quasi-compact.*
(2) *Every subset Y of X, with the induced topology, is a noetherian topological space.*
(3) *X is a finite union of irreducible closed subsets, say* $X = \bigcup_{i=1,\ldots,r} Z_i$*, such that* $Z_i \nsubseteq Z_j$ *for all* $i \neq j$*. The collection of closed sets* $\{Z_i\}_{i=1,\ldots,r}$ *is unique up to permutation.*
(4) *X is a finite disjoint union of closed connected subsets* $X = \coprod_{i=1,\ldots,r} X_i$*. The sets* X_i *are also open. The collection* $\{X_i\}_{i=1,\ldots,s}$ *is unique up to permutation.*

Exercise 5.50. Prove this proposition. (Hint: It is elementary but tricky. See [Har, Sec II.3] for proofs. Item (4) is a consequence of item (3).)

The irreducible closed subsets $\{Z_i\}_{i=1,\ldots,r}$ in item (3) of the proposition are called the *irreducible components* of X. The connected closed (and open) subsets $\{X_i\}_{i=1,\ldots,s}$ in item (4) of the proposition are called the *connected components* of X.

Definition 5.51. A scheme $(X, \mathcal{O}_X)$ is called an *irreducible scheme* (resp. a *connected scheme*, resp. a *quasi-compact scheme*) if the topological space X is irreducible (resp. connected, resp. quasi-compact).

comment: [(190522) next def improved - see Def 5.60 and Prop 5.61]

Definition 5.52. A scheme $(X, \mathcal{O}_X)$ is called a *noetherian scheme* if it is finite union of affine open subschemes $U_i = \operatorname{Spec}(A_i)$, s.t. each A_i is a noetherian ring.

Proposition 5.53. *Let* $(X, \mathcal{O}_X)$ *be a noetherian scheme. Then*

(1) *X is a noetherian topological space.*
(2) *Every irreducible closed subset* $Z \subseteq X$ *has a unique generic point.*

Exercise 5.54. Prove Proposition 5.53. (Hint: This is also elementary but tricky. The book [Har, Sec II.3] does not give a proof – it is an exercise there.)

Recall that a ring homomorphism $\phi : A \to B$ is called *flat* if B is a flat A-module. The ring homomorphism $\phi : A \to B$ is called *faithfully flat* if for every sequence

$$\boldsymbol{S} = \Big(\cdots M^i \xrightarrow{\phi^i} M^{i+1} \xrightarrow{\phi^{i+1}} M^{i+1} \cdots \Big)$$

in $\mathsf{Mod}\, A$, the sequence $\boldsymbol{S}$ is exact iff the sequence

$$\operatorname{Ind}_\phi(\boldsymbol{S}) := \Big(\cdots B \otimes_A M^i \xrightarrow{\operatorname{id} \otimes \phi^i} B \otimes_A M^{i+1} \xrightarrow{\operatorname{id} \otimes \phi^{i+1}} B \otimes_A M^{i+1} \cdots \Big)$$

in $\mathsf{Mod}\, B$ is exact. According to [Mat, Chapter 3 Section 7] the ring homomorphism $\phi : A \to B$ is faithfully flat iff it is flat, and the induced map of affine schemes

$$\operatorname{Spec}(\phi) : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$$

is surjective on the underlying sets.

Another fact that we need from commutative algebra is this: given ring homs $A \to B \to C$, if $A \to C$ flat and $B \to C$ is faithfully flat, then $A \to B$ is flat.

Lemma 5.55. *Let* $(X, \mathcal{O}_X)$ *be a scheme.*

(1) *Let* $V \subseteq U$ *be affine open sets in X. Then the ring homomorphism*

$$\Gamma(U, \mathcal{O}_X) \to \Gamma(V, \mathcal{O}_X)$$

is flat.

(2) *Let $U \subseteq X$ be an affine open set, and let $\{V_i\}_{i \in I}$ be a finite affine open covering of U. Then the ring homomorphism*

$$\Gamma(U, \mathcal{O}_X) \to \prod_{i \in I} \Gamma(V_i, \mathcal{O}_X)$$

is faithfully flat.

Proof.

comment: [(190515) The proof I gave in class was wrong. The assertion "or $\mathfrak{p} \notin V$, so $B \otimes_A A_{\mathfrak{p}} = 0$" is false. The new correct proof is a bit long – can it be simplified?]

The proof is done is steps.

Step 1. Let $A := \Gamma(U, \mathcal{O}_X)$ and $B := \Gamma(V, \mathcal{O}_X)$ as in item 1, and assume that V is a principal open set in U, namely $V = \mathrm{NZer}_U(s)$ for some $s \in A$. Then $B \cong A_s$ as A-rings, so it is flat.

Step 2. Here we are in the situation of item 2, $A := \Gamma(U, \mathcal{O}_X)$ and $B_i := \Gamma(V_i, \mathcal{O}_X)$. Let $B := \prod_i B_i$. Since $\mathrm{Spec}(B) = \coprod_i V_i$ and $U = \bigcup_i V_i$ we see that the map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective.

Step 3. Back to item 1: we can cover V by open sets $\{W_j\}_{j \in J}$, such that each W_j is a principal open set of U, i.e. $W_j = \mathrm{NZer}_U(s_j)$ for some $s_j \in A$. Because V is quasi-compact, we can assume that the indexing set J is finite. Note that W_j is also a principal open set in V, since $W_j = \mathrm{NZer}_V(s_j)$. Let $B := \Gamma(V, \mathcal{O}_X)$, $C_j := \Gamma(W_j, \mathcal{O}_X)$ and $C := \prod_{j \in J} C_j$. By step 1 we know that $A \to C_j$ and $B \to C_j$ are flat. So $A \to C$ and $B \to C$ are flat. By step 2 the homomorphism $B \to C$ is faithfully flat. It follows that $A \to B$ is flat. This establishes item 1.

Step 4. Now we prove item 2. Let $B_i := \Gamma(V_i, \mathcal{O}_X)$ and $B := \prod_i B_i$. By item (1) the ring homomorphism $A \to B$ is flat. Since $\mathrm{Spec}(B) = \coprod_i V_i$ and $U = \bigcup_i V_i$ we see that the map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective. □

Proposition 5.56. *Let $(X, \mathcal{O}_X)$ be a noetherian scheme. Then every affine open subscheme $(U, \mathcal{O}_X|_U)$ of $(X, \mathcal{O}_X)$ is a noetherian scheme.*

Proof. We need to prove that the ring $A := \Gamma(U, \mathcal{O}_X) = \Gamma(U, \mathcal{O}_X|_U)$ is noetherian.

Take a point $x \in U$. By Definition 5.52 there is an affine open set $V \subseteq X$ such that $x \in V$ and $B := \Gamma(V, \mathcal{O}_X)$ is a noetherian ring. There is an element $s \in B$ such that

$$x \in \mathrm{NZer}_V(s) = \mathrm{Spec}(B_s) \subseteq U.$$

The ring B_s is noetherian too. This shows that $U = \bigcup_{i \in I} V_i$, where each V_i is an affine open set, and $B_i := \Gamma(V_i, \mathcal{O}_X)$ is a noetherian ring. Because U is quasi-compact (see Proposition 5.49), we can assume that the indexing set I is finite.

Consider the ring homomorphism $\phi : A \to B$, where $B := \prod_{i \in I} B_i$. By Lemma 5.55, ϕ is faithfully flat. Suppose $\mathfrak{a}_0 \subsetneq \mathfrak{a}_1 \subsetneq \cdots$ is a strictly ascending chain of ideals in A. Then, letting $\mathfrak{b}_i := B \otimes_A \mathfrak{a}_i$, we get a strictly ascending chain of ideals $\mathfrak{b}_0 \subsetneq \mathfrak{b}_1 \subsetneq \cdots$ in B. This is where we need faithful flatness. But the ring B is noetherian, so this chain must be finite. □

Remark 5.57. Following Hartshorne [Har] we will often consider only noetherian schemes, in those cases where this assumption simplifies the discussion significanly. For instance, a finite type map $f : Y \to X$ between noetherian schemes (to be defined soon) is a very useful notion; but when the schemes are not noetherian, we have to look at *locally finitely presented maps*.

Recall that an *integral domain* is a nonzero ring A in which the only zero-divisor is 0. In other words, the zero ideal is a prime ideal of A.

Definition 5.58. A scheme $(X, \mathcal{O}_X)$ is called an *integral scheme* if it is nonempty, and if for every nonempty open set $U \subseteq X$, the ring $\Gamma(U, \mathcal{O}_X)$ is an integral domain.

Warning: The definitions of reduced and integral schemes in [Har, Sec II.3] are imprecise.

Proposition 5.59. *Let $(X, \mathcal{O}_X)$ be a scheme. The following two conditions are equivalent.*

(i) *$(X, \mathcal{O}_X)$ is an integral scheme.*
(ii) *$(X, \mathcal{O}_X)$ is reduced and irreducible.*

Proof.
(i) $\Rightarrow$ (ii): Assume Clearly integral implies reduced. If X is not an irreducible topological space, then there are closed strict subsets $X_1, X_2 \subsetneq X$ such that $X = X_1 \cup X_2$. Let $U_1 := X - X_2$ and $U_2 := X - X_1$, which are nonempty open subsets of X, and $U_1 \cap U_2 = \varnothing$. Define $U := U_1 \cup U_2$. We then have

$$\Gamma(U, \mathcal{O}_X) = \Gamma(U_1, \mathcal{O}_X) \times \Gamma(U_2, \mathcal{O}_X),$$

and this is not an integral domain.

comment: [(190515) The last part of the proof, below, wasn't done in class. Please read it at home.]

(ii) $\Rightarrow$ (i): Let $U \subseteq X$ be a nonempty open subset such that $\Gamma(U, \mathcal{O}_X)$ is not an integral domain. Then there exist nonzero elements $a, b \in \Gamma(U, \mathcal{O}_X)$ s.t. $a \cdot b = 0$. Define $Y := \mathrm{Zer}_U(a)$ and $Z := \mathrm{Zer}_U(b)$. These are closed subsets of U, by Prop 4.18.

We claim that $Y \subsetneq U$. Otherwise, if $\mathrm{Zer}_U(a) = U$, then for every nonempty affine open $V \subseteq U$ we would have $\mathrm{Zer}_V(a) = V$. Hence, taking $B := \Gamma(V, \mathcal{O}_X)$, we would have

$$\mathrm{Spec}(B_a) = \mathrm{NZer}_V(a) = \varnothing,$$

so $B_a = 0$. But the ring B is reduced, so a must be zero in B, i.e. $a|_V = 0$. This would imply $a = 0$, a contradiction.

For the same reasons we have $Z \subsetneq U$.

So we have strict closed subsets $Y, Z \subsetneq U$, and

$$Y \cup Z = \mathrm{Zer}_U(a) \cup \mathrm{Zer}_U(b) = \mathrm{Zer}_U(a \cdot b) = \mathrm{Zer}_U(0) = U.$$

Yet by Lem 5.47 the open set U is irreducible. We have a contradiction. □

Lecture 10, 22 May 2019

Please note that there was an error in my proof in class of Lem 5.55; the typed proof above is correct. I hope everybody read and understood the 2nd half of the proof of Prop 5.59.

Here is an abbreviation that we will use from now on (unless we need to be fully explicit): a scheme $(X, \mathcal{O}_X)$ will be be referred to as X; the structure sheaf $\mathcal{O}_X$ will be implicit.

We will start by improving upon Def 5.52. The subsequent prop will say that the two defs are equivalent.

Definition 5.60. A scheme X is called *noetherian* if it is quasi-compact, and for every affine open set $U \subseteq X$ the ring $A := \Gamma(U, \mathcal{O}_X)$ is noetherian.

Proposition 5.61. *TFAE for a scheme X :*

(i) *X is noetherian.*
(ii) *There is a finite affine open covering $X = \bigcup_{i \in I} U_i$, such that each of the rings $A_i := \Gamma(U_i, \mathcal{O}_X)$ is noetherian.*

Proof.

(i) ⇒ (ii): Since X is quasi-compact, there is a finite affine open covering $X = \bigcup_{i \in I} U_i$. By assumption the rings $A_i := \Gamma(U_i, \mathcal{O}_X)$ are noetherian.

(ii) ⇒ (i): Each affine open set U_i is quasi-compact, and a finite union of quasi-compacts is quasi-compact. Thus X is quasi-compact.

Take some affine open set $U \subseteq X$. We need to prove that the ring $A := \Gamma(U, \mathcal{O}_X)$ is noetherian. We can find a finite affine open covering $U = \bigcup_{j \in J} V_j$, where $V_j := \mathrm{NZer}_{U_i}(s_j)$ for some $i \in I$ and some $s_j \in \Gamma(U_i, \mathcal{O}_X)$. Now $B_j := \Gamma(V_j, \mathcal{O}_X) \cong (A_i)_{s_j}$ is noetherian. Let $B := \prod_{j \in J} B_j$, which is noetherian. The ring homomorphism $A \to B$ is faithfully flat. Hence, by faithfully flat descent (the argument used in the proof of Prop 5.56) we see that A is noetherian. □

The fact that a closed subscheme of an affine scheme is affine is something that I never understood well, and it always intrigued me. In Rem 5.27 I said that we need the concept of quasi-coherent sheaf to prove this result; but today I'll give a more direct proof; see Thm 5.66 below. It uses the criterion for affineness Thm 5.62, that appears as [Har, Exer II.2.17]. But I think it deserves to be a theorem.
comment: [(190522) Thm 5.66 will be next week...]

A sequence $(s_1, \ldots, s_n)$ of elements of a ring A is called a *covering sequence* if $A = \sum_i A \cdot s_i$. In other words, if

$$\mathrm{Spec}(A) = \bigcup_{i \in [1,n]} \mathrm{NZer}_{\mathrm{Spec}(A)}(s_i) = \bigcup_{i \in [1,n]} \mathrm{Spec}(A_{s_i}).$$

Theorem 5.62 (Criterion for Affineness)**.** *Let X be a scheme and $A := \Gamma(X, \mathcal{O}_X)$. Assume there is a covering sequence $(s_1, \ldots, s_n)$ of A such that for every i the open set $\mathrm{NZer}_X(s_i)$ is affine. Then X is an affine scheme.*

comment: [(190520) is there a reference for this thm? Maybe in [EGA] or in [SP]?]

We first need a lemma (it is [Har, Exer II.2.16]).

Lemma 5.63. *Let X be a scheme, with a finite affine open covering $\{U_i\}_{i \in I}$, such that every intersection $U_i \cap U_j$ is quasi-compact. Let $A := \Gamma(X, \mathcal{O}_X)$ and $s \in A$. Then*

$$\Gamma\big(\mathrm{NZer}_X(s), \mathcal{O}_X\big) \cong A_s$$

as A-rings.

Note that such an A-ring isomorphism is unique (if it exists).

Proof.
comment: [(190520) draw picture?]

For each pair of indices $i, j \in I$ let us choose a finite affine open covering

$$U_i \cap U_j = \bigcup_{k \in K_{i,j}} V_k.$$

Let $K := \coprod_{i,j} K_{i,j}$, $A_i := \Gamma(U_i, \mathcal{O}_X)$ and $B_k := \Gamma(V_k, \mathcal{O}_X)$. Note that

$$\Gamma(U_i \cap U_j, \mathcal{O}_X) \subseteq \prod_{k \in K_{i,j}} \Gamma(V_k, \mathcal{O}_X).$$

Hence, by the sheaf property, we have an exact sequence of A-modules

$$0 \to A \to \prod_{i \in I} A_i \to \prod_{k \in K} B_k. \tag{5.64}$$

These products are finite, so they are direct sums. Applying $A_s \otimes_A (-)$ to (5.64) we get the exact sequence of A_s-modules

$$0 \to A_s \to \prod_{i \in I} (A_s \otimes_A A_i) \to \prod_{k \in K} (A_s \otimes_A B_k). \tag{5.65}$$

Let's write $W := \mathrm{NZer}_X(s)$. Then $W \cap U_i = \mathrm{NZer}_{U_i}(s)$ and $W \cap V_k = \mathrm{NZer}_{V_k}(s)$ are affine open sets. We know that

$$\Gamma(W \cap U_i, \mathcal{O}_W) = \Gamma\big(\mathrm{NZer}_{U_i}(s), \mathcal{O}_{U_i}\big) = A_s \otimes_A A_i$$

and

$$\Gamma(W \cap V_k, \mathcal{O}_W) = \Gamma\big(\mathrm{NZer}_{V_k}(s), \mathcal{O}_{V_k}\big) = A_s \otimes_A B_k.$$

Also

$$W = \bigcup_{i \in I} (W \cap U_i)$$

and

$$(W \cap U_i) \cap (W \cap U_j) = \bigcup_{k \in K_{i,j}} (W \cap V_k).$$

Therefore, by the sheaf property of $\mathcal{O}_W$, the exact sequence (5.65) gives $\Gamma(W, \mathcal{O}_X) = A_s$. □

Proof of the theorem. We shall prove that the canonical map of schemes

$$f : X \to X' := \mathrm{Spec}(A)$$

is an isomorphism.

Let $U_i := \mathrm{NZer}_X(s_i)$, which by assumption is an affine scheme. For $i, j \in [1, n]$ we have

$$U_i \cap U_j = \mathrm{NZer}_X(s_i \cdot s_j) = \mathrm{NZer}_{U_i}(s_j),$$

which is affine and thus quasi-compact. This means that we can use Lemma 5.63, and it says $\Gamma(U_i, \mathcal{O}_X) = A_{s_i}$. Therefore the map of affine schemes

$$f|_{U_i} : U_i \to U'_i := \mathrm{Spec}(A_{s_i})$$

is an isomorphism. Let $g_i : U'_i \xrightarrow{\simeq} U_i$ be the inverse of $f|_{U_i}$. By the uniqueness of inverses we have $g_i|_{U'_i \cap U'_j} = g_j|_{U'_i \cap U'_j}$. Therefore, by Thm 1.22(gluing of maps), there is a map

$$g : X = \bigcup_{i \in I} U_i \to X' = \bigcup_{i \in I} U'_i$$

s.t. $g|_{U'_i} = g_i$. The map g is an inverse of f, so f is an isomorphism. □

Lecture 11, 29 May 2019

Theorem 5.66. *Let X be an affine scheme and $Y \subseteq X$ a closed subscheme. Then Y is affine.*

Proof.
comment: [(190520) draw picture?]

Let $A := \Gamma(X, \mathcal{O}_X)$ and $B := \Gamma(Y, \mathcal{O}_Y)$. So $X = \mathrm{Spec}(A)$; and eventually we will see that $Y = \mathrm{Spec}(B)$.

Take a point $y \in Y$. Let V' be some affine open neighborhood of y in Y. Since Y is a topological subspace of X, there is an open set $U' \subseteq X$ s.t. $V' = Y \cap U'$. Let $s \in A$ be s.t. $y \in \mathrm{NZer}_X(s) \subseteq U'$. Define $U := \mathrm{NZer}_X(s)$ and

$$V := Y \cap U = \mathrm{NZer}_Y(s) = \mathrm{NZer}_{V'}(s).$$

Then $U \subseteq X$ and $V \subseteq Y$ are open affine, and $y \in V \subseteq U$. Performing this for all points $y \in Y$ we get a collection $\{s_i\}_{i \in I}$ of elements of A, with indexing set $I := Y$, such that, letting $U_i := \mathrm{NZer}_X(s_i)$ and $V_i := \mathrm{NZer}_Y(s_i)$, each $U_i \subseteq X$ and $V_i \subseteq Y$ are affine open, and

$$Y = \bigcup_{i \in I} V_i \subseteq \bigcup_{i \in I} U_i. \tag{5.67}$$

Next consider the open set $X - Y$. There is a collection $\{s_j\}_{j \in J}$ of elements of A, indexed by a set J disjoint from I, such that, letting $U_j := \mathrm{NZer}_X(s_j)$, we have

$$X - Y = \bigcup_{j \in J} U_j. \tag{5.68}$$

Define the indexing set $K := I \sqcup J$. Then from (5.67) and (5.68) we obtain

$$X = Y \cup (X - Y) = \Big(\bigcup\nolimits_{i \in I} U_i\Big) \cup \Big(\bigcup\nolimits_{j \in J} U_j\Big) = \bigcup\nolimits_{k \in K} U_k. \tag{5.69}$$

The scheme X is quasi-compact, and therefore we can replace K by a finite subset K'. Let $I' := I \cap K'$ and $J' := J \cap K'$, so $K' = I' \sqcup J'$.

By definition (both for $k \in I'$ and for $k \in J'$) we have $U_k = \mathrm{NZer}_X(s_k)$. Thus

$$\mathrm{Spec}(A) = X = \bigcup\nolimits_{k \in K'} \mathrm{NZer}_X(s_k).$$

It follows that $\boldsymbol{s} := \{s_k\}_{k \in K'}$ is a covering sequence of A. Let $\bar{s}_k := s|_Y \in B = \Gamma(Y, \mathcal{O}_Y)$. Then $\bar{\boldsymbol{s}} := \{\bar{s}_k\}_{k \in K'}$ is a covering sequence of B. The reason is this: if $1_A = \sum_{k \in K'} a_k \cdot s_k$ for some elements $a_k \in A$, then $1_B = \sum_{k \in K'} \bar{a}_k \cdot \bar{s}_k$.

We claim that for every index $k \in K'$ the open set $\mathrm{NZer}_Y(\bar{s}_k) \subseteq Y$ is affine. Indeed, if $k \in I'$ then $\mathrm{NZer}_Y(\bar{s}_k)$ is the affine open set V_k from the second paragraph. And if $k \in J'$ then $\mathrm{NZer}_Y(\bar{s}_k) = \varnothing$. Now we can apply Thm 5.62 to deduce that Y is an affine scheme. □

Remark 5.70. In the situation of the theorem, let $A := \Gamma(X, \mathcal{O}_X)$ and $B := \Gamma(Y, \mathcal{O}_Y)$. We will prove later that the ring homomorphism $A \to B$ is surjective. Moreover, let $I := \mathrm{Ker}(A \to B)$, and let $\mathcal{I} := \mathrm{Ker}(\mathcal{O}_X \twoheadrightarrow \mathcal{O}_Y)$ be the ideal sheaf from Prop 5.22. Then $I = \Gamma(X, \mathcal{I})$. Thus closed subschemes of X correspond bijectively to ideals of B. The proof requires the use of quasi-coherent sheaves.

Exercise 5.71. Try to prove that the ring homomorphism $A \to B$ in the remark above is surjective, using only the methods we have now. I don't know if that's possible...

There are a few more properties of schemes that I want to mention now.

Definition 5.72. Let X be a scheme. A *subscheme* of X is a scheme Y that is a closed subscheme of an open subscheme of X.

Example 5.73. Let $\mathbb{K}$ be a ring and let X be a subscheme of the scheme $\mathbf{P}^n_{\mathbb{K}}$ for some $n \geq 0$. (We still did not define $\mathbf{P}^n_{\mathbb{K}}$ for $n \geq 2$, but will do so very soon.) Then X is called a *quasi-projective* $\mathbb{K}$*-scheme*.

A *finite chain of subsets* in a topological space X is a sequence

$$Z_0 \subsetneq Z_1 \subsetneq Z_2 \subsetneq \cdots \subsetneq Z_n$$

of subsets of X (with strict inclusions). The *length* of this chain is n.

Definition 5.74. Let X be a nonempty topological space. The *dimension of* X is the supremum of the lengths of finite chains of irreducible closed subsets of X. It is denoted by $\dim(X)$.

Exercise 5.75. Let X be a nonempty Hausdorff topological space. What is its dimension, in the sense of Def 5.74? (Hint: Use Lem 5.47.)

Definition 5.76. Let $(X, \mathcal{O}_X)$ be a nonempty scheme. The *dimension of* $(X, \mathcal{O}_X)$ is the dimension of the underlying topological space X, the sense of Def 5.74.

Exercise 5.77. Let $(X, \mathcal{O}_X)$ be a scheme. Prove that:

(1) If $(Y, \mathcal{O}_Y)$ a subscheme of $(X, \mathcal{O}_X)$, then $\dim(Y) \leq \dim(X)$.
(2) If $(X, \mathcal{O}_X)$ is a finite union of closed subschemes $(Y_i, \mathcal{O}_{Y_i})$, namely $X = \bigcup_{i \in I} Y_i$ as sets, then $\dim(X) = \sup \{\dim(Y_i)\}_{i \in I}$. Find a counterexample when the subschemes are not all closed.
(3) If $(X, \mathcal{O}_X) = \operatorname{Spec}(A)$, then $\dim(X) = \dim(A)$, where for a ring A we use *Krull dimension*.
(4) Compute the dimension of the scheme $\mathbf{A}^n_{\mathbb{K}}$, when $\mathbb{K}$ is a field. (If $\mathbb{K}$ is a noetherian ring of dimension m, then $\dim(\mathbf{A}^n_{\mathbb{K}}) = m + n$. This is harder to prove, See [Mat].)
(5) Compute the dimension of the scheme $\mathbf{P}^n_{\mathbb{K}}$, when $\mathbb{K}$ is a field. Use the fact that $\mathbf{P}^n_{\mathbb{K}}$ is irreducible, there is a closed embedding $\mathbf{P}^{n-1}_{\mathbb{K}} \rightarrowtail \mathbf{P}^n_{\mathbb{K}}$, an open embedding $\mathbf{A}^n_{\mathbb{K}} \rightarrowtail \mathbf{P}^n_{\mathbb{K}}$, s.t. $\mathbf{P}^n_{\mathbb{K}} = \mathbf{A}^n_{\mathbb{K}} \sqcup \mathbf{P}^{n-1}_{\mathbb{K}}$ as sets.

Definition 5.78. A noetherian scheme $(X, \mathcal{O}_X)$ is called *regular* if for every point $x \in X$ the local ring $\mathcal{O}_{X,x}$ is a regular local ring, in the sense of [Mat].

Proposition 5.79. *Let $(X, \mathcal{O}_X)$ be a noetherian scheme. TFAE:*

(i) *$(X, \mathcal{O}_X)$ is a regular scheme.*
(ii) *For every affine open set $U \subseteq X$ the ring $A := \Gamma(U, \mathcal{O}_X)$ is regular.*
(iii) *There is an affine open covering $X = \bigcup_{i \in I} U_i$ s.t. each $A_i := \Gamma(U_i, \mathcal{O}_X)$ is a regular ring.*

We leave out the easy proof.

Example 5.80. Let $\mathbb{K}$ be a finite dimensional regular noetherian ring. Then the scheme $(X, \mathcal{O}_X) := \mathbf{A}^n_{\mathbb{K}}$ is regular. This is because the polynomial ring $\mathbb{K}[t_1, \ldots, t_n]$ is regular, see [Mat].

6. Properties of Maps of Schemes

Recall that a map of schemes

$$(f, \phi) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X) \tag{6.1}$$

is a map of locally ringed spaces between schemes. I.e. Sch is a full subcategory of LRSp. There is a base ring $\mathbb{K}$, so we are really working in the category $\mathsf{Sch}/\mathbb{K}$. Bbut most of the time $\mathbb{K}$ won't be relevant, so we will suppress it from the notation.

Also a map of schemes (6.1) will be written as

$$f : Y \to X, \tag{6.2}$$

and when we need to mention the sheaf homomorphism component of f it will be

$$f^* : \mathcal{O}_X \to f_*(\mathcal{O}_Y). \tag{6.3}$$

Here is a relative notion of "affine".

Definition 6.4. A map of schemes $f : Y \to X$ is called an *affine map* if for every affine open set $U \subseteq X$, its preimage $V := f^{-1}(U) \subseteq Y$ is an affine open set.

Of course an affine map is quasi-compact.

Example 6.5. A closed embedding $Y \rightarrowtail X$ is an affine map. Indeed, given an affine open set $U \subseteq X$, the preimage $V := Y \cap U$ is an closed subscheme of U, and hence, by Thm 5.66, V is affine.

Example 6.6. The projection $f : \mathbf{A}^n_X \to X$ is an affine map. Indeed, for any affine open subset $U = \operatorname{Spec}(A) \subseteq X$ we have $f^{-1}(U) \cong \operatorname{Spec}(A[t_1, \ldots, t_n])$.

Exercise 6.7. Find an example of an open embedding which is affine, and another open embedding which is not affine.

As usual, the property of being an affine map is local on the target:

Theorem 6.8. *Let $f : Y \to X$ be a map of schemes. Assume there is an affine open covering $X = \bigcup_{i \in I} U_i$, s.t. for every i the preimage $V_i := f^{-1}(U_i) \subseteq Y$ is an affine open set. Then f is an affine map.*

We need a lemma first.

Lemma 6.9. *Let X be a scheme and $U, V \subseteq X$ affine open sets. Let $x \in U \cap V$ be some point. Then there are elements $s \in \Gamma(U, \mathcal{O}_X)$ and $t \in \Gamma(V, \mathcal{O}_X)$ such that*

$$x \in \operatorname{NZer}_U(s) = \operatorname{NZer}_V(t) \subseteq U \cap V.$$

Proof. Choose $s_0 \in \Gamma(U, \mathcal{O}_X)$ s.t. $U_0 := \operatorname{NZer}_U(s_0)$ satisfies $x \in U_0 \subseteq U \cap V$. Next choose $t \in \Gamma(V, \mathcal{O}_X)$ such that $x \in \operatorname{NZer}_V(t) \subseteq U_0$. But now $t|_{U_0} \in \Gamma(U_0, \mathcal{O}_X)$, and $\operatorname{NZer}_V(t) = \operatorname{NZer}_{U_0}(t|_{U_0})$. Since $\Gamma(U_0, \mathcal{O}_X) = \Gamma(U, \mathcal{O}_X)_{s_0}$, we can express $t|_{U_0}$ as a fraction $t|_{U_0} = a \cdot s_0^m$ for some $a \in \Gamma(U, \mathcal{O}_X)$ and $m \geq 1$. Define $s := a \cdot s_0 \in \Gamma(U, \mathcal{O}_X)$. Then a quick calculation shows that $\operatorname{NZer}_V(t) = \operatorname{NZer}_U(s)$. □

Proof of Thm 6.8. Given an affine open set $U \subseteq X$, we need to prove that $V := f^{-1}(U) \subseteq Y$ is an affine open set.

Take a point $x \in U$. Let $i_x \in I$ be an index such that $x \in U_{i_x}$. By Lem 6.9 there is an open set $U_x \subseteq X$ such that $x \in U_x \subseteq U \cap U_{i_x}$, and

$$U_x = \operatorname{NZer}_U(s_x) = \operatorname{NZer}_{U_{i_x}}(t_x)$$

for some $s_x \in \Gamma(U, \mathcal{O}_X)$ and $t_x \in \Gamma(U_{i_x}, \mathcal{O}_X)$. Now $V = f^{-1}(U)$ and $V_{i_x} = f^{-1}(U_{i_x})$, so

$$V_x := f^{-1}(U_x) = \operatorname{NZer}_V(f^*(s_x)) = \operatorname{NZer}_{V_{i_x}}(f^*(t_x)). \tag{6.10}$$

Since V_{i_x} is affine, and V_x is a principal open set in it, we see that V_x is affine. See Figure 8.

We have an open covering $U = \bigcup_{x \in U} U_x$. Because U is quasi-compact, we can pass to a finite subcovering. Let's change notation: we take a finite set J, and a function $J \to U$, $j \mapsto x_j$; then we write $U_j := U_{x_j}$, $s_j := s_{x_j}$ and $V_j := V_{x_j}$. So $U = \bigcup_{j \in J} U_j$, $U_j = \operatorname{NZer}_U(s_j)$ and $V_j = f^{-1}(U_j) = \operatorname{NZer}_V(f^*(s_j))$. The finite collection $\{s_j\}_{j \in J}$ is a covering collection of $A := \Gamma(U, \mathcal{O}_X)$. Therefore the collection $\{f^*(s_j)\}_{j \in J}$ is a covering collection of $B := \Gamma(V, \mathcal{O}_Y)$ – we saw this in the proof of Thm 5.66. Since every open set $\operatorname{NZer}_V(f^*(s_j)) = V_j \subseteq V$ is affine, Thm 5.62 tells us that V is affine. □

Lecture 12, 5 June 2019

We have three lectures left (today included). The plan for them is this:

- 5 June: Maps of schemes (continued): base change, fibers, quasi-compact, finite type, finite. (Some proofs)
- 12 June: Separated and proper maps. Explicit examples: affine maps, embeddings, $\mathbf{P}^n_X$. Valuative criteria. (No proofs).
- 19 June: Quasi-coherent and coherent sheaves. Serre's Theorem. Sheaf cohomology. (No proofs).

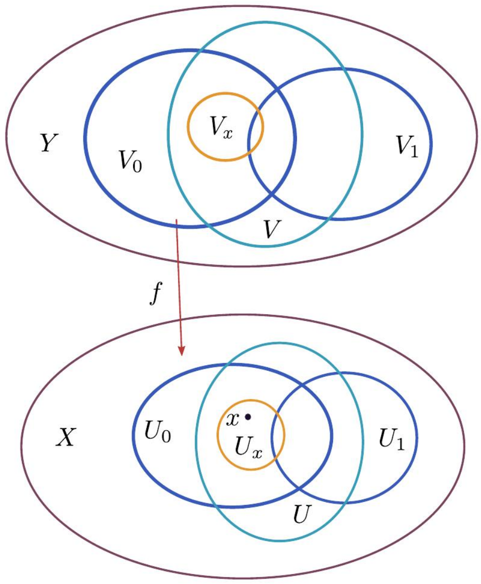


FIGURE 8.

Let X be a scheme and $x \in X$ a point. Choose some affine open neighborhood U of x in X, and let $A := \Gamma(U, \mathcal{O}_X)$. The the stalk $\mathcal{O}_{X,x}$ is canonically isomorphic to the local ring A_x. Thus there are ring homomorphisms

$$\Gamma(U, \mathcal{O}_X) \to \mathcal{O}_{X,x} \to \boldsymbol{k}(x).$$

Passing to schemes we get maps

$$\operatorname{Spec}(\boldsymbol{k}(x)) \to \operatorname{Spec}(\mathcal{O}_{X,x}) \to U \to X.$$

The resulting map

$$\operatorname{Spec}(\boldsymbol{k}(x)) \to X \tag{6.11}$$

is independent of the affine open set U.

Definition 6.12. Let $f : Y \to X$ be a map of schemes, and let $x \in X$ be a point. The *fiber of f above x* is the $\boldsymbol{k}(x)$-scheme

$$Y(x) := \operatorname{Spec}(\boldsymbol{k}(x)) \times_X Y,$$

the fibered product w.r.t. the maps (6.11) and f.

Proposition 6.13. *Let $f : Y \to X$ be a map of schemes, and let $x \in X$ be a point. Then the underlying topological space of the fiber $Y(x)$ is canonically homeomorphic to the topological subspace $f^{-1}(x) \subseteq Y$.*

Exercise 6.14. Prove this Prop. (Hint: replace X by an affine open neighborhood of x. See also Thm 1.13 of `https://www.math.bgu.ac.il/~amyekut/teaching/2017-18/comm-alg/final.pdf`.)

The next two propositions are similar to the previous one.

Proposition 6.15. *Let $f : X' \to X$ be a map of schemes, and let $U \subseteq X$ be an open subscheme. Consider the fibered product $U' := U \times_X X'$, with its projections $\mathrm{pr}_U : U' \to U$*

and $\mathrm{pr}_{X'} : U' \to X'$.

$$\begin{CD} U' @>{\mathrm{pr}_{X'}}>> X' \\ @V{\mathrm{pr}_U}VV @VV{f}V \\ U @>>> X \end{CD}$$

Then $\mathrm{pr}_{X'}$ *is an open embedding, and it is (isomorphic to) the open subscheme of* X' *on the open set* $f^{-1}(U) \subseteq X'$.

Exercise 6.16. Prove this Prop. (Hint: Examine the proofs of Thms 4.9 and 4.4.)

Proposition 6.17. *Let* $f : X' \to X$ *be a map of schemes, and let* $Z \subseteq X$ *be a closed subscheme. Consider the fibered product* $Z' := Z \times_X X'$, *with its projections* $\mathrm{pr}_Z : Z' \to Z$ *and* $\mathrm{pr}_{X'} : Z' \to X'$.

$$\begin{CD} Z' @>{\mathrm{pr}_{X'}}>> X' \\ @V{\mathrm{pr}_Z}VV @VV{f}V \\ Z @>>> X \end{CD}$$

Then $\mathrm{pr}_{X'}$ *is a closed embedding. The underlying topological space of* Z' *is (isomorphic to) the closed subset* $f^{-1}(Z) \subseteq X'$.

Exercise 6.18. Prove this prop. (Hint: First consider the affine case – i.e. the schemes X, Z and X' are affine. Then proceed like the proof of Prop 6.15.)

The next definition, a bit differently stated, is [Har, Exer II.3.2]. I want to upgrade it to a definition, and to improve it, following [EGA]. It is the relative version of the property of quasi-compact space.

Definition 6.19. A map of schemes $f : Y \to X$ is called *quasi-compact* if for every quasi-compact open set $U \subseteq X$, the preimage $f^{-1}(U) \subseteq Y$ is quasi-compact.

Lemma 6.20. *Let* X *be a quasi-compact scheme and* $s \in \Gamma(X, \mathcal{O}_X)$. *Then the open set* $\mathrm{NZer}_X(s)$ *is quasi-compact.*

Proof. Let $X = \bigcup_{i \in I} U_i$ be an affine open covering. Because X is quasi-compact, we can assume that the indexing set I is finite. For every i the open set $\mathrm{NZer}_{U_i}(s) \subseteq U_i$ is affine, and hence quasi-compact. Thus $\mathrm{NZer}_X(s) = \bigcup_{i \in I} \mathrm{NZer}_{U_i}(s)$ is quasi-compact. □

Proposition 6.21. *Let* $f : Y \to X$ *is a map of schemes with this property: there is a covering* $X = \bigcup_{i \in I} U_i$ *by affine open sets, such that each subset* $f^{-1}(U_i) \subseteq Y$ *is quasi-compact. Then* f *is quasi-compact.*

Exercise 6.22. Prove this Prop. (Hint: Use Lem 6.20, with the fact that for $s \in \Gamma(X, \mathcal{O}_X)$ we have $f^{-1}(\mathrm{NZer}_X(s)) = \mathrm{NZer}_Y(f^*(s))$. See also [SP, Lemma tag=01K4].)

Remark 6.23. If Y is a noetherian scheme than any map $f : Y \to X$ is quasi-compact. This follows easily from Prop 5.49. Perhaps that is the reason Hartshorne chose not to emphasize quasi-compactness.

An A-ring B is said to be a finite type A-ring, or a finitely generated A-ring, if B is a quotient of a polynomial ring $A[t_1, \ldots, t_n]$ for some $n \in \mathbb{N}$.

Definition 6.24. A map of schemes $f : Y \to X$ is called a *finite type map* if it is quasi-compact, and for every affine open set $U \subseteq X$ and every affine open set $V \subseteq f^{-1}(U)$, the ring homomorphism $f^* : \Gamma(U, \mathcal{O}_X) \to \Gamma(V, \mathcal{O}_Y)$ is finite type.

As often happens, the property of being a finite type map can be checked on a single covering:

Theorem 6.25. *Let $f : Y \to X$ be a map of schemes. Assume there is an affine open covering $X = \bigcup_{i \in I} U_i$, and for every i there is a finite affine open covering $f^{-1}(U_i) = \bigcup_{j \in J_i} V_j$, such that $\Gamma(V_j, \mathcal{O}_Y)$ is a finite type $\Gamma(U_i, \mathcal{O}_X)$-ring. Then f is of finite type.*

In the statement of the thm, we tacitly assume that the indexing sets J_i are disjoint; so there is a collection $\{V_j\}_{j \in J}$ of open subsets of Y, where $J := \coprod_i J_i$.

First a lemma.

Lemma 6.26. *Let $f : V \to U$ be a map of affine schemes, and let $V = \bigcup_{j \in J} V_j$ be a finite open covering, such that $V_j = \mathrm{NZer}_V(t_j)$ for some $t_j \in \Gamma(V, \mathcal{O}_V)$. Write $A := \Gamma(U, \mathcal{O}_U)$, $B := \Gamma(V, \mathcal{O}_V)$ and $B_j := \Gamma(V_j, \mathcal{O}_V)$. Assume that the ring homomorphisms $f^* : A \to B_j$ are finite type. Then the ring homomorphism $f^* : A \to B$ is finite type.*

Proof. The collection of elements $\{t_j\}_{j \in J}$ is a covering sequence of B, so there is a collection of elements $\{b_j\}_{j \in J}$ in B satisfying $1_B = \sum_j b_j \cdot t_j$. For each j let's choose a finite collection of elements $\{c_k\}_{k \in K_j}$ in B_j that generate $B_j = B_{t_j}$ as an A-ring. We can express these elements are fractions: $c_k = d_k \cdot t_j^{-m_k}$ for some $d_k \in B$ and $m_k \in \mathbb{N}$. Let B' be the A-subring of B generated by the finite collections of elements $\{t_j\}_{j \in J}$, $\{b_j\}_{j \in J}$ and $\{d_k\}_{k \in \coprod_j K_j}$.

We claim that $B' = B$. The proof is by faithful flatness. The collection of elements $\{t_j\}_{j \in J}$ is a covering sequence of B', since $1_{B'} = \sum_j b_j \cdot t_j$. Therefore the ring homomorphism $B' \to D'$, where $D' := \prod_{j \in J} B'_{t_j}$, is faithfully flat. It follows that the inclusion of B'-modules $B' \to B$ is bijective iff the inclusion of D'-modules

$$D' = D' \otimes_{B'} B' \rightarrowtail D' \otimes_{B'} B$$

is bijective. In each component j we need to look at the inclusion

$$B'_{t_j} \rightarrowtail B'_{t_j} \otimes_{B'} B = B_{t_j} = B_j .$$

But inside B'_{t_j} we have the fractions $c_k = d_k \cdot t_j^{-m_k}$, and these generate B_j as an A-ring, so $B'_{t_j} = B_j$. □

Proof of Thm 6.25.
comment: [(190609) small corrections to proof]

Proving that f is quasi-compact is quite easy, using Prop 6.21: for every i the set $f^{-1}(U_i)$ is a finite union of the quasi-compact sets V_j, $j \in J_i$, so it is quasi-compact.

Now let U and V be as in Def 6.24, and define $A := \Gamma(U, \mathcal{O}_X)$ and $B := \Gamma(V, \mathcal{O}_Y)$. We must prove that $f^* : A \to B$ is finite type. Let's write $A_i := \Gamma(U_i, \mathcal{O}_X)$ and $B_j := \Gamma(V_j, \mathcal{O}_Y)$.

Take a point $y \in V$ and let $x := f(y) \in U$. According to Lem 6.9 we can find an index i, an element $s \in A_i$ and an element $t \in A$ s.t.

$$x \in \mathrm{NZer}_{U_i}(s) = \mathrm{NZer}_U(t) \subseteq U_i \cap U. \tag{6.27}$$

Let write $U_x := \mathrm{NZer}_U(t)$, so

$$\Gamma(U_x, \mathcal{O}_X) \cong A_t \cong (A_i)_s \tag{6.28}$$

as A-rings.

Let $j \in J_i$ we an index such that $y \in V_j$. From (6.27) we see that $y \in V_y$, where we define $V_y := \mathrm{NZer}_{V_j}(f^*(s))$, so that $\Gamma(V_y, \mathcal{O}_U) = (B_j)_{f^*(s)}$. It is given that the ring homomorphism $A_i \to B_j$ is finite type, and hence $(A_i)_s \to (B_j)_{f^*(s)}$ is finite type. By formula (6.28) we see that $A_t \to (B_j)_{f^*(s)}$ is finite type; and hence $A \to (B_j)_{f^*(s)}$ is finite type. I.e. $A \to \Gamma(V_y, \mathcal{O}_V)$ is finite type.

Because V is quasi-compact, the open covering $\{V_y\}_{y \in V}$ has a finite subcovering, say $\{V_y\}_{y \in V_0}$ for a finite subset $V_0 \subseteq V$. In the previous paragraph we saw that each ring homomorphism $A \to \Gamma(V_y, \mathcal{O}_V)$ is finite type. Now we use Lem 6.26 to conclude that $A \to B$ is finite type. □

Remark 6.29. If we replace "finite type" by "essentially finite type", the prop becomes an open problem.

Example 6.30. A closed embedding $f : Y \to X$ is a finite type map. It is even finite. See Exa 6.41 below.

Example 6.31. The projection $f : \mathbf{A}^n_X \to X$ is a finite type map. Indeed, for any affine open subset $U = \operatorname{Spec}(A) \subseteq X$ we have $f^{-1}(U) \cong \operatorname{Spec}(A[t_1, \ldots, t_n])$, so the condition of Thm 6.25 is satisfied.

Example 6.32. A quasi-compact open embedding $f : Y \to X$ is a finite type map. Here is why: Take an affine open set $U = \operatorname{Spec}(A) \subseteq X$. We can cover $Y \cap U$ with affine open sets $V_i = \operatorname{NZer}_U(s_i) = \operatorname{Spec}(A_{s_i})$, $i \in I$. And $A_{s_i} = A[s_i^{-1}]$ is a finite type A-ring. Since f is quasi-compact, it follows that the set $Y \cap U = f^{-1}(U)$ is quasi-compact, so we can pass to a finite subset of I.

Lecture 13, 12 June 2019

We still need to say a few words on finiteness of maps of schemes. I will not present most of the proofs in class.

Noetherian schemes were defined in Def 5.60.

Lemma 6.33. *Let $X = \operatorname{Spec}(A)$ be an affine scheme. The scheme X is noetherian iff the ring A is noetherian.*

Proof. This is clear from Prop 5.61. □

Proposition 6.34. *Let $f : Y \to X$ be a finite type map of schemes. If X is a noetherian scheme then Y is a noetherian scheme.*

Proof. The scheme X is quasi-compact, so we can assume the indexing set I appearing in Thm 6.25 is finite. According to Prop 5.56, for every i the scheme $(U_i, \mathcal{O}_X|_{U_i})$ is noetherian. By the lemma above the ring $A_i := \Gamma(U_i, \mathcal{O}_X)$ is noetherian. By the Hilbert Basis Thm the ring $B_j := \Gamma(V_j, \mathcal{O}_Y)$ is noetherian for every $j \in J_i$. And $Y = \bigcup_{i \in I} \bigcup_{j \in J_i} V_j$ is a finite covering. So according to Prop 5.61 the scheme Y is noetherian. □

An A-ring B is said to be a *finite A-ring*, and the ring homomorphism $A \to B$ is to be a *finite ring homomorphism*, if B is finitely generated as an A-module.

Definition 6.35. A map of schemes $f : Y \to X$ is called a *finite map* if for every affine open set $U \subseteq X$ its preimage $V := f^{-1}(U)$ is an affine open subset of Y, and the ring homomorphism $f^* : \Gamma(U, \mathcal{O}_X) \to \Gamma(V, \mathcal{O}_X)$ is finite.

Proposition 6.36. *If $f : Y \to X$ is a finite map of schemes, then it is finite type.*

Exercise 6.37. Prove this prop.

Theorem 6.38. *Let $f : Y \to X$ be a map of schemes. Assume there is an affine open covering $X = \bigcup_{i \in I} U_i$, such that each $V_i := f^{-1}(U_i)$ is an affine open subset of Y, and the ring homomorphism $f^* : \Gamma(U_i, \mathcal{O}_X) \to \Gamma(V_i, \mathcal{O}_X)$ is finite. The f is a finite map.*

Proof. According to Thm 6.8 the map f is affine. By Thm 6.25 and Prop 6.36, for every affine open $U \subseteq X$, with $V := f^{-1}(U) \subseteq Y$, $A := \Gamma(U, \mathcal{O}_X)$ and $B := \Gamma(V, \mathcal{O}_X)$, the ring homomorphism $f^* : A \to B$ is finite type. On the other hand, the elements of B are all integral over A (i.e. every $b \in B$ satisfies a monic polynomial over A) – see exercise below. This implies that $A \to B$ is actually finite. □

Exercise 6.39. Prove that B is an integral A-ring, in the proof of the theorem above.

Remark 6.40. The similarity between the names "finite A-ring" and "finite type A-ring" is very confusing. The same for scheme maps. This confusion goes back at least to [EGA], and persists in [SP]. Some texts use the expression "module-finite" for a finite ring homomorphism. I am not sure this is optimal. Maybe we could invent better terminology?

Example 6.41. A closed embedding $f : Y \to X$ is a finite map. We already know that for every affine open $U \subseteq X$ its preimage $V := f^{-1}(U) = Y \cap U \subseteq Y$ is affine (by Thm 5.66). To prove that the ring homomorphism $f^* : \Gamma(U, \mathcal{O}_X) \to \Gamma(V, \mathcal{O}_Y)$ is surjective we need more. One way to do it is using quasi-coherent sheaves. Another is by the next thm.

Here is an extension of Thm 5.66. This is [Har, Exer II.3.11(b)*].

Theorem 6.42. *Let $X = \operatorname{Spec}(A)$ be an affine scheme, let $Y \subseteq X$ be a closed subscheme, and let $B := \Gamma(Y, \mathcal{O}_Y)$. Then:*

(1) *Y is an affine scheme.*
(2) *The ring homomorphism $A \to B$ is surjective.*

Proof.

(1) This is Thm 5.66.

(2) Sketch only: Consider the ideal $\mathfrak{a} := \operatorname{Ker}(A \to B)$, the ring $B' := A/\mathfrak{a}$ and the closed subscheme $Y' := \operatorname{Spec}(B') \subseteq X$, cf. Prop 5.25. We already know $Y = \operatorname{Spec}(B)$. The A-ring homomorphism $B' \to B$ gives rise to a map $Y \to Y'$ of closed subschemes of X. A calculation shows that $Y = Y'$ as closed subsets of X. Since $B' \to B$ is an injective ring homomorphism, another calculation, using the flatness of localization, shows that $\mathcal{O}_{Y'} \to \mathcal{O}_Y$ is an injective homomorphism of sheaves of rings. On the other hand, since these are quotients of $\mathcal{O}_X$, the homomorphism $\mathcal{O}_{Y'} \to \mathcal{O}_Y$ must be surjeictive. Thus $(Y, \mathcal{O}_Y) \to (Y', \mathcal{O}_{Y'})$ is an isomorphism of affine schemes, and this implies thst $B' \to B$ is an isomorphism of A-rings. □

7. Separated and Proper Maps of Schemes

The underlying topological space of a scheme is almost never Hausdorff. Yet it is very important to know when points in a scheme can be distinguished by functions, i.e. by local sections of the structure sheaf.

The next definition is the relative notion. Given a map of schemes $f : Y \to X$ we have the fibered product $Y \times_X Y$, with the two projections $\operatorname{pr}_i : Y \times_X Y \to Y$. There is a unique map

$$\operatorname{diag} = \operatorname{diag}_{Y/X} : Y \to Y \times_X Y \tag{7.1}$$

called the *diagonal embedding*, such that $\operatorname{pr}_i \circ \operatorname{diag} = \operatorname{id}_Y$.

Lemma 7.2. *If X and Y are affine then the map* diag *is a closed embedding.*

Proof. Say $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(B)$. Then $Y \times_X Y = \operatorname{Spec}(C)$, where $C := B \otimes_A B$. Let $\mathfrak{c} \subseteq C$ be the the kernel of the multiplication homomorphism $C \twoheadrightarrow B$. Then $\operatorname{Spec}(C/\mathfrak{c})$ is a closed subscheme of $\operatorname{Spec}(C)$, and the ring isomorphism $C/\mathfrak{c} \xrightarrow{\simeq} B$ gives the scheme isomorphism $\operatorname{diag} : \operatorname{Spec}(B) \xrightarrow{\simeq} \operatorname{Spec}(C/\mathfrak{c})$. □

Proposition 7.3. *The map* diag *is an embedding. Namely it induces an isomorphism from Y to a closed subscheme of an open subscheme of $Y \times_X Y$.*

Proof. Let $X = \bigcup_{i \in I} U_i$ be an affine open covering, and for each $i \in I$ let $f^{-1}(U_i) = \bigcup_{j \in J_i} V_j$ be an affine open covering. For every $j \in J_i$ the diagonal map $\operatorname{diag} : V_j \to V_j \times_{U_i} V_j$ is a closed embedding, by Lem 7.2. Now $V_j \times_{U_i} V_j$ is open in $Y \times_X Y$, and $\operatorname{diag}(Y) \cap (V_j \times_{U_i} V_j) = \operatorname{diag}(V_j)$. This shows that $\operatorname{diag}(Y)$ is a closed subscheme of the open subscheme

$$\bigcup_{i \in I} \bigcup_{j \in J_i} (V_j \times_{U_i} V_j) \subseteq Y \times_X Y.$$

□

Definition 7.4. A map of schemes $f : Y \to X$ is called *separated* if $\operatorname{diag}(Y)$ is closed in $Y \times_X Y$.

Corollary 7.5. *If $f : Y \to X$ is separated, then* $\operatorname{diag} : Y \to Y \times_X Y$ *is a closed embedding.*

Proof. Immediate from Prop 7.3. □

Definition 7.6. A scheme X is called *separated* if the map $X \to \operatorname{Spec}(\mathbb{Z})$ is separated.

Example 7.7. By Lem 7.2, a map between affine schemes is separated. In particular, every affine scheme is separated.

Remark 7.8. The notion of separatedness is based on a similar condition for a topological space X being Hausdorff.

Proposition 7.9. *The property of being a separated map is local on the target. Namely: a map of schemes $f : Y \to X$ is separated if there is an open covering $X = \bigcup_{i \in I} U_i$ s.t. the induced map $f_i : Y \times_X U_i \to U_i$ is separated for every i.*

Exercise 7.10. Prove this prop. (Hint: Examine the proofs of Thms 4.9 and 4.4.)

Exercise 7.11. Let X be a scheme and let $Y := \mathbf{A}^n_X$. Show that $Y \to X$ is separated.

Exercise 7.12. Let $\mathbb{K}$ be a field and let X be the affine line over $\mathbb{K}$ with the origin doubled. Namely we take $X_1 = X_2 := \mathbf{A}^1_K$, and $U_i \subseteq X_i$ is the complement of the origin $z \in X_i$. Then X is gotten by gluing X_1 with X_2 by $\operatorname{id} : U_1 \xrightarrow{\simeq} U_2$. Show that X is not separated over $\mathbb{K}$.

Proposition 7.13. *The property of being a separated map is transitive. Namely, if $f : Y \to X$ and $g : Z \to Y$ are separated, then $f \circ g : Z \to X$ is separated.*

Proof. Sketch only: play with fibered products, and use Cor 7.5 and Prop 6.17. □

Corollary 7.14. *If X is a separated $\mathbb{K}$-scheme, then X is a separated scheme.*

Proof. We are given that $X \to \operatorname{Spec}(\mathbb{K})$ is separated. By Lem 7.2 the map $\operatorname{Spec}(\mathbb{K}) \to \operatorname{Spec}(\mathbb{Z})$ is separated. Now use Prop 7.13. □

Theorem 7.15. *Let X be a separated scheme. If $U_1, U_2 \subseteq X$ are affine open sets, then $U_1 \cap U_2 \subseteq X$ is also an affine open set.*

Proof. Let $Z := \operatorname{diag}(X) \subseteq X \times_{\mathbb{Z}} X$. Then $Z \subseteq X \times_{\mathbb{Z}} X$ is a closed subscheme. An easy calculation shows that

$$U_1 \cap U_2 \cong Z \cap (U_1 \times_{\mathbb{Z}} U_2)$$

as schemes. So $U_1 \cap U_2$ is a closed subscheme of the affine scheme $U_1 \times_{\mathbb{Z}} U_2$. By Thm 5.66 we know that $U_1 \cap U_2$ is an affine scheme. □

Here is another important property of separated schemes. (It is analogous to a statement in topology regarding maps into a Hausdorff space.)

Theorem 7.16. *Let $f_1, f_2 : Y \to X$ be a map of $\mathbb{K}$-schemes. Assume X is separated, Y is reduced, and there is a dense open subscheme $V \subseteq Y$ s.t. $f_1|_V = f_2|_V$. Then $f_1 = f_2$.*

Proof. We know that $\operatorname{diag}(X)$ is a closed subscheme of $X \times_{\mathbb{K}} X$. There is a map of schemes

$$(f_1, f_2) : Y \to X \times_{\mathbb{K}} X,$$

and the preimage of $\operatorname{diag}(X)$ is a closed subscheme

$$Y' := Y \times_{X \times_{\mathbb{K}} X} \operatorname{diag}(X) \subseteq Y.$$

If $g : W \to Y$ is a map, then $f_1 \circ g = f_2 \circ g$ iff g factors through a map $g' : W \to Y'$. See exercise below. Note that such a factorization is unique (if it exists), since the closed embedding $Y' \rightarrowtail Y$ is a monomorphism. In particular, this implies that the open embedding $W \rightarrowtail Y$ factors through Y'. By plain topology we see that $Y' = Y$ as sets. Since $(Y, \mathcal{O}_Y)$ is reduced, we must have $(Y', \mathcal{O}_{Y'}) = (Y, \mathcal{O}_Y)$. We conclude that $f_1 = f_2$. □

Exercise 7.17. Prove the claim in the proof of the theorem.

It is time to define $\mathbf{P}^n_{\mathbb{K}}$. For it we need to know a bit about graded rings.

Consider the polynomial ring $\mathbb{K}[\boldsymbol{t}] := \mathbb{K}[t_0, \ldots, t_n]$ in $n + 1$ variables over the nonzero base ring $\mathbb{K}$. Each variable t_i has degree 1. This makes $\mathbb{K}[\boldsymbol{t}]$ into a graded ring. Next we look at the ring of Laurent polynomials

$$\mathbb{K}[\boldsymbol{t}, \boldsymbol{t}^{-1}] := \mathbb{K}[t_0, \ldots, t_n]_{t_0 \cdots t_n},$$

the localization w.r.t. the homogeneous element $t_0 \cdots t_n$. This is also a graded ring.

The degree 0 component $\mathbb{K}[\boldsymbol{t}, \boldsymbol{t}^{-1}]_0$ of $\mathbb{K}[\boldsymbol{t}, \boldsymbol{t}^{-1}]$ is a polynomial ring in n variables, but not canonically. For every $i, j \in [0, n]$ consider the element

$$t_j \cdot t_i^{-1} \in \mathbb{K}[\boldsymbol{t}, \boldsymbol{t}^{-1}]_0. \tag{7.18}$$

We define the sequence $\boldsymbol{t}_i$ in $\mathbb{K}[\boldsymbol{t}, \boldsymbol{t}^{-1}]_0$ as follows:

$$\boldsymbol{t}_i := \big(t_1 \cdot t_i^{-1}, \ldots, t_{i-1} \cdot t_i^{-1}, t_{i+1} \cdot t_i^{-1}, \ldots, t_n \cdot t_i^{-1}\big).$$

These elements are algebraically independent over $\mathbb{K}$, and

$$\mathbb{K}[\boldsymbol{t}, \boldsymbol{t}^{-1}]_0 = \mathbb{K}[\boldsymbol{t}_i].$$

Note that for $i \neq j$ there is equality of subrings

$$\mathbb{K}[\boldsymbol{t}_i]_{t_j \cdot t_i^{-1}} = \mathbb{K}[\boldsymbol{t}_j]_{t_i \cdot t_j^{-1}} \tag{7.19}$$

of the localization of the ring $\mathbb{K}[\boldsymbol{t}, \boldsymbol{t}^{-1}]_0$ w.r.t. the collection of elements $\{t_k \cdot t_l^{-1}\}_{k,l \in [0,n]}$.

For each i let us define

$$U_i := \operatorname{Spec}(\mathbb{K}[\boldsymbol{t}_i]) \cong \mathbf{A}^n_{\mathbb{K}}. \tag{7.20}$$

Then for each $j \neq i$ there is an affine open set

$$U_{i,j} := \operatorname{NZer}_{U_i}(t_j \cdot t_i^{-1}) \cong \operatorname{Spec}\big(\mathbb{K}[\boldsymbol{t}_i]_{t_j \cdot t_i^{-1}}\big). \tag{7.21}$$

Formula (7.19) defines an isomorphism of schemes

$$\phi_{i,j} : U_{i,j} \xrightarrow{\simeq} U_{j,i}, \tag{7.22}$$

and these isomorphisms satisfy the 1-cocycle condition.

Definition 7.23. Let $\mathbb{K}$ be a nonzero ring and $n \in \mathbb{N}$. The *n-dimensional projective space over* $\mathbb{K}$ is the $\mathbb{K}$-scheme $\mathbf{P}^n_{\mathbb{K}}$ obtained by gluing the affine schemes $U_0, \ldots, U_n$ from formula (7.21) along the isomorphisms $\phi_{i,j}$ from formula (7.22).

Next week we will prove that $\mathbf{P}^n_{\mathbb{K}}$ is separated. We will also give the functorial interpretation of $\mathbf{P}^n_{\mathbb{K}}$.

Then we will learn the defdinition of “proper”, and present without proofs the valuative criteria for separatedness and properness.

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Department of Mathematics, Ben Gurion University, Be'er Sheva 84105, Israel.

Email: `amyekut@math.bgu.ac.il`, *Web*: `http://www.math.bgu.ac.il/~amyekut`.