1. High Dimensional Local Fields

We work over a commutative base ring $\mathbb{k}$.

**Definition 1.1.** An $n$-dimensional local field over $\mathbb{k}$ is a field $K$, together with a sequence $\left(\mathcal{O}_1(K), \ldots, \mathcal{O}_n(K)\right)$ of complete DVRs, such that:

- The fraction field of $\mathcal{O}_1(K)$ is $K$.
- The residue field of $\mathcal{O}_i(K)$ is the fraction field of $\mathcal{O}_{i+1}(K)$. We denote this field by $k_i(K)$.
- All these rings and homomorphism are in the category of $\mathbb{k}$-rings, and $\mathbb{k} \to k_n(K)$ is finite.
1. High Dimensional Local Fields

A 0-dimensional local field over \( k \) is just a field \( K \) finite over \( k \).

This concept was introduced by Parshin [Pa1, Pa2] and Kato [Ka] in the 1970’s.

**Example 1.2.** Take \( k := \mathbb{Z} \). The fields \( \hat{\mathbb{Q}}_p \) and \( \mathbb{F}_p((t)) \) are 1-dimensional local fields over \( \mathbb{Z} \).

**Definition 1.3.** Let \( K \) and \( L \) be local fields over \( k \) of dimension \( n \geq 1 \).

A morphism of local fields \( f : K \to L \) is a \( k \)-ring homomorphism such that the following conditions hold:

- \( f \) restricts to a local homomorphism \( O_1(K) \to O_1(L) \).
- The induced homomorphism \( k_1(K) \to k_1(L) \) is a morphism of \( (n-1) \)-dimensional local fields.

Let us denote by \( \text{LF}^n(k) \) the category of \( n \)-dimensional local fields over \( k \).

It is not hard to show that any homomorphism \( K \to L \) in \( \text{LF}^n(k) \) is finite.

Actually one can talk about a morphism of local fields \( f : K \to L \) when \( \dim(K) < \dim(L) \); but the definition is a bit more complicated. We get a category \( \text{LF}(k) \), of which \( \text{LF}^n(k) \) is a full subcategory. See [Ye1] for details.

**Example 1.4.** If \( k \) is a field, then the field of Laurent series \( K := k((t)) \) is a 1-dimensional local field.

The field of iterated Laurent series \( L := K((t_1)) = k((t_2))((t_1)) \) is a 2-dimensional local field.

The inclusions \( k \to K \to L \) are morphisms in \( \text{LF}(k) \).

2. Topological Local Fields

From here on \( k \) is a perfect field. Therefore all our local fields are now of equal characteristics.

One of the reasons that we need this condition is as follows. Let \( K \) be an \( n \)-dimensional local field over \( k \), with last residue field \( k' := k_n(K) \).

Since \( k \) is perfect, the finite extension \( k \to k' \) is separable; or in other words, it is étale. An \( n \)-fold repeated application of formal lifting (also known as Hensel’s Lemma) shows that there is a canonical homomorphism \( k' \to K \) in the category of \( k \)-rings.

Let \( k' \) be a finite field extension of \( k \), and let \( t = (t_1, \ldots, t_n) \) be a sequence of variables.

Consider the field of iterated Laurent series

\[
\mathbb{F}_p((t)) = k'((t_1, \ldots, t_n)) := k'((t_n)) \cdots ((t_1)).
\]

The definition is recursive, adding one variable at a time.

The field \( \mathbb{F}_p((t)) \) has a canonical structure of \( n \)-dimensional local field over \( k \).

The DVRs are

\[
O_i(\mathbb{F}_p((t))) := k'((t_{i+1}, \ldots, t_n))[t_i],
\]

and the residue fields are

\[
k_i(\mathbb{F}_p((t))) := k'((t_{i+1}, \ldots, t_n)).
\]
The field $k'(t)$ has a canonical $k$-linear topology on it, coming from the discrete topology on $k'$, and then an $n$-fold zig-zag of inverse and direct limits.

When $n = 1$ this is the usual $t_1$-adic topology, and $k'(t_1)$ is a topological ring.

However, when $n \geq 2$, $k'(t)$ is not a topological ring, but only a semi-topological $k$-ring, as defined below.

**Definition 2.1.** A *semi-topological (ST) $k$-ring* is a $k$-ring $A$, equipped with a $k$-linear topology, such for any element $a \in A$ the multiplication function $a : A \to A$ is continuous.

We call $k'(t)$ the *standard $n$-dimensional topological local field* with last residue field $k' := k_n(K)$.

As topological $k$-module, $k'(t)$ is complete. The ring of Laurent polynomials is dense in it. When $n \geq 2$, $k'(t)$ is not metrizable.

The parametrization $f$ is not part of the structure of $K$; it is required to exist, but (as we shall soon see) there are many distinct parametrizations.

We use the abbreviation “TLF” for “topological local field”.

**Definition 2.2.** ([Ye1]) An *$n$-dimensional topological local field* over $k$ is a field $K$, together with:

(a) A structure $\{O_i(K)\}_{i=1}^n$ of $n$-dimensional local field over $k$.

(b) A topology, making $K$ a semi-topological $k$-ring.

The condition is this:

(P) There a bijection

$$f : k'(t) \overset{\sim}{\to} K$$

from the standard $n$-dimensional topological local field with last residue field $k' := k_n(K)$, such that:

(i) $f$ is an isomorphism in $\text{LF}^n(k)$ (i.e. it respects the valuations).

(ii) $f$ is an isomorphism of ST $k$-rings (i.e. it respects the topology).

Such an isomorphism $f$ is called a *parametrization* of $K$.

The next theorem tells us what are all the possible parametrizations of a TLF.

**Theorem 2.4.** ([Ye1]) Let $K$ be an $n$-dimensional TLF over $k$, let $(a_1, \ldots, a_n)$ be a system of uniformizers in $K$, let $k' := k_n(K)$, and let $\sigma : k' \to K$ be the canonical homomorphism.

Then $\sigma$ extends uniquely to a parametrization

$$f : k'(t_1, \ldots, t_n) \overset{\sim}{\to} K$$

such that $f(t_i) = a_i$. 
2. Topological Local Fields

**Definition 2.5.** Let $K$ and $L$ be $n$-dimensional TLFs.

A morphism of TLFs $f : K \to L$ is a morphism of local fields (Definition 1.3) which is also continuous.

We denote by $\text{TLF}^n(k)$ the category of $n$-dimensional TLFs over $k$.

There is a bigger category $\text{TLF}(k)$, that allows morphisms $f : K \to L$ with $\dim(K) < \dim(L)$. See Example 1.4. $\text{TLF}^n(k)$ is a full subcategory of $\text{TLF}(k)$.

Consider the functor $\text{TLF}^n(k) \to \text{LF}^n(k)$ that forgets the topology.

When $n \geq 2$ and $\text{char}(k) = 0$ this forgetful functor is far from being an equivalence.

In other words, in characteristic 0, any local field $K$ of dimension $\geq 2$ admits many distinct topologies, all satisfying condition (P). There is an example of this phenomenon in [Ye1].

However:

**Theorem 2.6.** ([Ye1]) If $\text{char}(k) = p > 0$, the forgetful functor $\text{TLF}^n(k) \to \text{LF}^n(k)$ is an equivalence.

The proof relies on the fact that for a TLF $K$ in characteristic $p$, any differential operator $\psi : K \to K$ is continuous. See [Ye1].

3. The Beilinson Completion

Recall that $k$ is a perfect field. Suppose $X$ is a finite type $k$-scheme.

By a *chain of points* in $X$ we mean a sequence $\xi = (x_0, \ldots, x_n)$ of points, such that $x_i$ is a specialization of $x_{i-1}$.

The chain $\xi$ is *saturated* if each $x_i$ is an immediate specialization of $x_{i-1}$.

The chain $\xi$ is *maximal* if it is saturated, $x_0$ is a generic point of $X$, and $x_n$ is a closed point.

In [Be], Beilinson defined a completion operation, which is a special case of his higher adeles.

In other words, given a quasi-coherent sheaf $\mathcal{M}$ on $X$, and a chain $\xi$, the *Beilinson completion* of $\mathcal{M}$ along $\xi$ is a $k$-module $\mathcal{M}_\xi$, gotten by an $n$-fold zig-zag of inverse and direct limits.

The completion $\mathcal{M}_\xi$ comes equipped with a $k$-linear topology.

The completion $\mathcal{O}_{X,\xi}$ of the structure sheaf $\mathcal{O}_X$ is a commutative ST $k$-ring.

There is a canonical $k$-ring homomorphism $\mathcal{O}_{X,x_0} \to \mathcal{O}_{X,\xi}$.

For any $\mathcal{M}$ there is a canonical isomorphism of $k$-modules

$$\mathcal{M}_\xi \cong \mathcal{O}_{X,\xi} \otimes_{\mathcal{O}_{X,x_0}} \mathcal{M}_{x_0}.$$ 

**Example 3.1.** If $n = 0$, so $\xi = (x_0)$, we get $\mathcal{O}_{X,\xi} = \mathcal{O}_{X,x_0}$, the $m_{x_0}$-adic completion of the local ring $\mathcal{O}_{X,x_0}$. Its topology is the $m_{x_0}$-adic topology.
3. The Beilinson Completion

If $X$ is an integral scheme, then its function field $k(X)$ can be viewed as a quasi-coherent sheaf (constant on $X$).

**Theorem 3.2.** ([Pa1], [Be], [Ye1]) Let $X$ be an integral finite type $k$-scheme of dimension $n$, and let $\xi$ be a maximal chain in $X$.

Then the Beilinson completion $k(X)_\xi$ is a finite product of $n$-dimensional TLFs.

The number of factors of the reduced artinian ring $k(X)_\xi$ depends on the singularities of the 1-dimensional local rings $O_{X_i,x_{i1}}$, where $\xi = (x_0, \ldots, x_n)$ and $X_i := \{x_i\}_{\text{red}}$.

4. The TLF Residue Functional

**Theorem 4.3.** ([Ye1]) Let $K$ be an $n$-dimensional TLF over $k$.

There is a unique $k$-linear homomorphism

$$\text{Res}_{\text{TLF}}^{K/k} : \Omega^n_{K/k, \text{sep}} \to k,$$

called the TLF residue functional, with these properties.

1. **Continuity:** the homomorphism $\text{Res}_{\text{TLF}}^{K/k}$ is continuous.

2. **Uniformization:** let $a = (a_1, \ldots, a_n)$ be a system of uniformizers for $K$, and let $k' \to K$ be the canonical homomorphism from the last residue field $k' := k_n(K)$ into $K$.

Then for any $b \in k'$ and any $i_1, \ldots, i_n \in \mathbb{Z}$ we have

$$\text{Res}_{\text{TLF}}^{K/k}(b \cdot a_1^{i_1} \cdots a_n^{i_n} \cdot d\log(a)) = \begin{cases} \text{tr}_{k'/k}(b) & \text{if } i_1 = \cdots = i_n = 0 \\ 0 & \text{otherwise} \end{cases}.$$

When $n = 1$ we recover the classical residue functional.

The residue functional has many nice properties.

Here is one of them: the residue induces a topological perfect pairing

$$\langle - , - \rangle_{\text{res}} : K \times \Omega^n_{K/k, \text{sep}} \to k, \quad \langle a, \alpha \rangle_{\text{res}} := \text{Res}_{\text{TLF}}^{K/k}(a \cdot \alpha).$$

Suppose $A = \prod_m A_m$ is a finite product of $n$-dimensional TLFs. (So $m$ runs over the finite set of maximal ideals of $A$.)

For instance, $A$ could arise as a Beilinson completion of a function field, as in Theorem 3.2.

We define

$$\text{Res}_{A/k}^{\text{TLF}} : \Omega^n_{A/k, \text{sep}} = \bigoplus_m \Omega^n_{A_m/k, \text{sep}} \to k.$$
4. The TLF Residue Functional

**Theorem 4.5.** (Residue Theorem, [Ye1, Ye3]) Let $X$ be a proper integral $\mathbb{k}$-scheme of dimension $n$, with function field $k(X)$. Take any $\alpha \in \Omega^n_{k(X)/\mathbb{k}}$. Then

$$\sum_{\xi} \text{Res}^{\text{TLF}}_{k(X)_{\xi}/\mathbb{k}}(\alpha) = 0,$$

where the sum is over all maximal chains $\xi$ in $X$.

When $n = 1$ this is just the classical Residue Theorem. For geometric applications of the TLF residue functional, mainly an explicit construction of the Grothendieck residue complex, see [Ye1, Ye2, Ye3].

The first attempt at a residue theory for higher local fields was by Parshin and his school. See the papers [Pa1], [Pa2], [Be], [Lo] and [Pa3]. However the concept of TLF was absent from their work, which resulted in incorrect statements.

5. The Geometric BT Residue Functional

In 1968 Tate [Ta] discovered a new approach to residues on curves. The construction had the flavor of functional analysis.

First, for a 1-dimensional local field $K$ over $\mathbb{k}$, he produced a residue functional

$$\text{Res}^{\text{Tate}}_{K/\mathbb{k}} : \Omega^1_{K/\mathbb{k}} \to \mathbb{k},$$

using algebraic manipulations (commutators and traces) of a certain ring of operators acting on $K$.

Then Tate proved a global residue theorem: for a proper 1-dimensional smooth $\mathbb{k}$-scheme $X$, the functionals $\text{Res}^{\text{Tate}}_{k(X)_{\xi}/\mathbb{k}}$ satisfy a formula like in Theorem 4.5.

Note that here the completion $k(X)_{\xi}$ along a maximal chain $\xi = (x_0, x_1)$ is just the local field at the closed point $x_1$.

The proof was original: he showed that

$$\sum_{\xi} \text{Res}^{\text{Tate}}_{k(X)_{\xi}/\mathbb{k}} = \text{Res}^{\text{Tate}}_{A(X)/\mathbb{k}},$$

where the latter is a residue functional that he associated to the ring of global adeles $\mathbb{A}(X)$.

The functional $\text{Res}^{\text{Tate}}_{\mathbb{A}(X)/\mathbb{k}}$ turns out to be zero because $\Gamma(X, \mathcal{O}_X)$ is finite over $\mathbb{k}$ (!)

Tate also proved that in dimension 1, his residue functional coincides with the classical one.

In 1980 Beilinson [Be] introduced higher adeles and the higher completion operation (that I already talked about).

He showed how to generalize the Tate residue construction to higher dimensions.

Suppose $X$ is an $n$-dimensional integral $\mathbb{k}$-scheme, with function field $k(X)$.

For any maximal chain $\xi$ in $X$, Beilinson produced a residue functional

$$\text{Res}^{\text{BT}}_{X,\xi} : \Omega^n_{k(X)_{\xi}/\mathbb{k}} \to \mathbb{k}. (5.1)$$

I refer to this as the geometric BT residue functional, where “BT” short for “Beilinson-Tate”.

When $n = 1$, this essentially coincides with Tate’s construction, and thus with the classical residue functional.

I will say more about $\text{Res}^{\text{BT}}_{X,\xi}$ later (in Section 9).
5. The Geometric BT Residue Functional

Beilinson asserted that when $X$ is proper, the functionals $\text{Res}_{X,\xi}^{BT}$ satisfy a global formula, as in Theorem 4.5.

As could be expected from such a short paper (2 pages), there were very few details in [Be], and no proofs.

The details of higher adeles were later worked out (with full proofs) by Huber in [Hu].

The higher local fields that arise from Beilinson completions were studied in [Ye1]. (This was already discussed in Section 4.)

However the geometric BT residue functional $\text{Res}_{X,\xi}^{BT}$ (in dimension $n \geq 2$) remained cryptic for many years.

Very recently there was renewed interest in the higher dimensional geometric BT residue.

There was a lot of progress on the local aspect of BT residues, by Braunling [Br1, Br2]. But even the local picture is not totally understood, as we shall see soon.

There is ongoing work on the global aspect of BT residues, by Braunling, Groechenig and Wolfson, including so far the papers [BGW1], [BGW2] and [BGW3].

There is still no proof of Beilinson’s assertion regarding a global residue formula for $\text{Res}_{X,\xi}^{BT}$; neither something extending Tate’s adelic proof in dimension 1, nor any other proof.

6. The Ring of Local Beilinson-Tate Operators

In the remainder of the talk I will outline a variant of the geometric BT residue functional. Details are in [Ye7].

This new construction applies to TLFs, and there is no geometry involved at all.

Thus it is more in the style of Tate than in the style of Beilinson (where the construction was intimately tied to the geometry).

The two conjectures that I will state, if proved, would imply most of the geometric assertions in [Be] regarding the functional $\text{Res}_{X,\xi}^{BT}$. This is because the corresponding assertions are known to hold for the functionals $\text{Res}_{k(X),\xi}^{TLF}/k$.

Conversely, as noted by Beilinson, the geometric properties of $\text{Res}_{X,\xi}^{BT}$ – if they can be proved directly – would imply our conjectures.

Let us fix an $n$-dimensional TLF $K$ over $k$, with last residue field $k' := k_n(K)$.

By a system of liftings for $K$ we mean a sequence

$$\sigma = (\sigma_1, \ldots, \sigma_n)$$

where each

$$\sigma_i : k_i(K) \hookrightarrow \mathcal{O}_i(K)$$

is a continuous $k$-ring lifting of the canonical surjection $\mathcal{O}_i(K) \twoheadrightarrow k_i(K)$.

Note that the standard TLF $K' = k'((t_1, \ldots, t_n))$ is equipped with a standard system of liftings. Hence any parametrization $f : K' \xrightarrow{\sim} K$ induces a system of liftings on $K$.

Let $\sigma = (\sigma_1, \ldots, \sigma_n)$ be a system of liftings of $K$. Then, if $n \geq 1$, the sequence

$$d_1(\sigma) := (\sigma_2, \ldots, \sigma_n)$$

is a system of liftings of $k_1(K)$. 
Suppose $M$ is a finite $K$-module. A lattice in $M$ is a finite $\mathcal{O}_1(K)$-submodule $L \subseteq M$ such that $M = K \cdot L$.

Let $M_1, M_2$ be finite $K$-modules. We denote by $\text{Lat}(M_1, M_2)$ the set of pairs $(L_1, L_2)$, where $L_i$ is a lattice in $M_i$.

Let $\phi : M_1 \to M_2$ be a $k$-linear homomorphism, and let

$$(L_1, L_2), (L'_1, L'_2) \in \text{Lat}(M_1, M_2).$$

We say that $(L'_1, L'_2)$ is a $\phi$-refinement of $(L_1, L_2)$ if

$$L'_1 \subset L_1, \ L_2 \subset L'_2, \ \phi(L'_1) \subset L_2 \quad \text{and} \quad \phi(L_1) \subset L'_2.$$
If \( n \leq 1 \) then there is only one choice for \( \sigma \), so \( E_\sigma(K) \) and \( E_\sigma(K)_{i,j} \) are trivially independent of \( \sigma \).

In fact, when \( n = 0 \) there is equality

\[ E(K) = \text{End}_k(K), \]

so there is nothing interesting.

When \( n = 1 \) it is not hard to show that

\[ E_\sigma(K) = \text{End}^\text{cont}_k(K), \]

the ring of continuous \( k \)-module endomorphisms of \( K \); and the subsets \( E_\sigma(K)_{i,j} \) are the two-sided ideals of the ring \( E_\sigma(K) \) that were considered by Tate in [Ta].

**But what about \( n \geq 2 \)?**

Let us denote by \( D^\text{cont}_{K/k} \) the ring of continuous differential operators of \( K \).

So there are inclusions of \( k \)-rings

\[ K \subset D^\text{cont}_{K/k} \subset \text{End}^\text{cont}_k(K) \subset \text{End}_k(K). \]

By induction of \( n \) we prove the next theorem.

**Theorem 6.5.** ([Ye7]) Let \( K \) be an \( n \)-dimensional TLF, with system of liftings \( \sigma \).

1. \( E_\sigma(K) \) is a subring of \( \text{End}_k(K) \).
2. Each \( E_\sigma(K)_{i,j} \) is a two-sided ideal of \( E_\sigma(K) \).
3. There are inclusions of \( k \)-rings

\[ D^\text{cont}_{K/k} \subset E_\sigma(K) \subset \text{End}^\text{cont}_k(K). \]

The main result of [Ye7] is:

**Theorem 6.6.** ([Ye7]) Let \( K \) be an \( n \)-dimensional TLF.

Let \( \sigma \) and \( \sigma' \) be two systems of liftings for \( K \).

Then

\[ E_\sigma(K) = E_{\sigma'}(K), \]

and for any \( i,j \) there is equality

\[ E_\sigma(K)_{i,j} = E_{\sigma'}(K)_{i,j}. \]

The proof of this result is quite difficult. It is interesting to note that the proof relies on Theorem 6.5.

Theorem 6.6 makes the following definition reasonable.

**Definition 6.7.** Let \( K \) be an \( n \)-dimensional TLF.

Let \( \sigma \) be any system of liftings for \( K \).

1. The **ring of local Beilinson-Tate operators** is the ring

\[ E(K) := E_\sigma(K). \]

2. For any \( i,j \) we define the two-sided ideal

\[ E(K)_{i,j} := E_\sigma(K)_{i,j} \]

of \( E(K) \).
7. Cubically Decomposed Rings of Operators

As before $k$ is a perfect field.

Let $M$ be a $k$-module. Following Tate, an operator $\phi \in \text{End}_k(M)$ is called finite potent if $\phi^l$ has finite rank for some positive integer $l$.

Here is a definition of Braunling [Br2], distilled from ideas in [Be]. The notation we use is very close to Tate’s original notation from [Ta].

If $A$ is any commutative $k$-ring, then there is a canonical embedding of $k$-rings $A \subset \text{End}_k(A)$.

**Definition 7.1.** ([Br2]) Let $A$ be a commutative $k$-ring.

An $n$-dimensional cubically decomposed ring of operators on $A$ is data $E = (E, \{E_{i,j}\})$ consisting of:

- A subring $E \subset \text{End}_k(A)$ containing $A$.
- Two-sided ideals $E_{i,j}$ of $E$, indexed by $i \in \{1, \ldots, n\}$ and $j \in \{1, 2\}$.

These are the conditions:

1. For every $i = 1, \ldots, n$ we have $E = E_{i,1} + E_{i,2}$.
2. Each operator $\phi \in \bigcap_{i=1}^n \bigcap_{j=1}^2 E_{i,j}$ is finite potent.

Consider an $n$-dimensional integral finite type $k$-scheme $X$, with function field $k(X)$. Let $\xi$ be a maximal chain in $X$.

We know (from Theorem 3.2) that the Beilinson completion $k(X)_\xi$ is a finite product of $n$-dimensional TLFs.

Beilinson gave a construction in [Be], expanded in [Br2], of a ring of operators $E_{X,\xi}(k(X)) \subset \text{End}_k(k(X))$, with two-sided ideals $E_{X,\xi}(k(X))_{i,j}$ in it. This construction is geometric, with an inductive aspect similar to Definition 6.2 (that mimics it).

**Theorem 7.2.** ([Be], [Br2]) With $X$ and $\xi$ as above, the data $E_{X,\xi}(k(X)) := \left( E_{X,\xi}(k(X)), \{E_{X,\xi}(k(X))_{i,j}\} \right)$ is an $n$-dimensional cubically decomposed ring of operators on $k(X)_\xi$.

On the other hand, suppose $A$ is a $k$-ring which is a finite product $A = \prod_m A_m$ of $n$-dimensional TLFs.

Define

(7.3) $E(A) := \prod_m E(A_m)$

and

(7.4) $E(A)_{i,j} := \prod_m E(A_m)_{i,j}$.

Here $E(A_m)$ is the ring of local BT operators from Definition 6.7.

**Theorem 7.5.** ([Ye7]) Let $A = \prod_m A_m$ be a finite product of $n$-dimensional TLFs.

The data $E(A) := \left( E(A), \{E(A)_{i,j}\} \right)$ is an $n$-dimensional cubically decomposed ring of operators on $A$. 
Here is our first conjecture.

**Conjecture 7.6.** Let $X$ be an integral $n$-dimensional finite type $\mathbb{k}$-scheme, with function field $k(X)$, and let $\xi$ be a maximal chain in $X$.

Then

$$E_{X,\xi}(k(X)) = E(k(X)_\xi),$$

as cubically decomposed rings of operators on the completion $k(X)_\xi$.

To emphasize: the left side is the ring of operators from Theorem 7.2, and it is defined geometrically. The right side is the ring of local BT operators from Theorem 7.5, and it depends only on the ring $k(X)_\xi$, which is a finite product of TLFs.

Note that for $n = 0$ this is a trivial fact.

For $n = 1$ this is quite easy to check, using the results of [Ta].

But for $n \geq 2$ the conjecture seems to be challenging.

As mentioned before, Beilinson [Be] discovered how to extend Tate’s local operator theoretic residue (from [Ta]) to higher dimensions, using Lie algebra extensions. This was made precise in [Br2]:

**Theorem 8.1.** ([Be], [Br2]) Suppose $A$ is a commutative $\mathbb{k}$-ring, with an $n$-dimensional cubically decomposed ring of operators $E = (E, \{E_{ij}\})$.

Then there is an induced $\mathbb{k}$-linear functional

$$\text{Res}^E_{A/\mathbb{k}} : \Omega^n_{A/\mathbb{k}} \to \mathbb{k},$$

with explicit formulas.

Finally we can explain how the geometric BT residue functional $\text{Res}^\text{BT}_{X,\xi}$ from (5.1) is constructed.

Here $X$ is an $n$-dimensional integral scheme, $\xi$ is a maximal chain, and $k(X)$ is the function field of $X$.

According to Theorem 7.2, there is an $n$-dimensional cubically decomposed ring of operators $E_{X,\xi}(k(X))$ on the completion $k(X)_\xi$. It is defined geometrically.

Beilinson’s geometric construction of the higher residue (in [Be], and elaborated in [Br2]) is this:

$$\text{Res}^\text{BT}_{X,\xi} := \text{Res}^E_{k(X)_\xi/\mathbb{k}} : \Omega^n_{k(X)_\xi/\mathbb{k}} \to \mathbb{k},$$

with $E := E_{X,\xi}(k(X))$.

Now we go to the “analytic” setup.

Take a $\mathbb{k}$-ring $A$ which is a finite product of $n$-dimensional TLFs. It comes with the cubically decomposed ring of local BT operators $E(A)$; see Theorem 7.5.

Using Theorem 8.1 we define the local BT residue functional

$$\text{Res}^\text{BT}_{A/\mathbb{k}} := \text{Res}^E_{A/\mathbb{k}} : \Omega^n_{A/\mathbb{k}} \to \mathbb{k},$$

with $E := E(A)$.

On the other hand we have the TLF residue functional

$$\text{Res}^\text{TLF}_{A/\mathbb{k}} : \Omega^n_{A/\mathbb{k}} \to \mathbb{k},$$

from Theorem 4.3 and formula (4.4).

By definition there is a canonical surjection

$$\text{can} : \Omega^n_{A/\mathbb{k}} \to \Omega^n_{A/\mathbb{k}}^{\text{sep}},$$
Here is our second conjecture.

**Conjecture 8.2.** ([Ye7]) Let $A$ be a finite product of $n$-dimensional TLFs over $k$.

Then the following diagram is commutative:

\[
\begin{array}{ccc}
\Omega^n_A & \xrightarrow{\text{can}} & \Omega^{n,\text{sep}}_A \\
\text{Res}_{A/k} & \downarrow & \downarrow \text{Res}_{A/k} \\
\Omega^n_A & \rightarrow & \text{Res}_{A/k} \\
\end{array}
\]

Again, for $n \leq 1$ all is understood. The challenge is only for $n \geq 2$.

\[\sim \text{END} \sim\]


