Local and Geometric Beilinson-Tate Operators

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0. Introduction

In 1968, Tate [Ta] introduced a new approach to residues on algebraic curves, based on a certain ring of operators that acts on the completion at a point of the function field of the curve.

This approach was generalized to higher dimensional algebraic varieties by Beilinson [Be] in 1980. We refer to the relevant ingredients in Beilinson's work as the *ring of geometric BT operators* and the *BT residue functional*. Throughout, "BT" stands for "Beilinson-Tate".

Beilinson's paper had very few details, and his operator-theoretic construction of residues remained cryptic for many years. In particular, several important assertions regarding the global geometric behavior of the BT residue functional were never proved.

In recent years, the team of young researchers Braunling, Groechenig and Wolfson (BGW) wrote several papers on Beilinson's higher residues and related topics. See the bibliography.

Influenced by the work of BGW, I made an attempt to clarify Beilinson's approach to higher residues, by comparing it to my work on topological local fields (TLFs) and residues (in the 1992 paper [Ye1]).

The global geometric properties of TLF residues are well understood. If the TLF residues could be effectively compared to the BT residues, this would establish the missing global properties of the BT residues.

To make this comparison, in my recent paper [Ye4] I introduced a new ring of operators, called the *ring of local BT operators*, that has an algebro-analytic nature (closer in spirit to Tate's original construction).

In this talk I will explain the concepts mentioned above, and two conjectures that are needed to finish the comparison between the BT and the TLF approaches.

1. High Dimensional Local Fields

Throughout this talk we fix a perfect base field $\Bbbk$.

Working over a perfect field is not needed everywhere; but it greatly simplifies the presentation.

Definition 1.1. An *n-dimensional local field* over $\Bbbk$ is a field K, together with a sequence

$$\big(\mathcal{O}_1(K), \ldots, \mathcal{O}_n(K)\big)$$

of complete DVRs, such that:

- The fraction field of $\mathcal{O}_1(K)$ is K.
- The residue field $\boldsymbol{k}_i(K)$ of $\mathcal{O}_i(K)$ is also the fraction field of $\mathcal{O}_{i+1}(K)$.

- All these rings and homomorphism are in the category of $\Bbbk$-rings.
- The homomorphism $\Bbbk \to \boldsymbol{k}_n(K)$ is finite.

Here is the picture for $n = 2$.

$$\begin{tikzcd}
 & & \mathcal{O}_1(K) \arrow[r, tail] \arrow[d, two heads] & K \\
\Bbbk \arrow[urr, bend left] \arrow[r] \arrow[dr, bend right, "\text{finite}"'] & \mathcal{O}_2(K) \arrow[r, tail] \arrow[d, two heads] & \boldsymbol{k}_1(K) & \\
 & \boldsymbol{k}_2(K) & &
\end{tikzcd}$$

The definition above was introduced by Parshin [Pa1, Pa2] and Kato [Ka] in the 1970's.

Let $\Bbbk'$ be a finite extension field of $\Bbbk$, and let $(t_1, \ldots, t_n)$ be a sequence of variables.

The field of iterated Laurent series

$$K = \Bbbk'((t_1, \ldots, t_n)) := \Bbbk'((t_n)) \cdots ((t_1))$$

is an n-dimensional local field over $\Bbbk$.

Its first DVR is

$$\mathcal{O}_1(K) = \Bbbk'((t_2, \ldots, t_n))[[t_1]],$$

the first residue field is

$$\boldsymbol{k}_1(K) = \Bbbk'((t_2, \ldots, t_n)),$$

and so on. The last residue field is

$$\boldsymbol{k}_n(K) = \Bbbk'.$$

Now let K be any n-dimensional local field over $\Bbbk$.

Because $\Bbbk$ is perfect, the last residue field $\Bbbk' := \boldsymbol{k}_n(K)$ is a finite separable extension of $\Bbbk$.

Using Hensel's Lemma n times, we see that there is a *canonical $\Bbbk$-ring homomorphism* $\Bbbk' \to K$.

The homomorphism $\Bbbk' \to K$ can be extended *noncanonically* to an isomorphism of n-dimensional local fields

$$f : \Bbbk'((t_1, \ldots, t_n)) \xrightarrow{\simeq} K \tag{1.2}$$

from the field of iterated Laurent series.

Let $a_i := f(t_i) \in K$. The sequence

$$\boldsymbol{a} = (a_1, \ldots, a_n)$$

in K is called a *system of uniformizers* of K.

2. Topological Local Fields

A *semi-topological* $\Bbbk$*-ring* is a commutative $\Bbbk$-ring A, with a $\Bbbk$-linear topology, such that for any element $a \in A$ the multiplication homomorphism $a : A \to A$ is continuous.

Suppose A is a nonzero semi-topological $\Bbbk$-ring.

The ring of power series in one variable

$$A[[t]] = \lim_{\leftarrow i} A[t]/(t^i)$$

is given the $\lim_{\leftarrow}$ topology.

The ring of Laurent series

$$A((t)) = \lim_{j \to} t^{-j} \cdot A[[t]]$$

is given the $\lim_{\to}$ topology.

It turns out that $A((t))$ is also a semi-topological $\Bbbk$-ring.

Warning: the stronger property of being a topological ring is not preserved under passage to the ring of Laurent series. This is why we must work with semi-topological rings.

Let $\Bbbk'$ be a finite field extension of $\Bbbk$.

For any $n \geq 1$, the discrete topology on $\Bbbk'$ extends recursively, by the procedure above, to a $\Bbbk$-linear topology on the field of iterated Laurent series

$$\Bbbk'((\boldsymbol{t})) := \Bbbk'((t_1, \ldots, t_n))$$

in the sequence of variables $\boldsymbol{t} = (t_1, \ldots, t_n)$.

We call $\Bbbk'((\boldsymbol{t}))$ the *standard n-dimensional TLF with last residue field* $\Bbbk'$.

Recall that "TLF" is an abbreviation for "topological local field"

Definition 2.1. ([Ye1]) An *n-dimensional TLF* over $\Bbbk$ is a field K, together with:

(a) A structure $\{\mathcal{O}_i(K)\}_{i=1}^n$ of n-dimensional local field over $\Bbbk$.

(b) A $\Bbbk$-linear topology, making K a semi-topological ring.

The condition is this:

(P) There an isomorphism of $\Bbbk$-rings

$$f : \Bbbk'((\boldsymbol{t})) \xrightarrow{\simeq} K$$

from the standard n-dimensional TLF with last residue field $\Bbbk' := \boldsymbol{k}_n(K)$, such that:

- (i) f is an isomorphism of n-dimensional local fields (i.e. it respects the valuations).
- (ii) f is an isomorphism of semi-topological rings (i.e. it respects the topologies).

Such an isomorphism f is called a *parametrization* of K.

The parametrization f is not part of the structure of K; it is required to exist, but there are many distinct parametrizations.

Remark 2.2. Assume $\Bbbk$ has characteristic 0 and $n \geq 2$. Let K be an n-dimensional TLF over $\Bbbk$.

There exist (many) automorphisms of K as n-dimensional local field that are not continuous. This was discovered in [Ye1].

Therefore, a meaningful theory must take the topology into account as part of the structure.

In dimension $n = 1$ there are no surprises.

Strangely, when $\operatorname{char} \Bbbk = p > 0$, any automorphism of K as n-dimensional local field is continuous. This was proved in [Ye1].

The key to understanding these subtle facts (not noticed by earlier researchers) is *differential operators*.

3. The TLF Residue Functional

Let K be an n-dimensional TLF over $\Bbbk$, with last residue field $\Bbbk' = \boldsymbol{k}_n(K)$.

The module of differentials $\Omega^n_{K/\Bbbk}$ is equipped with a $\Bbbk$-linear topology.

The *module of separated differential n-forms* of K is

$$\Omega^{n,\mathrm{sep}}_{K/\Bbbk} := \Omega^n_{K/\Bbbk} \,/\, \{\text{closure of } 0\}. \tag{3.1}$$

We know that $\Omega^{n,\mathrm{sep}}_{K/\Bbbk}$ is a free semi-topological K-module of rank 1.

There is a canonical surjection

$$\mathrm{can} : \Omega^n_{K/\Bbbk} \twoheadrightarrow \Omega^{n,\mathrm{sep}}_{K/\Bbbk}. \tag{3.2}$$

In characteristic 0, this surjection has an enormous kernel. But in characteristic $p > 0$ it is bijective.

The theory of TLF residues is encapsulated in Theorem 3.4 below.

A system of uniformizers $\boldsymbol{a} = (a_1, \ldots, a_n)$ in K gives rise to a nonzero element

$$\mathrm{dlog}(\boldsymbol{a}) := \mathrm{dlog}(a_1) \wedge \cdots \wedge \mathrm{dlog}(a_n) \in \Omega^{n,\mathrm{sep}}_{K/\Bbbk}, \tag{3.3}$$

where

$$\mathrm{dlog}(a_i) := a_i^{-1} \cdot \mathrm{d}(a_i) \in \Omega^{1,\mathrm{sep}}_{K/\Bbbk}.$$

The ring of Laurent polynomials $\Bbbk'[a_1^{\pm 1}, \ldots, a_n^{\pm 1}]$ is dense in K.

It follows that a continuous $\Bbbk$-linear functional on $\Omega^{n,\mathrm{sep}}_{K/\Bbbk}$ is determined by its values on the forms

$$b \cdot a_1^{i_1} \cdots a_n^{i_n} \cdot \mathrm{dlog}(\boldsymbol{a}) \in \Omega^{n,\mathrm{sep}}_{K/\Bbbk}$$

for $b \in \Bbbk'$ and $i_1, \ldots, i_n \in \mathbb{Z}$.

Theorem 3.4. *([Ye1]) Let K be an n-dimensional TLF over $\Bbbk$.*

There is a unique $\Bbbk$-linear homomorphism

$$\mathrm{Res}^{\mathrm{TLF}}_{K/\Bbbk} : \Omega^{n,\mathrm{sep}}_{K/\Bbbk} \to \Bbbk,$$

called the TLF residue functional*, with these properties.*

1. *Continuity*: the homomorphism $\mathrm{Res}^{\mathrm{TLF}}_{K/\Bbbk}$ is continuous.
2. *Uniformization*: let $\boldsymbol{a} = (a_1, \ldots, a_n)$ be a system of uniformizers for K, and let $\Bbbk' := \boldsymbol{k}_n(K)$ be the last residue field.

   Then for any $b \in \Bbbk'$ and any $i_1, \ldots, i_n \in \mathbb{Z}$ we have

$$\mathrm{Res}^{\mathrm{TLF}}_{K/\Bbbk}\big(b \cdot a_1^{i_1} \cdots a_n^{i_n} \cdot \mathrm{dlog}(\boldsymbol{a})\big) = \begin{cases} \mathrm{tr}_{\Bbbk'/\Bbbk}(b) & \text{if } i_1 = \cdots = i_n = 0 \\ 0 & \text{otherwise .} \end{cases}$$

When $n = 1$ we recover the classical residue functional.

The TLF residue functional has many nice properties.

For geometric applications of the TLF residue functional, mainly an *explicit construction of the Grothendieck residue complex*, see the papers [Ye1], [Ye2] and [Ye3].

The first attempt at a residue theory for higher local fields was by Parshin and his school. See the papers [Pa1], [Pa2], [Be], [Lo] and [Pa3].

However the concept of TLF was absent from their work, and they were not aware of the subtleties mentioned in Remark 2.2.

Since the residue functional is not well defined on untopologized local fields (there are counterexamples), some of the papers by the Parshin school contain incorrect statements.

4. The BT Residue Functional

Let M be a $\Bbbk$-module. Following Tate [Ta], an operator $\phi \in \operatorname{End}_{\Bbbk}(M)$ is called *finite potent* if ϕ^m has finite rank for some positive integer m.

The definition and theorem below are taken from Braunling's paper [Br2]. They are distilled from ideas in [Be]. However, the notation we use is closer to Tate's original notation from [Ta].

If A is any commutative $\Bbbk$-ring, then there is a canonical embedding of $\Bbbk$-rings $A \subseteq \operatorname{End}_{\Bbbk}(A)$.

Definition 4.1. ([Ta], [Be], [Br2]) Let A be a commutative $\Bbbk$-ring.

An *n-dimensional cubically decomposed ring of operators on A* is this data:

- A subring $E \subseteq \operatorname{End}_{\Bbbk}(A)$ containing A.
- Two-sided ideals $E_{i,j}$ of E, indexed by $i \in \{1, \ldots, n\}$ and $j \in \{1, 2\}$.

These are the conditions:

(i) For every $i = 1, \ldots, n$ we have $E = E_{i,1} + E_{i,2}$.

(ii) Each operator $\phi \in \bigcap_{i=1}^{n} \bigcap_{j=1}^{2} E_{i,j}$ is finite potent.

Theorem 4.2. *([Ta], [Be], [Br2]) Suppose A is a commutative $\Bbbk$-ring, with an n-dimensional cubically decomposed ring of operators E.*

Then there is an induced $\Bbbk$-linear functional

$$\operatorname{Res}^{\mathrm{BT}}_{A/\Bbbk;E} : \Omega^n_{A/\Bbbk} \to \Bbbk$$

with explicit formulas, called the BT residue functional.

The functional $\operatorname{Res}^{\mathrm{BT}}_{A/\Bbbk;E}$ can be effectively described in terms of Lie algebra cohomology.

For $n = 1$ this is just Tate's original construction.

Note that the ring A has no topology on it, and the functional $\operatorname{Res}^{\mathrm{BT}}_{A/\Bbbk;E}$ is defined on the algebraic module of differentials $\Omega^n_{A/\Bbbk}$.

5. The Ring of Local BT Operators

As mentioned in the Introduction, in order to gain a better understanding of Beilinson's work in [Be], I introduced (in the paper [Ye4] from 2014) a new ring of operators, that acts on a TLF K.

The construction is inspired by the ideas in [Ta] and [Be], and by their interpretation in the recent work of Braunling [Br1] and [Br2].

Indeed, the ring of operators $\mathrm{E}^{\mathrm{loc}}(K)$ discussed below is designed to mimic (in algebro-analytic terms) the original ring of operators $\mathrm{E}^{\mathrm{geo}}(K)$, that was constructed in [Be] by geometric means. The latter will be discussed in Section 6.

Let's fix an n-dimensional TLF K.

Suppose M is a finite K-module. A *lattice* in M is a finite $\mathcal{O}_1(K)$-submodule $L \subseteq M$ such that $M = K \cdot L$.

The set of lattices in M is denoted by $\mathrm{Lat}(M)$.

If $L' \subseteq L$ are nested lattices in M, then the quotient $\bar{M} := L/L'$ is a finite length module over $\mathcal{O}_1(K)$. We call $\bar{M}$ a *residue module* of M.

Suppose we are given a continuous $\Bbbk$-ring lifting

$$\sigma_1 : \boldsymbol{k}_1(K) \to \mathcal{O}_1(K)$$

of the canonical surjection $\mathcal{O}_1(K) \twoheadrightarrow \boldsymbol{k}_1(K)$.

The lifting σ_1 allows us to view the residue module $\bar{M}$ as a finite module over the residue field $\boldsymbol{k}_1(K)$.

This observation opens the way for inductive definitions.

Definition 5.1. Let M_1 and M_2 be finite K-modules, and let $\phi : M_1 \to M_2$ be a $\Bbbk$-linear homomorphism.

Given a pair of lattices

$$(L_1, L_2) \in \mathrm{Lat}(M_1) \times \mathrm{Lat}(M_2),$$

a *ϕ-refinement* of (L_1, L_2) is a pair

$$(L_1^{\mathrm{s}}, L_2^{\mathrm{b}}) \in \mathrm{Lat}(M_1) \times \mathrm{Lat}(M_2)$$

satisfying these conditions:

- $L_1^{\mathrm{s}} \subseteq L_1$ and $L_2 \subseteq L_2^{\mathrm{b}}$.
- $\phi(L_1^{\mathrm{s}}) \subseteq L_2$ and $\phi(L_1) \subseteq L_2^{\mathrm{b}}$.

Notice that in this situation, there is an induced $\Bbbk$-linear homomorphism

$$\bar{\phi} : L_1/L_1^{\mathrm{s}} \to L_2^{\mathrm{b}}/L_2$$

between the residue modules.

A *system of liftings* for K is a sequence

$$\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$$

where each σ_j is a continuous $\Bbbk$-ring lifting

$$\sigma_j : \boldsymbol{k}_j(K) \to \mathcal{O}_j(K)$$

of the canonical surjection $\mathcal{O}_j(K) \twoheadrightarrow \boldsymbol{k}_j(K)$.

Systems of liftings exist. They are closely related to parametrizations (Definition 2.1).

Given a system of liftings $\boldsymbol{\sigma}$ for K, its truncation

$$\mathrm{d}_1(\boldsymbol{\sigma}) := (\sigma_2, \ldots, \sigma_n)$$

is a system of liftings for the residue field $\boldsymbol{k}_1(K)$.

Definition 5.2. Let $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ be a system of liftings for K.

Let M_1 and M_2 be finite K-modules, and let $\phi : M_1 \to M_2$ be a $\Bbbk$-linear homomorphism.

We call ϕ a *local BT operator* if the following conditions hold:

(0) If $n = 0$ there is no condition.

(1) If $n \geq 1$ there are two conditions:

- Every pair of lattices (L_1, L_2) in $\mathrm{Lat}(M_1) \times \mathrm{Lat}(M_2)$ has some ϕ-refinement $(L_1^{\mathrm{s}}, L_2^{\mathrm{b}})$.
- For every pair (L_1, L_2), and every ϕ-refinement $(L_1^{\mathrm{s}}, L_2^{\mathrm{b}})$ of it, the induced homomorphism
  $$\bar{\phi} : L_1/L_1^{\mathrm{s}} \to L_2^{\mathrm{b}}/L_2$$
  between the residue modules is a BT operator over the TLF $\boldsymbol{k}_1(K)$.

  Here the residue modules are $\boldsymbol{k}_1(K)$-modules via σ_1, and $\boldsymbol{k}_1(K)$ is equipped with the system of liftings $\mathrm{d}_1(\boldsymbol{\sigma})$.

Definition 5.3. In the situation of Definition 5.2, the set of BT operators $\phi : M_1 \to M_2$ is denoted by $\mathrm{Hom}^{\mathrm{BT}}_{\Bbbk,\boldsymbol{\sigma}}(M_1, M_2)$.

Theorem 5.4. *([Ye1]) Let K be an n-dimensional TLF, with system of liftings* $\boldsymbol{\sigma}$.

Then the set

$$\mathrm{E}^{\mathrm{loc}}_{\boldsymbol{\sigma}}(K) := \mathrm{Hom}^{\mathrm{BT}}_{\Bbbk,\boldsymbol{\sigma}}(K, K)$$

is an n-dimensional cubically decomposed ring of operators on K.

It is somewhat troubling to have a dependence on $\boldsymbol{\sigma}$.

The next theorem, which is the main result of [Ye1], takes care of that.

Theorem 5.5. *Let K be an n-dimensional TLF, with systems of liftings* $\boldsymbol{\sigma}$ *and* $\boldsymbol{\sigma}'$.

Then there is equality

$$\mathrm{E}^{\mathrm{loc}}_{\boldsymbol{\sigma}}(K) = \mathrm{E}^{\mathrm{loc}}_{\boldsymbol{\sigma}'}(K)$$

of n-dimensional cubically decomposed ring of operators on K.

The proof is pretty hard. It uses continuous differential operators.

Definition 5.6. We write $\mathrm{E}^{\mathrm{loc}}(K) := \mathrm{E}^{\mathrm{loc}}_{\boldsymbol{\sigma}}(K)$, where $\boldsymbol{\sigma}$ is any system of liftings for K. We call $\mathrm{E}^{\mathrm{loc}}(K)$ the *ring of local BT operators* on the TLF K.

According to Theorem 4.2, the cubically decomposed ring of operators $\mathrm{E}^{\mathrm{loc}}(K)$ gives rise to a residue functional, the BT residue

$$\mathrm{Res}^{\mathrm{BT}}_{K/\Bbbk;\mathrm{E}^{\mathrm{loc}}(K)} : \Omega^n_{K/\Bbbk} \to \Bbbk.$$

To simplify notation we denote this functional by $\mathrm{Res}^{\mathrm{BT}}_{K/\Bbbk}$.

Recall that there is also the TLF residue functional

$$\mathrm{Res}^{\mathrm{TLF}}_{K/\Bbbk} : \Omega^{n,\mathrm{sep}}_{K/\Bbbk} \to \Bbbk$$

from Theorem 3.4.

Comparing these two residue functionals is one of our goals.

Conjecture 5.7. ([Ye4]) *Let K be an n-dimensional TLF over* $\Bbbk$.

Then the following diagram is commutative:

$$\begin{tikzcd} \Omega^n_{K/\Bbbk} \arrow[r, "\mathrm{can}"] \arrow[dr, "\mathrm{Res}^{\mathrm{BT}}_{K/\Bbbk}"'] & \Omega^{n,\mathrm{sep}}_{K/\Bbbk} \arrow[d, "\mathrm{Res}^{\mathrm{TLF}}_{K/\Bbbk}"] \\ & \Bbbk \end{tikzcd}$$

The horizontal arrow labelled “can” is the canonical surjection from (3.2).

For $n = 1$ this was proved by Tate in [Ta].

But for $n \geq 2$ it is an open problem.

The main difficulty is *proving that* $\mathrm{Res}^{\mathrm{BT}}_{K/\Bbbk}$ *is continuous*.

6. The Ring of Geometric BT Operators

Suppose X is a finite type $\Bbbk$-scheme.

By a *chain of points* in X we mean a sequence $\xi = (x_0, \ldots, x_n)$ of points, such that x_i is a specialization of x_{i-1}.

The chain ξ is *saturated* if each x_i is an immediate specialization of x_{i-1}.

The chain ξ is *maximal* if it is saturated, x_0 is a generic point of X, and x_n is a closed point.

In [Be], Beilinson defined a completion operation, which is a special case of his higher adeles.

Given a quasi-coherent sheaf $\mathcal{M}$ on X, and a chain ξ, the *Beilinson completion* of $\mathcal{M}$ along ξ is a $\Bbbk$-module $\mathcal{M}_\xi$, gotten by an n-fold zig-zag of inverse and direct limits.

The completion $\mathcal{M}_\xi$ comes equipped with a $\Bbbk$-linear topology.

Example 6.1. Assume $n = 0$, so that $\xi = (x_0)$. Let $\mathcal{M}$ be a coherent sheaf on X.

The Beilinson completion here is $\mathcal{M}_\xi = \widehat{\mathcal{M}}_{x_0}$, the $\mathfrak{m}_{x_0}$-adic completion of the stalk $\mathcal{M}_{x_0}$.

The topology is the $\mathfrak{m}_{x_0}$-adic topology.

If X is an integral scheme, then its function field $\boldsymbol{k}(X)$ can be viewed as a quasi-coherent sheaf (constant on X).

Theorem 6.2. *([Pa1], [Be], [Ye1]) Let X be an integral finite type $\Bbbk$-scheme of dimension n, and let ξ be a maximal chain in X.*

Then the Beilinson completion $\boldsymbol{k}(X)_\xi$ is a finite product of n-dimensional TLFs.

The number of factors of the reduced artinian ring $\boldsymbol{k}(X)_\xi$ depends on the singularities of the 1-dimensional local rings $\mathcal{O}_{X_i, x_{i+1}}$, where $\xi = (x_0, \ldots, x_n)$ and $X_i := \overline{\{x_i\}}_{\mathrm{red}}$.

The Beilinson completion does more:

Theorem 6.3. *([Ta], [Be], [Br2]) Let X be an integral finite type $\Bbbk$-scheme of dimension n, and let ξ be a maximal chain in X.*

There is an n-dimensional cubically decomposed ring of operators $\mathrm{E}^{\mathrm{geo}}(\boldsymbol{k}(X)_\xi)$ on the completion $\boldsymbol{k}(X)_\xi$, called the ring of geometric BT operators.

The operators in $\mathrm{E}^{\mathrm{geo}}(\boldsymbol{k}(X)_\xi)$ are defined in terms of lattices, like the local BT operators (see Section 5).

But here the lattices arise from quasi-coherent sheaves on the scheme X, and the inductive process is on the length of chains.

We know that the completion $\boldsymbol{k}(X)_\xi$ is a finite product of n-dimensional TLFs, say $\boldsymbol{k}(X)_\xi = \prod_j K_j$.

Let us define the ring of local BT operators on $\boldsymbol{k}(X)_\xi$ to be

$$\mathrm{E}^{\mathrm{loc}}(\boldsymbol{k}(X)_\xi) := \prod_j \mathrm{E}^{\mathrm{loc}}(K_j).$$

Here is another conjecture from [Ye4]:

Conjecture 6.4. *Let X be an integral n-dimensional finite type $\Bbbk$-scheme, with function field $\boldsymbol{k}(X)$, and let ξ be a maximal chain in X.*

Then

$$\mathrm{E}^{\mathrm{geo}}(\boldsymbol{k}(X)_\xi) = \mathrm{E}^{\mathrm{loc}}(\boldsymbol{k}(X)_\xi),$$

as n-dimensional cubically decomposed rings of operators on the completion $\boldsymbol{k}(X)_\xi$.

In the preprint [BGW5], the authors announced a proof of Conjecture 6.4.

When planning to give this talk, I wanted to explain this proof.

However, after reading the paper [BGW5] carefully, I still do not understand some of the arguments. So the discussion of the proof shall have to wait to another lecture...

$\sim$ *END* $\sim$

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